Convergence quotient maps

by

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Introduction. In this paper we study quotient maps for convergence spaces. The first three theorems describe the behaviour of the decomposition series of a convergence space under continuous maps and quotient maps. The last two theorems show the rather surprising equivalence of certain types of convergence quotient maps with such topological notions as pseudo-open maps, almost open maps, and bi-quotient maps.

1. Preliminaries. An extensive discussion of convergence spaces can be found in [4], [5], and [6]; however, a brief summary of essential results will be repeated here.

A *convergence structure* q on a set S is a mapping from the set $F(S)$ of all filters on S into the set of all subsets of S which satisfies the following conditions: (1) $x \in q(\dot{x})$, all $x \in S$, where $\dot{x}$ is the principal ultrafilter containing $\{x\}$; (2) if $\mathcal{F}$ and $\mathcal{J}$ are in $F(S)$ and $\mathcal{F} \leqslant \mathcal{J}$, then $q(\mathcal{F}) \subset q(\mathcal{J})$; (3) if $x \in q(\mathcal{F})$, then $x \in q(\mathcal{F} \cap \dot{x})$. The pair (S, q) is called a *convergence space*, and $x \in q(\mathcal{F})$ is interpreted "$\mathcal{F}$ q-converges to x". The filter $\mathcal{V}_q(x)$ obtained by intersecting all filters which q-converge to x is called the *q-neighborhood filter at x*. If $\mathcal{V}_q(x)$ q-converges to x for each point x in S, then q is called a *pretopology*, and (S, q) a *pretopological space*.

Starting with a convergence space (S, q), we define for each ordinal number α a set function I_q^α defined recursively for every subset A of S as follows:

$$I_q^0(A) = A;$$

$$I_q^1(A) = \{x \in A\colon\ A \in \mathcal{V}_q(x)\};$$

$$I_q^\alpha(A) = I_q^1\big(I_q^{\alpha-1}(A)\big), \text{ if } \alpha-1 \text{ exists};$$

$$I_q^\alpha(A) = \cap\, \{I_q^\sigma(A)\colon\ \sigma < \alpha\}, \text{ if } \alpha \text{ is a limit ordinal.}$$

It is not difficult to show that

$$I_q^\alpha(A \cap B) = I_q^\alpha(A) \cap I_q^\alpha(B)$$

for all ordinals α and all choices of A and B. Indeed, I_q^α has all of the properties of a topological interior operator except idempotency. Since any transfinite process of shrinking subsets of S must eventually terminate, there is a least ordinal number $\alpha = \gamma_q$ such that $I_q^\alpha(A) = I_q^{\alpha+1}(A)$ for all subsets A of S; γ_q is called the *length of the decomposition series* for (S, q).

Let $\lambda(q) = \{U \subset S\colon I_q^1(U) = U\}$. Then $\lambda(q)$ is a topology on S coarser than q, and indeed the finest such topology. The set $\{I_q^1(A)\colon A \subset S\}$ forms the base for a topology which we call $\varphi(q)$; $\varphi(q)$ is obviously finer than $\lambda(q)$, but $\varphi(q)$ may not be comparable with q itself.

Next we introduce, for each point x in S and each ordinal number α, the filter $\mathcal{V}_q^\alpha(x) = \{A \subset S\colon x \in I_q^\alpha(A)\}$. The pretopology which has $\mathcal{V}_q^\alpha(x)$ as its neighborhood filter at each point x in S denoted $\pi^\alpha(q)$. It can be shown that $\mathcal{V}_q^1(x) = \mathcal{V}_q(x)$; thus $\pi^1(q)$ (more often written merely as $\pi(q)$) is the finest pretopology coarser than q. Some propositions concerning the various topologies and pretopologies on S related to q are listed below.

1. $\lambda(q) \leqslant \pi(q) \leqslant \varphi(q)$.
2. If $1 \leqslant \alpha < \beta \leqslant \gamma_q$, then $\lambda(q) \leqslant \pi^\alpha(q) < \pi^\beta(q) \leqslant \pi(q)$.
3. $\pi^{\gamma_q}(q) = \lambda(q)$.
4. If α is a limit ordinal, then $\mathcal{V}_q^\alpha(x) = \bigcap \{\mathcal{V}_q^\sigma(x)\colon \sigma < \alpha\}$.

