We will now show that the sublattice $L'$, generated by $X$ is exactly the set of all finite sums of finite products of elements of $X$. For this, it is sufficient to prove:

$$
\left( \sum_{i=1}^{n} A_i \right) \left( \sum_{j=1}^{m} B_j \right) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} A_i B_j.
$$

Assume, without loss of generality, that $L$ has a least element which is contained in $X$. Then (6) implies (7) for $n = 1$. Suppose (7) is true for $n = q$; then,

$$
\left( \sum_{i=1}^{q+1} A_i \right) \left( \sum_{j=1}^{m} B_j \right) = \left( \sum_{i=1}^{q} A_i + A_{q+1} \right) \left( \sum_{j=1}^{m} B_j \right) = \left( \sum_{i=1}^{q} A_i \right) \left( \sum_{j=1}^{m} B_j \right) + A_{q+1} \left( \sum_{j=1}^{m} B_j \right).
$$

Finally, since (7) implies that $L'$ is distributive, the proof is complete.

**Randomly hamiltonian digraphs**

by

G. Chartrand (Kalamazoo, Mich.), H. V. Kronk (Binghamton, N. Y.),

Don R. Lick (Kalamazoo, Mich.)

**Introduction.** In [1] a randomly hamiltonian graph was defined as a graph $G$ for which a hamiltonian cycle always results upon starting at any vertex of $G$ and successively proceeding to any adjacent vertex not yet encountered, with the final vertex adjacent to the initial vertex. These graphs were characterized in [1] as complete graphs, cycles, and regular complete bipartite graphs. In this article we define and characterize in an analogous manner randomly hamiltonian directed graphs. Furthermore, the characterization given in [1] is shown to be a corollary of the result obtained here.

**Definitions and notation.** A directed graph (or simply digraph) $D$ is called hamiltonian if there exists a (directed) cycle containing all vertices of $D$; such a cycle is also referred to as hamiltonian. A digraph $D$ is randomly hamiltonian if a hamiltonian cycle automatically results upon starting at any vertex and successively proceeding to any vertex which has not yet been visited and which is adjacent from the preceding vertex, where also the final vertex is adjacent to the initial vertex.

By way of notation, we represent the complete symmetric digraph having $p$ vertices and $p(p-1)$ arcs by $K_p$. Also we denote the cycle with $p$ vertices (and $p$ arcs) by $C_p$ and the symmetric cycle (with $2p$ arcs) by $S_p$. By $D(n, k)$ we mean the digraph whose vertex set $V$ can be expressed as the disjoint union $\bigcup_{i=1}^{n} V_i$, where $|V_i| = k$, $1 \leq i \leq n$, and $se$ is an arc of $D$ if and only if $u \in V_i, v \in V_j$, and $j-i \equiv 1 \pmod{n}$. We note that the digraph $D(p, 1)$ is the cycle $C_p$. The digraphs $K_p, S_p$, and $D(3, 2)$, each of which is randomly hamiltonian, are shown in Figure 1.

Throughout this article, wherever we refer to a randomly hamiltonian (and therefore hamiltonian) digraph $D$ we shall assume the existence of some fixed hamiltonian cycle $C$ whose $p$ vertices are labeled consecutively $v_1, v_2, \ldots, v_p$. A path $P: v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ (the subscripts expressed modulo $p$), $n \geq 2$, together with the arc $v_{i_k}v_{i_1}$ is referred to as an outer $n$-cycle or simply outer cycle if the length $n$ is not relevant.
that each of the arcs $v_{t+1}v_t$ must belong to $D$, implying that $D$ is the symmetric cycle $S_p$.

Thus, without loss of generality, we assume henceforth that $D$ contains an arc which joins two non-consecutive vertices of $C$, implying the existence of outer transitive cycles.

If $D$ contains an outer transitive triangle (which is necessarily an outer transitive cycle of minimum length), then by the lemma, $D$ contains all arcs of the type $v_{j+1}v_j$. In this case, $D$ is the complete symmetric digraph $K_p$. To see this, let $s_1, s_2$ be any two distinct vertices of $D$, where $j \neq i + 1$ (modulo $p$). We show here that $v_{i,j}$ is an arc of $D$. Begin a path $P$ with the vertex $v_{i,j}$ and proceed along $C$ in the order: $v_{i,j}, v_{i+1,j}, \ldots, v_{j-1,j}$.

We therefore assume that $n + 2$ is the length of the smallest outer orientable cycle of $D$, where $n + 2 \geq 4$. Let $v_{i+1,j}$ be a transitive arc of an outer transitive $(n+2)$-cycle. Now show that $D$ contains the arc $v_{i+1,j}, v_{i,j}$, if and only if $r = i + 2$. By considering any path which begins with $v_{i+1,j}$, proceeds along $C$ to the vertices $v_{i+1,j}, v_{i,j}, v_{i,j}$, and then encounters $v_{i,j}$, we see that $D$ contains an arc of the aforementioned type. The arc $v_{i+1,j}$ is not in $D$, but otherwise we could construct the following path $P$: $v_{i,j}, v_{i+1,j}, \ldots, v_{i+1,j}$.

Since $D$ contains all arcs of $D$, $v_{i+1,j}$ would belong to $D$ implying the existence of an outer transitive triangle. If $n + 2 = 4$, we have the desired result; if not, suppose that $D$ contains an arc $v_{i+1,j}$, where $i + 2 < s < i + n + 1$. Consider now a path $P$ containing all vertices of $D$, which begins as follows: $v_{i+1,j}, v_{i+1,j}, \ldots, v_{i,j}$, $v_{i,j}$. Since $D$ is randomly Hamiltonian and $v_{i+1,j}$ is a transitive arc of an outer transitive $(n+2)$-cycle, the final vertex of $P$ is necessarily $v_{i,j}$, which implies that $v_{i,j}$ must be adjacent to a vertex $v_{i,j}$, where $i + 1 < s < s$. A contradiction is now reached by considering a path $Q$ which begins as $v_{i,j}, v_{i,j}, v_{i,j}, \ldots, v_{i,j}, v_{i,j}$.

