A note on distributive sublattices of a modular lattice*

by

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In [2], B. Jónsson used Zorn's Lemma to show that a non-empty subset X of a modular lattice generates a distributive sublattice if and only if

$$(1)\qquad \Big(\sum_{i=1}^{m} a_i\Big)\prod_{j=1}^{n} b_j = \sum_{i=1}^{m}\Big(a_i \prod_{j=1}^{n} b_j\Big) \quad \text{whenever} \quad a_1, \ldots, a_m, b_1, \ldots, b_n \in X .$$

The purpose of this paper is to prove this theorem without recourse to the axiom of choice or any of its equivalent formulations.

In order to simplify notation, we establish the following conventions. Let X be a non-empty subset of a modular lattice L. The elements of X will be denoted by $a_1, a_2, \ldots, b_1, b_2, \ldots,$ and finite products of elements of X will sometimes be denoted by $A, B, C, \ldots$ Also, we will abbreviate $\prod_{i=1}^{n} a_i$ by $\prod_1^n a_i$ whenever possible. For the following two lemmas, we assume (1), and note that

$$(2)\qquad \prod_1^m a_i + \sum_1^n b_j = \prod_{i=1}^{m}\Big(a_i + \sum_1^n b_j\Big) \qquad ([2],\ \text{p. } 683)\,.$$

LEMMA 1. *If m, n are positive integers, then*

$$(3)\qquad A\Big(\prod_1^n b_j + \sum_1^m a_i\Big) = A\prod_1^n b_j + \sum_1^m A a_i .$$

Proof. For $n = 1$, (3) reduces to (1). Assume (3) holds for $n = q$. Then

$$B\Big(\prod_1^{q+1} b_j + \sum_1^m a_i\Big) = B\Big(b_{q+1} + \sum_1^m a_i\Big)\prod_1^q\Big(b_j + \sum_1^m a_i\Big)$$
$$= B\Big(b_{q+1} + \sum_1^m a_i\Big)\Big(\prod_1^q b_j + \sum_1^m a_i\Big)$$

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$$= \Big(Bb_{q+1} + \sum_1^m Ba_i\Big)\Big(B\prod_1^q b_j + \sum_1^m Ba_i\Big)$$

$$= Bb_{q+1}\Big(B\prod_1^q b_j + \sum_1^m Ba_i\Big) + \sum_1^m Ba_i$$

$$= Bb_{q+1}\Big(\prod_1^q b_j + \sum_1^m a_i\Big) + \sum_1^m Ba_i$$

$$= B\prod_1^{q+1} b_j + \sum_{i=1}^m Bb_{q+1}a_i + \sum_1^m Ba_i$$

$$= B\prod_1^{q+1} b_j + \sum_1^m Ba_i\,.$$

LEMMA 2. *Let r, s be positive integers. For each positive integer t, let $P(r, s, t)$ be the statement:*

$$\prod_1^t a_i + \sum_1^r B_j + \sum_1^s c_k = \prod_{i=1}^t \Big(a_i + \sum_1^r B_j + \sum_1^s c_k\Big) \tag{4}$$

and

$$A\Big(\prod_1^t a_i + \sum_1^r B_j + \sum_1^s c_k\Big) = A\prod_1^t a_i + \sum_1^r AB_j + \sum_1^s Ac_k\,. \tag{5}$$

Then $P(r, s, 1), \ldots, P(r, s, q) \Rightarrow P(r, s, q+1)$ *for each positive integer q.*

Proof. Assume the validity of $P(r, s, 1), \ldots, P(r, s, q)$:

$$\prod_{i=1}^{q+1}\Big(a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= \Big(a_{q+1} + \sum_1^r B_j + \sum_1^s c_k\Big)\prod_{i=1}^q\Big(a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= \Big(a_{q+1} + \sum_1^r B_j + \sum_1^s c_k\Big)\Big(\prod_1^q a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= a_{q+1}\Big(\prod_1^q a_i + \sum_1^r B_j + \sum_1^s c_k\Big) + \sum_1^r B_j + \sum_1^s c_k$$

$$= \prod_1^{q+1} a_i + \sum_{j=1}^r a_{q+1}B_j + \sum_{k=1}^s a_{q+1}c_k + \sum_1^r B_j + \sum_1^r c_k$$

$$= \prod_1^{q+1} a_i + \sum_1^r B_j + \sum_1^s c_k\,.$$

We use this to prove (5), as follows:

$$B\Big(\prod_1^{q+1} a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= B\Big(a_{q+1} + \sum_1^r B_j + \sum_1^s c_k\Big)\prod_{i=1}^q\Big(a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= B\Big(a_{q+1} + \sum_1^r B_j + \sum_1^s c_k\Big)\Big(\prod_1^q a_i + \sum_1^r B_j + \sum_1^s c_k\Big)$$

$$= \Big(Ba_{q+1} + \sum_1^r BB_j + \sum_1^s Bc_k\Big)\Big(B\prod_1^q a_i + \sum_1^r BB_j + \sum_1^s Bc_k\Big)$$

$$= B\prod_1^q a_i\Big(Ba_{q+1} + \sum_1^r BB_j + \sum_1^s Bc_k\Big) + \sum_1^r BB_j + \sum_1^s Bc_k$$

$$= B\prod_1^q a_i\Big(a_{q+1} + \sum_1^r B_j + \sum_1^s c_k\Big) + \sum_1^r BB_j + \sum_1^s Bc_k$$

$$= B\prod_1^{q+1} a_i + \sum_{j=1}^r\Big(B\prod_1^q a_i\Big)B_j + \sum_{k=1}^s\Big(B\prod_1^q a_i\Big)c_k + \sum_1^r BB_j + \sum_1^s Bc_k$$

$$= B\prod_1^{q+1} a_i + \sum_1^r BB_i + \sum_1^s Bc_k\,.$$

We can now prove the main result.