The diagram given below is called the *decomposition series for* (S, q):

$$(S, q) \to (S, \pi(q)) \to \ldots \to (S, \pi^\alpha(q)) \to \ldots \to (S, \lambda(q));$$

the arrows can be regarded as representing the identity mapping on S.

2. Continuous maps. Throughout this section, f will represent a function mapping a convergence space (S, q) onto another convergence space (T, p). If $\mathcal{F} \in F(S)$, then $f(\mathcal{F})$ will denote the filter on T generated by $\{f(F)\colon F \in \mathcal{F}\}$. If $\mathcal{G} \in F(T)$, then $f^{-1}(\mathcal{G}) = \{f^{-1}(G)\colon G \in \mathcal{G}\}$ is a filter on S.

DEFINITION 1. The function f is *continuous* if $f(\mathcal{F})$ p-converges to $f(x)$ whenever $\mathcal{F}$ q-converges to x.

PROPOSITION 1. (a) *If f is a continuous function, then $f(\mathcal{V}_q(x)) \geqslant \mathcal{V}_p(f(x))$ for all x in S.*

(b) *If p is a pretopology and $f(\mathcal{V}_q(x)) \geqslant \mathcal{V}_p(f(x))$ for all $x \in S$, then f is continuous.*

Proof. (a) $f(\mathcal{V}_q(x)) = f(\bigcap \{\mathcal{F}\colon\ x \in q(\mathcal{F})\}) = \bigcap \{f(\mathcal{F})\colon\ x \in q(\mathcal{F})\} \geqslant \mathcal{V}_p(f(x))$.

(b) Let $x \in q(\mathcal{F})$; then $\mathcal{F} \geqslant \mathcal{V}_q(x)$ and $f(\mathcal{F}) \geqslant f(\mathcal{V}_q(x)) \geqslant \mathcal{V}_p(f(x))$. Thus $f(\mathcal{F})$ p-converges to x, and f is continuous.

PROPOSITION 2. *For each ordinal number α, the following statements are equivalent:*

(a) *For each x in S, $f(\mathcal{V}_q^\alpha(x)) \supset \mathcal{V}_p^\alpha(f(x))$;*

(b) *For each $A \subset T$, $f^{-1}(I_p^\alpha(A)) \subset I_q^\alpha(f^{-1}(A))$.*

Proof. First, assume that (a) is true, and let $x \in f^{-1}(I_p^\alpha(A))$. Then

$$f(x) \in I_p^\alpha(A) \Rightarrow A \in \mathcal{V}_p^\alpha(f(x)) \Rightarrow f^{-1}(A) \in f^{-1}(\mathcal{V}_p^\alpha(f(x)))$$
$$\Rightarrow f^{-1}(A) \in \mathcal{V}_q^\alpha(x) \Rightarrow x \in I_q^\alpha(f^{-1}(A)).$$

Next, assume that (b) is true and let $A \in \mathcal{V}_p^\alpha(f(x))$. Then

$$f(x) \in I_p^\alpha(A) \Rightarrow x \in f^{-1}(I_p^\alpha(A)) \Rightarrow x \in I_q^\alpha(f^{-1}(A))$$
$$\Rightarrow f^{-1}(A) \in \mathcal{V}_q^\alpha(x) \Rightarrow A \in f(\mathcal{V}_q^\alpha(x)).$$

THEOREM 1. *If f is continuous, then for each $x \in S$ and each ordinal number α,*

$$f(\mathcal{V}_q^\alpha(x)) \geqslant \mathcal{V}_p^\alpha(f(x)).$$

Proof. Induction on α. For $\alpha = 1$, the result follows from Proposition 1. If α has an immediate predecessor, $\alpha-1$, then $V \in \mathcal{V}_p^\alpha(f(x))$ implies $f(x) \in I_p^\alpha(V) = I_p(I_p^{\alpha-1}(V))$. Thus $I_p^{\alpha-1}(V) \in \mathcal{V}_p(f(x))$, which implies that $f^{-1}(I_p^{\alpha-1}(V)) \in \mathcal{V}_q(x)$. By Proposition 2 and the induction hypothesis, $I_q^{\alpha-1}(f^{-1}(V)) \in \mathcal{V}_q(x)$. Thus $x \in I_q(I_q^{\alpha-1}(f^{-1}(V))) = I_q^\alpha(f^{-1}(V))$, and so $f^{-1}(V) \in \mathcal{V}_q^\alpha(x)$, as desired. Finally, assume that α is a limit ordinal. Let $V \in \mathcal{V}_p^\alpha(f(x))$; then $V \in \mathcal{V}_p^\sigma(f(x))$ for all $\sigma < \alpha$, and by the induction hypothesis, $f^{-1}(V) \in \mathcal{V}_q^\sigma(x)$ for all $\sigma < \alpha$. But $f^{-1}(V) \in \bigcap \{\mathcal{V}_q^\sigma(x)\colon \sigma < \alpha\} = \mathcal{V}_q^\alpha(x)$, and the proof is complete.

