We will now show that the sublattice L', generated by X is exactly the set of all finite sums of finite products of elements of X. For this, it is sufficient to prove:

$$\Big(\sum_1^n A_i\Big)\Big(\sum_1^m B_j\Big) = \sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} A_i B_j\,. \tag{7}$$

Assume, without loss of generality, that L has a least element which is contained in X. Then (6) implies (7) for $n=1$. Suppose (7) is true for $n=q$; then,

$$\begin{aligned}
\Big(\sum_1^{q+1} A_i\Big)\Big(\sum_1^m B_j\Big) &= \Big(\sum_1^{q+1} A_i\Big)\Big(\sum_1^q A_i + \sum_1^m B_j\Big)\Big(\prod_1^m B_j\Big)\\
&= \Big[\sum_1^q A_i + A_{q+1}\Big(\sum_1^q A_i + \sum_1^m B_j\Big)\Big]\Big[\sum_1^m B_j\Big]\\
&= \Big(\sum_1^q A_i + \sum_{i=1}^q A_{q+1}A_i + \sum_{j=1}^m A_{q+1}B_j\Big)\Big(\sum_1^m B_j\Big)\\
&= \Big(\sum_1^q A_i + \sum_{j=1}^m A_{q+1}B_j\Big)\Big(\sum_1^m B_j\Big)\\
&= \Big(\sum_1^q A_i\Big)\Big(\sum_1^m B_j\Big) + \sum_{j=1}^m A_{q+1}B_j\\
&= \sum_{\substack{1\leqslant i\leqslant q\\ 1\leqslant j\leqslant m}} A_i B_j + \sum_{j=1}^m A_{q+1}B_j = \sum_{\substack{1\leqslant i\leqslant q+1\\ 1\leqslant j\leqslant m}} A_i B_j\,.
\end{aligned}$$

Finally, since (7) implies that L' is distributive, the proof is complete.

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Randomly hamiltonian digraphs

by

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Introduction. In [1] a randomly hamiltonian graph was defined as a graph G for which a hamiltonian cycle always results upon starting at any vertex of G and successively proceeding to any adjacent vertex not yet encountered, with the final vertex adjacent to the initial vertex. These graphs were characterized in [1] as complete graphs, cycles, and regular complete bipartite graphs. In this article we define and characterize in an analogous manner randomly hamiltonian directed graphs. Furthermore, the characterization given in [1] is shown to be a corollary of the result obtained here.

Definitions and notation. A directed graph (or simply digraph) D is called *hamiltonian* if there exists a (directed) cycle containing all vertices of D; such a cycle is also referred to as *hamiltonian*. A digraph D is *randomly hamiltonian* if a hamiltonian cycle automatically results upon starting at any vertex and successively proceeding to any vertex which has not yet been visited and which is adjacent from the preceding vertex, where also the final vertex is adjacent to the initial vertex.

By way of notation, we represent the complete symmetric digraph having p vertices and $p(p-1)$ arcs by K_p. Also we denote the cycle with p vertices (and p arcs) by C_p and the symmetric cycle (with $2p$ arcs) by S_p. By $D(n,k)$ we mean the digraph whose vertex set V can be expressed as the disjoint union $\bigcup_{i=1}^{n} V_i$, where $|V_i| = k$, $1\leqslant i\leqslant n$, and uw is an arc of D if and only if $u\in V_i$, $w\in V_j$, and $j-i\equiv 1 \pmod n$. We note that the digraph $D(p,1)$ is the cycle C_p. The digraphs K_4, S_5, and $D(3,2)$, each of which is randomly hamiltonian, are shown in Figure 1.

