Multiplicative extension operators and locally connected continua

by

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1. Introduction. Let X be a compact metric space. Denote by $C_+(X)$ the space of all continuous non-negative functions on X with the metric:

$$\varrho(f,g) = \sup_{x\in X} |f(x)-g(x)| \quad \text{for} \quad f,g \in C_+(X).$$

We shall write $f_n \rightrightarrows f$ if $\varrho(f_n, f)\to 0$.

Unless otherwise stated, by a space we shall mean a compact metric space. "A map" will mean "a continuous function".

If A and B are closed subsets of a space X, then we define:

$$\operatorname{dist}(A,B) = \sup_{a\in A} d(a,B) + \sup_{b\in B} d(b,A).$$

Let Y be a subspace of a space X. A map $M\colon C_+(Y)\to C_+(X)$ is said to be a *multiplicative extension operator* (*meo*) if:

(i) $(Mf)(y) = f(y)$ for every $y\in Y$ and for every $f\in C_+(Y)$,

(ii) $M(f\cdot g) = Mf\cdot Mg$ for every $f,g\in C_+(Y)$.

A space Y is said to be an $\mathcal{AM}_+$-*space* if for every space $X\supset Y$ there exists a meo from $C_+(Y)$ into $C_+(X)$.

The purpose of the present paper is to prove the following

1.1. THEOREM. *A space Y is an $\mathcal{AM}_+$-space if and only if it is a locally connected continuum.*

Let us mention some known results related to Theorem 1.1.

First we recall two theorems on the existence of operators of extension (here $C(X)$ denotes the space of all real-valued continuous functions on X):

1.2. *For every closed subset Y of an arbitrary metric space X there exists a linear extension operator $L\colon C(Y)\to C(X)$* (Borsuk [2], Dugundji [6]).

1.3. *If* $Y \subset X$ *are topological compact spaces, then there exists a linear multiplicative extension operator from* $C(Y)$ *into* $C(X)$ *if and only if* Y *is a retract of* X (Yoshizava [11]).

For other results on extension operators we refer the reader to [8].

Observe that 1.2 and 1.3 remain valid if we replace the spaces $C(X)$ and $C(Y)$ by $C_+(X)$ and $C_+(Y)$ and "linear operator" by "affine operator".

On the other hand, in the case of multiplicative extension operators the situation becomes entirely different. For instance there exists no multiplicative extension operator from $C(S_1)$ to $C(I^2)$ where I denotes a closed interval and S_1 the boundary of the square I^2 (cf. [9]).

Secondly let us mention other facts, equivalent to the fact that Y is a locally connected continuum:

1.4. Y *is a continuous image of the unit interval.*

1.5. *The hyperspace* 2^Y *is an* AR*-space* (Wojdysławski [10]).

1.6. Y *admits a convex metric* (Bing [1]).

The idea of the proof of Theorem 1.1. The proof of necessity is based upon a result of Bourgin [5] on the representation of multiplicative functionals on $C_+(Y)$. The proof of sufficiency is more sophisticated. It consists of three steps:

1°. We construct (in 3.1 and 3.2) a meo M: $C_+(S_1) \to C_+(I^2)$. This construction is based upon the following result, due to Bott [4]: the third symmetric potency of the circumference is homeomorphic to the three-dimensional Euclidean sphere.

2°. By a theorem of Borsuk, there exists an AR-space Y'' containing Y such that $X = Y'' - Y$ is a polytope. Since Y is a locally connected continuum, a result of Kuratowski implies that there exists a retraction of $Y \cup X_1$ on Y (X_1 denotes the one-dimensional skeleton of X). This enables us to construct a meo N: $C_+(Y) \to C_+(Y \cup X_1)$.

3°. Using the meo M of 1°, we construct a meo from $C_+(X_1)$ to $C_+(X)$, which together with the meo N of 2° enables us to define a meo from $C_+(Y)$ to $C_+(Y'')$. Since Y'' is an AR-space, this implies (cf. Proposition 2.7) that Y is an $\mathcal{AM}_+$-space.