COROLLARY. *If f is continuous, then the decomposition series for (S, q) is mapped continuously by f onto the decomposition series for (T, p). In other words, in the diagram below, where the horizontal arrows represent the identity maps and the vertical arrows represent f, all mappings are continuous.*

$$\begin{array}{ccccccccc}
(S, q) & \to & (S, \pi(q)) & \to \ldots \to & (S, \pi^\alpha(q)) & \to \ldots \to & (S, \lambda(q)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(T, p) & \to & (T, \pi(p)) & \to \ldots \to & (T, \pi^\alpha(p)) & \to \ldots \to & (T, \lambda(p)).
\end{array}$$

3. Quotient maps. As in the preceding section, f will always represent a mapping from (S, q) onto (T, p); these spaces will be assumed to be convergence spaces unless otherwise designated.

DEFINITION 2. The mapping f is called a *convergence quotient map* if p is the finest convergence structure on T relative to which f is continuous.

DEFINITION 3. If (S, q) and (T, p) are both pretopological spaces and p is the finest pretopology on T relative to which f is continuous, then f is called a *pretopological quotient map*.

If "pretopology" is replaced by „topology" in Definition 2, then one obtains the usual definition of a *topological quotient mapping*. It is clear that quotient maps, in any of the three senses described above, are always continuous.

PROPOSITION 3. *The following statements about f are equivalent:*

(a) *f is a convergence quotient map;*

(b) *A filter $\mathcal{F}$ p-converges to y in T if and only if there is x in $f^{-1}(y)$ and $\mathcal{I} \in F(S)$ such that $x \in q(\mathcal{I})$ and $\mathcal{F} \geqslant f(\mathcal{I})$.*

Proof. Assume that f is a quotient map and let $y \in p(\mathcal{F})$. Assume further that for all $x \in f^{-1}(y)$ there is no $\mathcal{I} \in F(S)$ such that $x \in q(\mathcal{I})$ and $\mathcal{F} \geqslant f(\mathcal{I})$. Let r be the function mapping $F(T)$ into the set of all subsets of T defined as follows: if $y \notin p(\mathcal{K})$, then $r(\mathcal{K}) = p(\mathcal{K})$; if $y \in p(\mathcal{K})$, then $r(\mathcal{K}) = p(\mathcal{K})$ if $\mathcal{K} \geqslant f(\mathcal{I})$ for some filter $\mathcal{I}$ in $F(S)$ which q-converges to some element x in $f^{-1}(y)$, and otherwise $r(\mathcal{K}) = p(\mathcal{K}) - \{y\}$.

First we verify that r is a convergence structure. In checking the three conditions which define convergence structures, we can restrict our attention to those filters which r-converge to y, since r-convergence agrees with p-convergence otherwise.

(1) $y \in r(\mathring{y})$, since $x \in q(\mathring{x})$ for all $x \in f^{-1}(y)$, and $f(\mathring{x}) = \mathring{y}$.

(2) Let $\mathcal{K}_1 \leqslant \mathcal{K}_2$ in $F(T)$ and $y \in r(\mathcal{K}_1)$. Then there is $\mathcal{I} \in F(S)$ such that $\mathcal{I}$ q-converges to some x in $f^{-1}(y)$ and $f(\mathcal{I}) \leqslant \mathcal{K}_1$. But then $f(\mathcal{I}) \leqslant \mathcal{K}_2$, and $y \in r(\mathcal{K}_2)$.

(3) This condition follows immediately from the fact that $f(\mathcal{I}) \leqslant \mathcal{K}$ implies $f(\mathcal{I} \cap \mathring{x}) \leqslant \mathcal{K} \cap \mathring{y}$ for all $x \in f^{-1}(y)$.

It is clear from the construction of r that f maps (S, q) continuously onto (T, r). It is equally clear that r is strictly finer that p. But these facts contradict the original assumption that f is a quotient map from (S, q) onto (T, p).