Throughout this article, wherever we refer to a randomly hamiltonian (and therefore hamiltonian) digraph D we shall assume the existence of some fixed hamiltonian cycle C whose p vertices are labeled consecutively $v_1, v_2, \ldots, v_p$. A path P: $v_i, v_{i+1}, \ldots, v_{i+n-1}$ (the subscripts expressed modulo p), $n\geqslant 2$, together with the arc $v_{i+n-1}v_i$ is referred to as an *outer n-cycle* or simply *outer cycle* if the length n is not relevant,

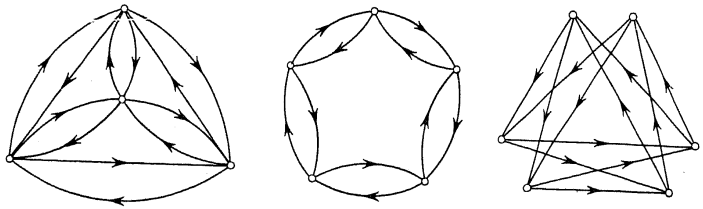

Fig. 1. Three randomly hamiltonian digraphs

while P together with the arc v_iv_{i+n-1} will be called an *outer transitive n-cycle* or *outer transitive cycle*. In such a case, we call $v_{i+n-1}v_i$ a *cyclic arc* and v_iv_{i+n-1} a *transitive arc*. For $n = 3$ the terminology outer triangle and outer transitive triangle is employed.

The following lemma will be useful in the proof of our main result.

LEMMA. *If C' is an outer transitive cycle of minimum length in a randomly hamiltonian digraph D and v_iv_j is the corresponding transitive arc, then $v_{i+k}v_{j+k}$ is an arc of D for $0 < k < p$, where $i+k$ and $j+k$ are expressed modulo p.*

Proof. Consider a path P which begins at the vertex v_i and proceeds along the arc v_iv_j to v_j. We then follow along the hamiltonian cycle C in the order $v_{j+1}, v_{j+2}, \ldots, v_{i-1}$. Since D is randomly hamiltonian, v_i is already on P, and v_iv_j belongs to an outer transitive cycle of minimum length, this implies that v_{j-1} is necessarily the vertex of P which follows v_{i-1}. In particular, this implies that $v_{i-1}v_{j-1}$ is an arc of D. If we now start with the arc $v_{i-1}v_{j-1}$ and proceed in exactly the same way as before, we can conclude that $v_{i-2}v_{j-2}$ is an arc of D (where, as always, the subscripts are expressed modulo p). Thus, beginning with v_iv_j and proceeding as indicated $p-k$ times, we arrive at the fact that $v_{i-(p-k)}v_{j-(p-k)} = v_{i+k}v_{j+k}$ is an arc of D.

THEOREM. *A digraph D with p ($\geqslant 2$) vertices is randomly hamiltonian if and only if it is one of the following*: (i) *the symmetric cycle S_p*, (ii) *the complete symmetric digraph K_p*, (iii) *the digraph $D(n, k)$ for some n and k, where $nk = p$.*

Proof. That each of the digraphs S_p, K_p, and $D(n, k)$ is randomly hamiltonian is easily verified. Hence, let D be any randomly hamiltonian digraph with p ($\geqslant 2$) vertices, and let C be a fixed hamiltonian cycle whose vertices are labeled consecutively $v_1, v_2, \ldots, v_p$.

If D has no arcs other than those of C, then D is the digraph $C_p = D(p, 1)$. If D has no arcs which join non-consecutive vertices of C but contains an arc $v_{j+1}v_j$, then by beginning a path with v_{j+1}, v_j we see

that each of the arcs $v_{i+1}v_i$ must belong to D, implying that D is the symmetric cycle S_p.

Thus, without loss of generality, we assume henceforth that D contains an arc which joins two non-consecutive vertices of C, implying the existence of outer transitive cycles.