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2. Preliminaries. A map m from $C_+(X)$ into non-negative reals is said to be a *multiplicative functional* if $m(f \cdot g) = m(f) \cdot m(g)$ for every $f, g \in C_+(X)$. The following result is due to Bourgin [5]:

2.1. *If* m *is a non-constant multiplicative functional on* $C_+(X)$, *then there is a closed, non-empty set* $S(m) \subset X$ *such that*

$$m(f) = \prod_{x \in S(m)} f(x)^{a(x)}$$

where $a(x) > 0$ *for* $x \in S(m)$ *and* $\sum_{x \in S(m)} a(x) < \infty$.

We see that:

(1) if m is a non-constant multiplicative functional, then: $m(1) = 1$, $m(0) = 0$ and $m(10) > 1$ (we identify real numbers with correspondent constant functions).

By $\widetilde{m}$ we shall denote the purely atomic regular measure defined by

$$\widetilde{m}(U) = \sum_{x \in U \cap S(m)} a(x) \quad \text{for every set } U \subset X .$$

2.2. Theorem. *If* M: $C_+(Y) \to C_+(X)$ *is a function such that for every* x *in* X *the functional* M_x *defined by*

$$M_x(f) = (Mf)(x)$$

is multiplicative and non-constant, then M *is a multiplicative operator.*

Proof. Clearly $M(f \cdot g) = Mf \cdot Mg$ for every $f, g \in C_+(Y)$, thus we need to prove only the continuity of M. Since X is compact, it is enough to show that if $f_n \rightrightarrows f$ and $x_n \to x$ then $M_{x_n}(f_n) \to M_x(f)$.

Put $m_n = M_{x_n}$, $m = M_x$ and $S_n = S(m_n)$, $S = S(m)$. Since Mg is continuous for every $g \in C_+(Y)$, we have:

$$m_n(g) \to m(g) \quad \text{for every } g \in C_+(Y) . \tag{2}$$

Now observe that:

$$\operatorname{dist}(S_n, S) \to 0 . \tag{3}$$

Suppose to the contrary that (3) does not hold. Then there are two possibilities:

1° there exist $y_0 \in S$ and $\varepsilon > 0$ such that $K(y_0, \varepsilon) \cap S_n = \emptyset$ for infinitely many n.

2° there exists an $\varepsilon > 0$ such that $S_n - K(S, \varepsilon) \neq \emptyset$ for infinitely many n.

(We use the notation: $K(A, \varepsilon) = \{x \in X: d(x, A) < \varepsilon\}$ for $A \subset X$).

Define $g \in C_+(Y)$ such that:

$$g(y) = \begin{cases} 1 & \text{for } y \in Y - K(y_0, \varepsilon), \\ 0 & \text{for } y = y_0, \end{cases} \qquad \text{in case } 1°;$$

$$g(y) = \begin{cases} 1 & \text{for } y \in K(S, \varepsilon/2), \\ 0 & \text{for } y \in Y - K(S, \varepsilon), \end{cases} \qquad \text{in case } 2° .$$

It is clear that: in case 1°: $m(g) = 0$ and $m_n(g) = 1$ for infinitely many n, in case 2°: $m(g) = 1$ and $m_n(g) = 0$ for infinitely many n; thus in both cases we have got a contradiction of (2). This establishes (3).

Next we will show that:

(4) For every open set $U \subset Y$ we have $\widetilde{m}_n(U) \geqslant \frac{1}{2}\widetilde{m}(U)$ for almost all n.

Indeed, pick a compact set $Z \subset U$ and a $g \in C_+(Y)$ so that:

$$\widetilde{m}(Z) \geqslant \tfrac{3}{4}\widetilde{m}(U),$$

$$1 \leqslant g(y) \leqslant 10 \text{ for all } y \in Y \quad \text{and} \quad g(y) = \begin{cases} 10 & \text{for } y \in Z, \\ 1 & \text{for } y \in Y - U. \end{cases}$$

Then $\log m_n(g) \leqslant \widetilde{m}_n(U)$ and $\widetilde{m}(Z) \leqslant \log m(g)$ (where $\log a$ denotes the decimal logarithm of a). Thus, by (2) and the definition of Z, we get (4).