Conversely, assume condition (b); then the continuity of f is obvious. If there were a convergence structure r on T such that the mapping by f of (S, q) onto (T, r) were continuous for $r \geqslant p$, then it is easy to see that any filter which p-converges to y would necessarily r-converge to y, and we would have $r = p$. Thus, given (b), p is the finest convergence structure on T relative to which f is continuous.

PROPOSITION 4. *If f is a convergence quotient map then, for each $y \in T$,*

$$\mathcal{V}_p(y) = \bigcap \{f(\mathcal{V}_q(x)) : x \in f^{-1}(y)\}.$$

Proof. The inequality $\leqslant$ follows from Proposition 1. To establish the other direction of the inequality, let $A \in f(\mathcal{V}_q(x))$ for each $x \in f^{-1}(y)$. If $\mathcal{I}$ p-converges to y, then by Proposition 3 there is $z \in f^{-1}(y)$ and $\mathcal{K}$ q-converging to z such that $\mathcal{I} \geqslant f(\mathcal{K})$. Since $\mathcal{K} \geqslant \mathcal{V}_q(z)$, we have $f^{-1}(A) \in \mathcal{K}$, which implies that $A \in f(\mathcal{K}) \subset \mathcal{I}$. Since each filter $\mathcal{I}$ which p-converges to y is finer than $\bigcap \{f(\mathcal{V}_q(x)) : x \in f^{-1}(y)\}$, then the result holds for $\mathcal{V}_p(y)$, which is the intersection of such filters.

PROPOSITION 5. *Let (S, q) and (T, p) be pretopological spaces. Then a continuous function f is a convergence quotient map if and only if, for each $y \in T$, there is $x \in f^{-1}(y)$ such that $f(\mathcal{V}_q(x)) = \mathcal{V}_p(y)$.*

Proof. That $\mathcal{F}$ is a convergence quotient function under the given condition is an immediate consequence of Proposition 3. If the condition fails, then no filter $\mathcal{F}$ which q-converges to $x \in f^{-1}(y)$ could map on $\mathcal{V}_p(y)$, and so, by Proposition 3, $\mathcal{V}_p(y)$ could not p-converge to y.

PROPOSITION 6. *Let (S, q) and (T, p) be pretopologies. Then f is a pretopological quotient map if and only if f is continuous and, in addition, $V \in \mathcal{V}_p(y)$ whenever $f^{-1}(V) \in \mathcal{V}_q(x)$ for all $x \in f^{-1}(y)$.*

Proof. Assume the given condition and let r be a pretopology on T such that f maps (S, p) continuously onto (T, r). If $V \in \mathcal{V}_r(y)$, then $f^{-1}(V) \in \mathcal{V}_q(x)$, all $x \in f^{-1}(y)$. Thus $\mathcal{V}_p(y) \geqslant \mathcal{V}_r(y)$ for all $y \in T$; this implies that $p \geqslant r$, and so f is a pretopological quotient map.

Conversely, if f is a pretopological quotient map, then f certainly is continuous. Consider $V \subset T$ such that $f^{-1}(V) \in \mathcal{V}_q(x)$ for all $x \in f^{-1}(y)$. If $\mathcal{V}(y)$ is the filter generated by such sets, and r the pretopology on T obtained by setting $\mathcal{V}_r(y) = \mathcal{V}(y)$ for all y, then it follows from Proposition 1 that f mapping (S, q) onto (T, r) is continuous, and, since f is a pretopological quotient map onto (T, p), $\mathcal{V}_p(y) \geqslant \mathcal{V}_r(y)$. But each member of $\mathcal{V}_p(y)$ is clearly in $\mathcal{V}_r(y)$, and so $p = r$, and the proof is complete.

THEOREM 2. *If f is a quotient map, then $f\colon (S, \pi(q)) \to (T, \pi(p))$ is a pretopological quotient map, and $f\colon (S, \lambda(q)) \to (T, \lambda(p))$ is a topological quotient map.*

Proof. Let us first consider the pretopological case. If r is a pretopology on T such that $f\colon (S, \pi(q)) \to (T, r)$ is continuous and $V \in \mathcal{V}_r(y)$ for some $y \in T$, then $x \in f^{-1}(y)$ implies $f^{-1}(V) \in \mathcal{V}_q(x)$. Thus $V \in f(\mathcal{V}_q(x))$, all $x \in f^{-1}(y)$ and, by Proposition 4, $V \in \mathcal{V}_p(y)$. Hence $\pi(p) \geqslant r$, and $\pi(p)$ is the finest pretopology on T for which the above mapping is continuous.