If D contains an outer transitive triangle (which is necessarily an outer transitive cycle of minimum length), then by the lemma, D contains all arcs of the type v_iv_{i+2}. In this case, D is the complete symmetric digraph K_p. To see this, let v_i and v_j be any two distinct vertices of D, where $j \neq i+1$ (modulo p). We show here that v_iv_j is an arc of D. Begin a path P with the vertex v_{i+1} and proceed along C in the order: $v_{i+2}, v_{i+3}, \ldots, v_{j-1}$. We then take v_{j+1} via the transitive arc $v_{j-1}v_{j+1}$ and follow this with $v_{j+2}, v_{j+3}, \ldots, v_i$. (It is possible that $v_{j+1} = v_i$.) Since D is randomly hamiltonian and the portion of P thus far constructed fails only to contain v_j, the arc v_iv_j belongs to P and so also to D.

We therefore assume that $n+2$ is the length of the smallest outer aransitive cycle of D, where $n+2 \geqslant 4$. Let v_iv_{i+n+1} be a transitive arc of an outer transitive $(n+2)$-cycle. We now show that D contains the arc $v_{i+n+1}v_r$, $i < r < i+n+1$, if and only if $r = i+2$. By considering any path which begins with v_{i+n+2}, proceeds along C to the vertices $v_{i+n+3}, v_{i+n+4}, \ldots, v_i$, and then encounters v_{i+n+1}, we see that D contains an arc of the aforementioned type. The arc $v_{i+n+1}v_{i+1}$ is not in D, for otherwise we could construct the following path P: $v_{i+n+2}, v_{i+n+3}, \ldots, v_i$, $v_{i+n+1}, v_{i+1}, v_{i+2}, \ldots, v_{i+n}$. Since P contains all the vertices of D, $v_{i+n}v_{i+n+2}$ would belong to D implying the existence of an outer transitive triangle. If $n+2 = 4$, we have the desired result; if not, suppose that D contains an arc $v_{i+n+1}\, v_s$, where $i+2 < s < i+n+1$. Consider now a path Q containing all vertices of D which begins as follows: $v_{i+n+2}, v_{i+n+3}, \ldots, v_i$, v_{i+n+1}, v_s, $v_{s+1}, \ldots, v_{i+n}$. Since D is randomly hamiltonian and v_iv_{i+n+1} is a transitive arc of an outer transitive $(n+2)$-cycle, the final vertex of Q is necessarily v_{i+1} which implies that v_{i+n} must be adjacent to a vertex v_m, where $i+1 < m < s$. A contradiction is now reached by considering a path Q' which begins as $v_i, v_{i+1}, v_{i+n+2}, v_{i+n+3}, \ldots, v_{i-1}$, v_{i+n}, v_m. Since D is randomly hamiltonian, Q' must be the initial portion of a hamiltonian cycle so that there must be an arc v_tv_{i+n+1}, where $i+1 < t < i+n$, but this determines an outer transitive cycle of length less than $n+2$. Hence, for each transitive arc v_iv_{i+n+1}, there is a corresponding cyclic arc $v_{i+n+1}v_{i+2}$.

We now show that D is the digraph $D(n, k)$, where $p = nk$. In order to prove this, we show that v_iv_j is an arc of D if and only if $j - i \equiv 1 \pmod{n}$. Assume first that $j = qn+i+1$ for some q. (Of course, we already know v_iv_j is an arc of D for $q = 0$ or 1.) Employing the results just obtained, we consider the following path P: $v_j, v_{j+1}, \ldots, v_{i-1}$, $v_{i+n}, v_{i+1}, v_{1+2}, \ldots$