Now observe that $\widetilde{m}_n(Y) = \log m_n(10)$; thus, by (2), $\widetilde{m}_n(Y) \to \widetilde{m}(Y)$. Hence:

(5) $$\widetilde{m}_n(Y) \leqslant \widetilde{m}(Y) + 1 \quad \text{for almost all } n.$$

Let $\|g\|_A = \sup_{t \in A} |g(t)|$ for every $A \subset Y$, $g \in C(Y)$. Let us consider two cases:

1° $m(f) = 0$. Then there exists a point $x_0 \in S$ such that $f(x_0) = 0$. Let U be a neighbourhood of x_0 such that $\|f\|_U < 1$. Since $f_n \rightrightarrows f$, we may assume without loss of generality that $\|f_n\|_U < 1$. It is easy to see that:

$$m_n(f_n) \leqslant \|f_n\|_U^{\widetilde{m}_n(U)} \cdot \|f_n\|_{Y-U}^{\widetilde{m}_n(Y-U)} \leqslant \|f_n\|_U^{\widetilde{m}_n(U)} \cdot (\|f_n\|_Y + 1)^{\widetilde{m}_n(Y)}.$$

Since $\|f_n\|_U < 1$, we have, by (4) and (5),

$$m_n(f_n) \leqslant \|f_n\|_U^{\frac{1}{2}\widetilde{m}(U)} \cdot (\|f_n\|_Y + 1)^{\widetilde{m}(Y)+1} \quad \text{for almost all } n.$$

Since $\widetilde{m}(U) \geqslant \widetilde{m}(x_0) > 0$ and the sequence $\{f_n\}$ is uniformly bounded, there exists a constant b (not dependent on U) such that:

(6) $$m_n(f_n) \leqslant \|f_n\|_U^{\frac{1}{2}\widetilde{m}(x_0)} \cdot b \quad \text{for almost all } n.$$

Clearly if $\operatorname{diam}(U) \to 0$, then $\|f\|_U \to 0$. Since $\|f_n\|_U \to \|f\|_U$ for every U, by (6), we infer that $m_n(f_n) \to 0 = m(f)$.

2°. $m(f) > 0$. Then $f(t) > 0$ for every $t \in S$. Since S is compact, there exist a compact neighbourhood U of S and $d > 0$ such that $f(t) > d$ for $t \in U$. Since $f_n \rightrightarrows f$, we may assume without loss of generality that $f_n(t) > d$ for $t \in U$ and for all n.

Suppose now that $m_n(f_n) \geqslant m_n(f)$. Let us observe that:

$$\frac{f_n(x)}{f(x)} \leqslant 1 + \|f_n - f\|_Y \big(\inf_{t \in U} f(t)\big)^{-1} \leqslant 1 + \frac{\|f_n - f\|_Y}{d} \quad \text{for every } x \in U.$$

By (3), we may assume without loss of generality that $S_n \subset U$; thus:

$$|m_n(f_n) - m_n(f)| \leqslant \left[\left(1 + \frac{\|f_n - f\|_Y}{d}\right)^{\widetilde{m}_n(Y)} - 1\right] \cdot m_n(f).$$

Similarly, if $m_n(f_n) \leqslant m_n(f)$, then

$$|m_n(f_n) - m_n(f)| \leqslant \left[\left(1 + \frac{\|f_n - f\|_Y}{d}\right)^{\widetilde{m}_n(Y)} - 1\right] \cdot m_n(f_n).$$

Since $m_n(f_n) \leqslant \|f_n\|_Y^{\widetilde{m}_n(Y)}$, it is clear that, by (5):

$$|m_n(f_n) - m_n(f)| \to 0.$$

This, by (2), completes the proof of 2.2. □

A meo M: $C_+(Y) \to C_+(X)$ will be called *regular* if $Mc = c$ for every constant function c, i.e. if $\widetilde{M}_x(Y) = 1$ for every $x \in X$. The next proposition shows that it is enough to consider only the regular meo's.

2.3. Proposition. *If N: $C_+(Y) \to C_+(X)$ is a meo, then there exists a regular meo M: $C_+(Y) \to C_+(X)$.*

Proof. Let $X' = \{x \in X\colon N_x \text{ is non-constant}\}$. Then, by (1):

$$X' = \{x \in X\colon [N(10)](x) > 1\} = \{x \in X\colon [N(1)](x) = 1 \text{ and } [N(0)](x) = 0\}.$$

Thus X' is a closed-open set. Hence the formula $\varphi(x) = [\log N_x(10)]^{-1}$ defines a continuous function on X'.