THEOREM 3. *Let X be a non-empty subset of a modular lattice L. Then (1) is necessary and sufficient for the sublattice generated by X to be distributive.*

Proof. We will first show that

$$A\Big(\sum_1^n B_j + \sum_1^m c_k\Big) = \sum_1^n AB_j + \sum_1^m Ac_k \quad \text{for all } A, m\,. \tag{6}$$

For $n = 1$, (6) is a restatement of (3). Assume now that

$$A\Big(\sum_1^n B_j + \sum_1^m c_k\Big) = \sum_1^n AB_j + \sum_1^m Ac_k \quad \text{for all } A, m\,.$$

In particular, if s is positive,

$$A\Big(a_1 + \sum_1^n B_j + \sum_1^s c_k\Big) = Aa_1 + \sum_1^n AB_j + \sum_1^s Ac_k\,.$$

Thus in the notation of Lemma 2, $P(n, s, 1)$ is true. So by Lemma 2 and induction $P(n, s, t)$ is true for all s, t. Hence

$$A\Big(B_{n+1} + \sum_1^n B_j + \sum_1^m c_k\Big) = \sum_1^{n+1} AB_j + \sum_1^m Ac_k\,.$$

This completes the proof of (6).

We will now show that the sublattice L', generated by X is exactly the set of all finite sums of finite products of elements of X. For this, it is sufficient to prove:

$$\Big(\sum_1^n A_i\Big)\Big(\sum_1^m B_j\Big)=\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} A_iB_j\,. \tag{7}$$

Assume, without loss of generality, that L has a least element which is contained in X. Then (6) implies (7) for $n=1$. Suppose (7) is true for $n=q$; then,

$$\begin{aligned}\Big(\sum_1^{q+1} A_i\Big)\Big(\sum_1^m B_j\Big) &= \Big(\sum_1^{q+1} A_i\Big)\Big(\sum_1^q A_i+\sum_1^m B_j\Big)\Big(\prod_1^m B_j\Big)\\
&=\Big[\sum_1^q A_i+A_{q+1}\Big(\sum_1^q A_i+\sum_1^m B_j\Big)\Big]\Big[\sum_1^m B_j\Big]\\
&=\Big(\sum_1^q A_i+\sum_{i=1}^q A_{q+1}A_i+\sum_{j=1}^m A_{q+1}B_j\Big)\Big(\sum_1^m B_j\Big)\\
&=\Big(\sum_1^q A_i+\sum_{j=1}^m A_{q+1}B_j\Big)\Big(\sum_1^m B_j\Big)\\
&=\Big(\sum_1^q A_i\Big)\Big(\sum_1^m B_j\Big)+\sum_{j=1}^m A_{q+1}B_j\\
&=\sum_{\substack{1\leqslant i\leqslant q\\ 1\leqslant j\leqslant m}} A_iB_j+\sum_{j=1}^m A_{q+1}B_j=\sum_{\substack{1\leqslant i\leqslant q+1\\ 1\leqslant j\leqslant m}} A_iB_j\,.\end{aligned}$$

Finally, since (7) implies that L' is distributive, the proof is complete.

References

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Randomly hamiltonian digraphs

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Introduction. In [1] a randomly hamiltonian graph was defined as a graph G for which a hamiltonian cycle always results upon starting at any vertex of G and successively proceeding to any adjacent vertex not yet encountered, with the final vertex adjacent to the initial vertex. These graphs were characterized in [1] as complete graphs, cycles, and regular complete bipartite graphs. In this article we define and characterize in an analogous manner randomly hamiltonian directed graphs. Furthermore, the characterization given in [1] is shown to be a corollary of the result obtained here.

Definitions and notation. A directed graph (or simply digraph) D is called *hamiltonian* if there exists a (directed) cycle containing all vertices of D; such a cycle is also referred to as *hamiltonian*. A digraph D is *randomly hamiltonian* if a hamiltonian cycle automatically results upon starting at any vertex and successively proceeding to any vertex which has not yet been visited and which is adjacent from the preceding vertex, where also the final vertex is adjacent to the initial vertex.

By way of notation, we represent the complete symmetric digraph having p vertices and $p(p-1)$ arcs by K_p. Also we denote the cycle with p vertices (and p arcs) by C_p and the symmetric cycle (with $2p$ arcs) by S_p. By $D(n,k)$ we mean the digraph whose vertex set V can be expressed as the disjoint union $\bigcup_{i=1}^n V_i$, where $|V_i|=k$, $1\leqslant i\leqslant n$, and uw is an arc of D if and only if $u\in V_i$, $w\in V_j$, and $j-i\equiv 1\,(\mathrm{mod}\,n)$. We note that the digraph $D(p,1)$ is the cycle C_p. The digraphs K_4, S_5, and $D(3,2)$, each of which is randomly hamiltonian, are shown in Figure 1.

Throughout this article, wherever we refer to a randomly hamiltonian (and therefore hamiltonian) digraph D we shall assume the existence of some fixed hamiltonian cycle C whose p vertices are labeled consecutively $v_1, v_2, \ldots, v_p$. A path P: $v_i, v_{i+1}, \ldots, v_{i+n-1}$ (the subscripts expressed modulo p), $n\geqslant 2$, together with the arc $v_{i+n-1}v_i$ is referred to as an *outer n-cycle* or simply *outer cycle* if the length n is not relevant,