We now show that $D$ is the digraph $D(n,k)$, where $p = nk$. In order to prove this, we show that $v_{i,j}$ is an arc of $D$ if and only if $j = i + 1$ (mod $n$).

Assume first that $j = i + 1$ for some $q$. (Of course, we already know $v_{i,j}$ is an arc of $D$ for $q = 0$ or 1.) Employing the results just obtained, we consider the following path $P$: $v_{i,j}, v_{i+1,j}, \ldots, v_{i+1,j}, v_{i,j}$.

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Continuing in this way, we 
arrive at the vertex \( v_{1+m} = v_1 \), from which we proceed to \( v_{1+m+1}, \ldots, v_{1+2m} \). The path \( P \) thus far contains all vertices of \( D \) with the exception of \( v_1 \) so that \( D \) contains the arcs \( v_1 \rightarrow v_i \) and \( v_i \rightarrow v_1 \). Conversely, suppose \( \{v_i\} \) is an arc of \( D \) and \( j - i \neq 1 \) (mod \( n \)). We then construct a path \( P' \) which begins as follows: \( v_1, v_i, v_{i+1}, \ldots, v_1 \). We then continue as before until we reach the final vertex of the type \( v_{i+m} \) which is not thus far on \( P' \). The next vertices of \( P' \) would then be \( v_{i+m}, v_{i+m+1}, \ldots, v_{i+m} \). Since \( j \neq i + (i+1) \), \( n+1 \), the vertex of \( P' \) following \( v_{i+m} \), necessarily defines an outer transitive cycle of length less than \( n+1 \), and this is a contradiction. Because \( v_n \) obviously belongs to \( D \), we have \( 1 \equiv p = 1 \) (mod \( n \)), or there exists an integer \( k \) such that \( p = nk \). If for each \( i, 1 \leq i \leq n \), we let \( V_i = \{v_i | x = i \) (mod \( n \)}), \( D \) is seen to be the digraph \( D(n, k) \). This completes the proof.

Each randomly hamiltonian graph may be considered a randomly hamiltonian digraph (obtained by replacing each edge by a symmetric arc), but among the randomly hamiltonian digraphs with \( p \) vertices, only \( S_p, k_p, \) and \( D(3, p/2) \) are (ordinary) graphs. Thus, we obtain as a corollary the result presented in [1].

**Corollary.** A graph is randomly hamiltonian if and only if it is a cycle, a complete graph, or a regular complete bipartite graph.

References


\[ \begin{align*}
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\end{align*} \]

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**Extended operations and relations on the class of ordinal numbers**

by

Arthur L. Rubin and Jean E. Rubin (Lafayette, Ind.)

§1. Introduction. This is intended as a sequel to the paper As extended arithmetic of ordinal numbers by John Doner and Alfred Tarski. Thus, our notation is the same as theirs. For the sake of convenience we shall repeat several of their definitions. When referring to a theorem, lemma, etc. in the Doner–Tarski paper we shall prefix the numeral by the symbol "D-T".

Lower case greek letters \( \alpha, \beta, \gamma, \ldots \) represent ordinal numbers and the class of all ordinal numbers is denoted by \( \Omega \).

**Definition 1.** For each \( \gamma \in \Omega \), \( O_\gamma \) is a binary operation from \( \Omega \times \Omega \) to \( \Omega \) such that for all \( \alpha, \beta \in \Omega \),

\[
(i) \quad aO_\gamma b = a + \beta, \text{ if } \gamma = 0;
\]

\[
(ii) \quad aO_\gamma b = \bigcup_{\gamma \leq \gamma, \gamma \neq \gamma} ([aO_\gamma 0] \cup a), \text{ if } \gamma > 1.
\]

**Definition 2.** For each \( \gamma \in \Omega \), \( R_\gamma \) and \( L_\gamma \) are relations such that

\[
(i) \quad R_\gamma, L_\gamma \subseteq \Omega \times \Omega;
\]

\[
(ii) \quad \text{For all }\alpha, \beta \in \Omega
\]

\[
aR_\gamma a \iff (\exists \delta)(\delta \neq 0 \text{ and } aO_\gamma \delta = \beta);
\]

\[
aL_\gamma a \iff (\exists \delta)(\delta \neq 0 \text{ and } \delta O_\gamma a = \beta).
\]

(For \( \gamma = 0, 1 \), \( R_\gamma \) and \( L_\gamma \) have been described in Rubin [3].)

Our results include the following: If \( \Delta = \{a \mid aR_\gamma \} \) for some \( \gamma \in \Omega \), \( \beta > 0 \), and \( \Theta \neq \subseteq \Delta \) then \( \bigcup \Delta \notin \Delta \). If \( \gamma \) is a limit ordinal and \( \Omega \setminus \Omega \sim \Omega \), then \( \Omega^\prime, \Omega \) is a complete lattice. Moreover, for \( \gamma \) a limit ordinal we have obtained necessary and sufficient conditions for \( O_\gamma \) to be commutative and associative. Also, for \( \alpha, \beta, \gamma \in \Omega \) we have obtained necessary and sufficient conditions on \( \alpha' \) such that \( aO_\gamma \beta = a' O_\gamma \beta \).

We shall assume the traditional arithmetic of ordinal numbers. (Sierpinski [5] is an excellent reference.) We frequently use the following well-known result.

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