Let x_0 be any point of Y. Put (for $f \in C_+(Y)$):

$$M_x(f) = \begin{cases} N_x(f)^{\varphi(x)} & \text{if } x \in X', \\ f(x_0) & \text{if } x \in X - X'. \end{cases}$$

It is obvious that $Mf \in C_+(X)$ if $f \in C_+(Y)$ and that M_x is a multiplicative functional for every $x \subset X$; thus, by 2.2, M is a multiplicative operator. Obviously it is a regular meo. □

In the sequel "meo" will always mean "regular meo".

We have the following obvious proposition:

2.4. Proposition. *If r: $X \to Y$ is a retraction, then the formula*

$$Mf = f \circ r \quad \textit{for} \quad f \in C_+(Y)$$

determines a meo. □

Let $Y \subset X$. Define for $x_1, x_2 \in X$, $0 \leqslant t_1, t_2 \leqslant 1$:

$$(x_1, t_1) \sim (x_2, t_2) \quad \text{if} \quad (x_1, t_1) = (x_2, t_2) \text{ or } t_1 = t_2 = 0 \text{ or } t_1 = t_2 = 1 .$$

Obviously $\sim$ is an equivalence relation. The pair of quotient spaces

$$\mathfrak{Z}(X) = X \times I/\sim , \quad \mathfrak{Z}(Y) = Y \times I/\sim$$

will be called the *suspension* of the pair (X, Y) (by I we denote the unit interval).

2.5. PROPOSITION. *If $N\colon C_+(Y) \to C_+(X)$ is a meo, then there exists a meo $M\colon C_+(\mathfrak{Z}(Y)) \to C_+(\mathfrak{Z}(X))$.*

Proof. If $f \in C_+(\mathfrak{Z}(Y))$, then f may be regarded as a function on $Y \times I$ such that the restrictions $f|Y \times \{0\}$ and $f|Y \times \{1\}$ are constant functions. Put $f_t(y) = f(y, t)$ and $(Mf)(x, t) = (Nf_t)(x)$.

Observe that Mf is a continuous function on $X \times I$. Indeed, if $(x_n, t_n) \to (x, t)$ in $X \times I$, then $f_{t_n} \rightrightarrows f_t$. By the continuity of N, $(Nf_{t_n})(x_n) \to (Nf_t)(x)$; hence $Mf \in C_+(X \times I)$.

By the regularity of N, Nf_0 and Nf_1 are constant functions; hence $Mf \in C_+(\mathfrak{Z}(X))$. Thus, by Theorem 2.2, M is a meo. □

Let us notice the following simple property of meo's:

2.6. PROPOSITION. *If $M\colon C_+(Y) \to C_+(X)$ is a meo and Y is connected, then*

$$(Mf)(X) = f(Y) \quad \textit{for every } f \in C_+(X) .$$

Proof: Let $M_x(f) = \prod_i f(y_i)^{a_i}$ for a point $x \in X$. By the regularity of M, we have $\sum a_i = 1$. Thus

$$\inf_{t \in Y} f(t) \leqslant \prod_i f(y_i)^{a_i} \leqslant \sup_{t \in Y} f(t) .$$

But Y is connected and thus there exists a point $t \in Y$ such that

$$M_x(f) = f(t). \; □$$

We have the following characterisation of $\mathcal{AM}_+$-spaces (cf. Proposition 6.2 in [8]).

2.7. PROPOSITION. *The following conditions are equivalent:*

1° *Y is an $\mathcal{AM}_+$-space,*

2° *There exist an AR-space X and a meo $M\colon C_+(Y) \to C_+(X)$,*

3° *There exist an $\mathcal{AM}_+$-space X and a meo $M\colon C_+(Y) \to C_+(X)$.*

Proof: Implication 1°→2° is obvious. By 2.4, every AR-space is an $\mathcal{AM}_+$-space; thus implication 2°→3° is also trivial. Now let 3° be satisfied and let Z be any space such that $Y \subset Z$. Without loss of generality one may assume that $X \cap Z = Y$. Since X is an $\mathcal{AM}_+$-space, there exists a meo $N\colon C_+(X) \to C_+(X \cup Z)$ (topology in $X \cup Z$ is induced by X and Z). Let us define $L_z = (N \circ M)_z$ for $z \in Z$. It is obvious that $L\colon C_+(Y) \to C_+(Z)$ is a meo; thus 3°→1°. □

3. Proof of theorem 1.1. We recall that the *third symmetric potency* of a space Y is the family $Y^{(3)}$ of all at most three-point subsets of Y with the dist metric (cf. Introduction). The formula $J(x) = \{x\}$ defines the natural embedding $J\colon Y \to Y^{(3)}$.