$\ldots, v_{i+n-1}, v_{i+2n}, v_{i+n+1}, v_{i+n+2}, \ldots, v_{i+2n-1}$. Continuing in this way, we arrive at the vertex $v_{i+qn} = v_{j-1}$, from which we proceed to $v_{j-n}, v_{j-n+1}, \ldots$ $\ldots, v_{j-2}$. The path P thus far contains all vertices of D with the exception of v_i so that D contains the arcs $v_{j-2}v_i$ and v_iv_j. Conversely, suppose v_iv_j is an arc of D and $j-i \not\equiv 1 \pmod{n}$. We then construct a path P' which begins as follows: $v_i, v_j, v_{j+1}, \ldots, v_{i-1}, v_{i+n}, v_{i+1}, v_{i+2}, \ldots, v_{i+n-1}, v_{i+2n}$. We then continue as before until we reach the final vertex of the type v_{i+tn} which is not thus far on P'. The next vertices of P' would then be $v_{i+tn}, v_{i+(t-1)n-1}, v_{i+(t-1)n}, \ldots, v_{i+tn-1}$. Since $j \neq i+(t+1)\ n+1$, the vertex of P' following v_{i+tn-1} necessarily defines an outer transitive cycle of length less than $n+2$, and this is a contradiction. Because v_pv_1 obviously belongs to D, we have $1-p \equiv 1 \pmod{n}$, or there exists an integer k such that $p = nk$. If for each i, $1 \leqslant i \leqslant n$, we let $V_i = \{v_s|\ s \equiv i \pmod{n}\}$, D is seen to be the digraph $D(n, k)$. This completes the proof.

Each randomly hamiltonian graph may be considered a randomly hamiltonian digraph (obtained by replacing each edge by a symmetric pair of arcs), but among the randomly hamiltonian digraphs with p vertices, only S_p, K_p, and $D(2, p/2)$ are (ordinary) graphs. Thus, we obtain as a corollary the result presented in [1].

Corollary. *A graph is randomly hamiltonian if and only if it is a cycle, a complete graph, or a regular complete bipartite graph.*

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Extended operations and relations on the class of ordinal numbers

by

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§ 1. Introduction. This is intended as a sequel to the paper *An extended arithmetic of ordinal numbers* by John Doner and Alfred Tarski. Thus, our notation is the same as theirs. For the sake of convenience we shall repeat several of their definitions. When referring to a theorem, lemma, etc. in the Doner–Tarski paper we shall prefix the numeral by the symbol "D–T".

Lower case greek letters $\alpha, \beta, \gamma, \ldots$ represent ordinal numbers and the class of all ordinal numbers is denoted by Ω.

Definition 1. For each $\gamma \in \Omega$, O_γ is a *binary operation* from $\Omega \times \Omega$ to Ω such that for all $\alpha, \beta \in \Omega$,

(i) $\alpha O_\gamma \beta = \alpha + \beta$, if $\gamma = 0$;

(ii) $\alpha O_\gamma \beta = \bigcup\limits_{\eta<\beta, \zeta<\gamma} [(\alpha O_\gamma \eta) O_\zeta \alpha]$, if $\gamma \geqslant 1$.

Definition 2. For each $\gamma \in \Omega$, R_γ and L_γ are *relations* such that

(i) $R_\gamma, L_\gamma \subseteq \Omega \times \Omega$;

(ii) For all $\alpha, \beta \in \Omega$

$$\alpha R_\gamma \beta \quad \text{iff} \quad (\exists \delta)(\delta \neq 0 \text{ and } \alpha O_\gamma \delta = \beta),$$
$$\alpha L_\gamma \beta \quad \text{iff} \quad (\exists \delta)(\delta \neq 0 \text{ and } \delta O_\gamma \alpha = \beta).$$

(For $\gamma = 0, 1$, R_γ and L_γ have been described in Rubin [3].)

Our results include the following: If $A = \{\alpha\colon\ \alpha R_\gamma \beta\}$ for some $\beta, \gamma \in \Omega$, $\beta > 0$, and $\emptyset \neq X \subseteq A$ then $\bigcup X \in A$. If γ is a limit ordinal and $\Omega' = \Omega \sim\{0\}$, then $\langle \Omega', R_\gamma \rangle$ is a complete lattice. Moreover, for γ a limit ordinal we have obtained necessary and sufficient conditions for O_γ to be commutative and associative. Also, for $\alpha, \beta, \gamma \in \Omega$ we have obtained necessary and sufficient conditions on α' such that $\alpha O_\gamma \beta = \alpha' O_\gamma \beta$.

We shall assume the traditional arithmetic of ordinal numbers. (Sierpiński [5] is an excellent reference.) We frequently use the following well-known result.