3.1. THEOREM. *Let $Y \subset X$. If there exists a map $\Phi\colon X \to Y^{(3)}$ such that $\Phi(y) = J(y)$ for every $y \in Y$, then there exists a meo $M\colon C_+(Y) \to C_+(X)$.*

Proof. Put for $i = 1, 2, 3$:

$$f_i(a_1, a_2, a_3) = \frac{1}{2} - \frac{a_i}{2(a_1 + a_2 + a_3)} \quad \text{for} \quad a_1, a_2, a_3 > 0 .$$

Let $c(x) = \operatorname{card}\Phi(x)$ and let $\Phi(x) = \bigcup_{i \leqslant c(x)} \{y_i\}$ where $y_i = y_i(x)$. Let us put

$$\psi_x(y_i) = \begin{cases} 1 & \text{if} \quad c(x) = 1 , \\ \frac{1}{2} & \text{if} \quad c(x) = 2 , \\ f_i(d(y_2, y_3), d(y_1, y_3), d(y_1, y_2)) & \text{if} \quad c(x) = 3 , \end{cases}$$

where $d(.,.)$ denotes the metric of Y.

Put:

$$M_x(f) = \prod_{y \in \Phi(x)} f(y)^{\psi_x(y)} \quad \text{for} \quad f \in C_+(Y) ,\; x \in X .$$

Since $\Phi(y) = J(y)$ for $y \in Y$, we have $M_y(f) = f(y)$. Thus, by 2.2, it remains only to prove for each f in $C_+(Y)$ the continuity of Mf, i.e. to show that if $x_n \to x$, then $M_{x_n}(f) \to M_x(f)$.

Write $m_n = M_{x_n}$, $m = M_x$. Pick open sets $K_i \subset Y$ for $i \leqslant c(x)$ so that: $y_i \in K_i$, $K_i \cap K_j = \emptyset$ if $i \neq j$. Put $A_{i,n} = K_i \cap \Phi(x_n)$. We shall prove that

$$\tilde{m}_n(K_i) \to \tilde{m}(K_i) \quad \text{for} \quad i \leqslant c(x) . \tag{7}$$

Indeed, since Φ is continuous, $\operatorname{dist}(\Phi(x_n), \Phi(x)) \to 0$. Thus without loss of generality we may assume that:

$$\bigcup_j A_{j,n} = \Phi(x_n) \quad \text{and} \quad A_{i,n} \neq \emptyset \quad \text{for} \quad i \leqslant c(x) \;\; (n = 1, 2, \ldots) .$$

Hence if $c(x) = 1$, then $\Phi(x_n) \subset K_1$. Thus

$$\tilde{m}_n(K_1) = \tilde{m}_n(\Phi(x_n)) = 1 = \tilde{m}(K_1) .$$

Now suppose that $c(x)=2$. If $c(x_n)=2$, then $\widetilde{m}_n(K_i)=\frac{1}{2}=\widetilde{m}(K_i)$ for $i=1,2$. Therefore we may restrict our attention to the case where $c(x_n)=3$. Without loss of generality one may assume that $A_{1,n}=\{u_n\}$ and $A_{2,n}=\{t_n,z_n\}$. Since $\operatorname{dist}(\Phi(x_n),\Phi(x))\to 0$, obviously:

$$d(t_n,z_n)\to 0\,,\qquad d(t_n,u_n)\to \operatorname{diam}\Phi(x)\neq 0\,,\qquad d(z_n,u_n)\to \operatorname{diam}\Phi(x).$$

Thus

$$\widetilde{m}_n(K_1)=\frac{1}{2}-\frac{d(t_n,z_n)}{2\big(d(t_n,z_n)+d(u_n,z_n)+d(u_n,t_n)\big)}\to\frac{1}{2}=\widetilde{m}(K_1)$$

and

$$\widetilde{m}_n(K_2)=1-\widetilde{m}_n(K_1)\to\tfrac{1}{2}=\widetilde{m}(K_2)\,.$$

Finally, suppose that $c(x)=3$. Let $A_{i,n}=\{y_n^i\}$ for $i=1,2,3$. Obviously $y_n^i\to y_i$ for $i=1,2,3$.

Since the functions f_i are continuous, it is easy to see that:

$$\widetilde{m}_n(y_n^i)\to\psi_x(y_i)\,,\qquad \text{i.e.}\qquad \widetilde{m}_n(K_i)\to\widetilde{m}(K_i)\qquad\text{for}\qquad i=1,2,3\,.$$

Thus (7) is proved.

Since $d(y_i,A_{i,n})\to 0$, the continuity of f and (7) imply

$$\lim_{n=\infty}\prod_{y\in A_{i,n}} f(y)^{\widetilde{m}_n(y)}=f(y_i)^{\widetilde{m}(y_i)}\,.$$

But

$$m_n(f)=\prod_{i\leqslant c(x)}\prod_{y\in A_{i,n}} f(y)^{\widetilde{m}_n(y)}\qquad\text{and}\qquad m(f)=\prod_{i\leqslant c(x)} f(y_i)^{\widetilde{m}(y_i)}\,.\ \square$$

Let I^n denote the n-dimensional cell and let S_{n-1} denote the boundary of I^n, the $(n-1)$-dimensional sphere.

3.2. Corollary. *For every $n\geqslant 2$ there exists a meo M: $C_+(S_{n-1})\to\to C_+(I^n)$.*

Proof. Bott showed in [4] that $S_1^{(3)}$ is homeomorphic to S_3; thus the set $J(S_1)$ is contractible in $S_1^{(3)}$. This obviously implies that there exists a map Φ: $I^2\to S_1^{(3)}$ such that $\Phi(t)=J(t)$ for every $t\in S_1$. Thus, by Theorem 3.1, there exists a meo M: $C_+(S_1)\to C_+(I^2)$.

Observe now that the pair (I^{n+1},S_n) may be regarded as the suspension of the pair (I^n,S_{n-1}). Hence, by 2.5 and a standard induction procedure, we obtain 3.2. $\square$

Now we are ready for the proof of Theorem 1.1.

Necessity. Suppose first that Y is not connected. Let A be a proper closed-open subset of Y. Put

$$f(x)=\begin{cases}0 & \text{if}\quad x\in A\,,\\ 1 & \text{if}\quad x\in Y-A\,.\end{cases}$$

Let $X\supset Y$ be any connected space, suppose that M: $C_+(Y)\to C_+(X)$ is a meo. Since $f^2=f$, we have $(Mf)^2=Mf$ and thus $Mf=0$ or $Mf=1$, a contradiction with the assumption that Mf is an extension of f.

Now suppose that Y is not locally connected. Let $p\in Y$ be a point of local unconnectedness. Let $b>0$ be such that:

(8) if $D\subset Y$, $\operatorname{diam}D<2b$ and $p\in\operatorname{Int}D$, then D is not connected.

Put $U=\overline{K(p,b)}$. Let us notice that for every $n>b^{-1}$ there exists a closed-open set A_n in U such that $p\in A_n$ and $d(p,U-A_n)<1/n$, i.e. $K(p,1/n)-A_n\neq\emptyset$.

Indeed, suppose that for some $n>b^{-1}$ we have $K(p,1/n)\subset B$, where B denotes the intersection of all neighbourhoods of the point p closed-open in U. By definition, B is the quasi-component of the point p in U; since U is compact, B is connected. But $B\subset U$, $p\in K(p,1/n)\subset\operatorname{Int}B$, a contradiction with (8).

For $n>b^{-1}$ pick a point $x_n\in U-A_n$ and a function $f_n\in C_+(Y)$ so that:

$$d(p,x_n)<1/n\qquad\text{and}\qquad f_n(y)=\begin{cases}0 & \text{for}\quad y\in U-A_n\,,\\ 1 & \text{for}\quad y\in A_n\,.\end{cases}$$

Now assume that Y is a subspace of a Hilbert cube Q and suppose that M: $C_+(Y)\to C_+(Q)$ is a meo.

Let y_n be a point of the interval $\langle p,x_n\rangle$ in Q such that $(Mf_n)(y_n)=\frac{1}{2}$. Obviously:

$$S(M_{y_n})-U\neq\emptyset\,. \tag{9}$$

Pick an $f\in C_+(Y)$ so that $f(p)=1$ and $f(y)=0$ for $y\in Y-U$. By (9): $(Mf)(y_n)=0$, obviously $(Mf)(p)=1$. Since $y_n\to p$, Mf is not continuous, a contradiction. $\square$

Sufficiency. Since Y is a compact space, by Theorem (6.2) in [3], ch. V., there exists an AR-space Y'' such that $Y\subset Y''$ and $X=Y''-Y$ is a polytope (for the definition—see [3], ch. III., 2, p. 72), having a null-triangulation, say T (T is called null-triangulation if it is countable and the diameters of its simplexes converge to 0). We shall construct a meo N: $C_+(Y)\to C_+(Y'')$. By 2.7, this will imply that Y is an $\mathcal{AM}_+$-space.

Let X_k denote the k-dimensional skeleton of the polytope X; similarly let τ_k denote the k-dimensional skeleton of a simplex $\tau\in T$. Since Y is a locally connected continuum and $\dim(X_1)=1$, by Theorem 1′ in [7], there exists a retraction r: $Y\cup X_1\to Y$. By 2.4, this retraction determines a meo M_1: $C_+(Y)\to C_+(Y\cup X_1)$.

By 3.2, for every n-dimensional simplex τ with $n \geqslant 2$ there exists a regular meo M_τ: $C_+(\tau_{n-1}) \to C_+(\tau)$. For $n \geqslant 2$ define M_n: $C_+(X_{n-1} \cup Y) \to C_+(X_n \cup Y)$ by

$$(M_n f)(x) = \begin{cases} f(x) & \text{for} \quad x \in X_{n-1} \cup Y, \\ (M_\tau f)(x) & \text{for} \quad x \in \tau \subset X_n - X_{n-1}. \end{cases}$$

Now for each $f \in C_+(Y)$ we define a function Nf on Y'' by:

$$(Nf)(x) = \begin{cases} f(x) & \text{for} \quad x \in Y, \\ [(M_n \circ M_{n-1} \circ \ldots \circ M_2 \circ M_1)f](x) & \text{for} \quad x \in X_n \text{ with } n \geqslant 1. \end{cases}$$

Since N_x is a multiplicative functional for every $x \in Y''$, in order to prove that N is a meo from $C_+(Y)$ to $C_+(Y'')$, it is enough (by 2.2) to show that $Nf \in C_+(Y'')$ whenever $f \in C_+(Y)$.

Since the restriction $Nf|\tau$ is continuous for each τ in T, Nf is continuous on X (cf. Theorem 2.5, ch. III. in [3]). Thus it remains only to prove that if $x \in Y$, $x_n \in X$ and $x_n \to x$ then $(Nf)(x_n) \to f(x)$. Observe first that, by the regularity of each M_τ and 2.6,

$$(Nf)(\tau) = (M_1 f)(\tau_1) \text{ for every } \tau \in T. \tag{10}$$

Let $x_n \in \tau^n$. It is clear that no simplex $\tau \in T$ appears infinitely many times in the sequence $\{\tau^n\}$. Since T is a null-triangulation, $\operatorname{diam}\tau^n \to 0$. Thus, since $x_n \in \tau^n$ and $x_n \to x$,

$$d(x, \tau^n) \to 0, \quad \text{whence} \quad d(x, r\tau_1^n) \to 0.$$

We have $(M_1 f)(\tau_1^n) = f[r(\tau_1^n)]$, thus, by the continuity of f:

$$d[f(x), (M_1 f)(\tau_1^n)] \to 0. \tag{11}$$

By (10): $(Nf)(x_n) \in (M_1 f)(\tau_1^n)$, whence, by (11): $(Nf)(x_n) \to f(x)$. □

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