Turning to the topological case, we begin by showing continuity. If $U \in \lambda(p)$ and $x \in f^{-1}(U)$, then $f(x) = y \in U$, and $U \in \mathcal{V}_p(y)$. Thus $U \in f(\mathcal{V}_q(x))$, which implies $f^{-1}(U) \in \mathcal{V}_q(x)$, and $f^{-1}(U)$ is $\lambda(q)$-open. Furthermore, let $\tau \geqslant \lambda(p)$ be a topology on T such that $f\colon (S, \lambda(q)) \to (T, \tau)$ is continuous, and let V be τ-open. If $y \in V$ and $x \in f^{-1}(y)$, then $f^{-1}(V) \in \mathcal{V}_q(x)$, and $V \in f(\mathcal{V}_q(x))$; since this is true for all $x \in f^{-1}(y)$,

$$V \in \bigcap \{f(\mathcal{V}_q(x))\colon x \in f^{-1}(y)\} = \mathcal{V}_p(y)\,,$$

and V is $\lambda(p)$-open.

EXAMPLE 1. Let $S = \{w, x, y, z\}$ and let $T = \{a, b, c\}$. Let q be the topology on S with neighborhood filters specified as follows:

$$\mathcal{V}_q(x) = \mathring{x}\,, \quad \mathcal{V}_q(y) = [y, z]\,, \quad \mathcal{V}_q(z) = \mathring{z}\,, \quad \mathcal{V}_q(w) = [w, z]\,.$$

(Here $[y, z]$ denotes the filter on S generated by the set $\{y, z\}$, etc.) Let p be the pretopology on T with neighborhood filters given by:

$$\mathcal{V}_p(a) = \mathring{a}\,, \quad \mathcal{V}_p(b) = [b, c]\,, \quad \mathcal{V}_p(c) = [a, c]\,.$$

Let f be the function specified as follows:

$$f(x) = a\,, \quad f(y) = b\,, \quad f(z) = f(w) = c\,.$$

The following facts can be easily verified.

(1) $f\colon (S, q) \to (T, p)$ is a convergence quotient map.

(2) $f\colon (S, \varphi(q)) \to (T, \varphi(p))$ is not continuous.

(3) $f\colon (S, \lambda(q)) \to (T, \lambda(p))$ is a topological quotient map but not a pretopological quotient map.

(4) $\gamma_q < \gamma_p$.

The third conclusion of Example 1 shows that $f\colon (S, \pi^\alpha(q)) \to (T, \pi^\alpha(p))$ is not in general a pretopological quotient map for $\alpha > 1$. Concerning the fourth conclusion, we can readily see that the inequality $\gamma_q > \gamma_p$ is possible for convergence quotient maps by considering the projection of an arbitrary convergence space onto the trivial space constisting of a single point; thus we conclude that convergence quotient maps do not yield any predictable relationship between γ_q and γ_p unless further conditions are imposed. Our next theorem displays such a condition. However we first need to establish a preliminary proposition, the proof of which is similar to that of Proposition 2 and will be omitted.

PROPOSITION 7. *For each ordinal number α, the following statements about f are equivalent:*

(a) $f(\mathcal{V}_q^\alpha(x)) \leqslant \mathcal{V}_p^\alpha(f(x))$, *all* $x \in S$;

(b) *For each* $A \subset T$, $f^{-1}(I_p^\alpha(A)) \supset I_q^\alpha(f^{-1}(A))$;

(c) *For each* $B \subset S$, $f(I_q^\alpha(B)) \subset I_p^\alpha(f(B))$.

DEFINITION 4. A convergence quotient map q is said to be *neighborhood-preserving* if, for each $x \in S$, $f(\mathcal{V}_q(x)) = \mathcal{V}_p(f(x))$.

THEOREM 3. *If f is neighborhood-preserving, then:*

(1) $f\colon (S, \pi^\alpha(q)) \to (T, \pi^\alpha(p))$ *is neighborhood-preserving for all ordinal numbers* α;

(2) $f\colon (S, \varphi(q)) \to (T, \varphi(p))$ *is a continuous function.*

(3) $\gamma_p \leqslant \gamma_q$.

Proof. (1) Induction on α. For $\alpha = 1$, the result follows immediately from Definition 3. Next, assume that α is an ordinal number with an immediate predecessor, and assume that the result holds for $\alpha - 1$. Since f is continuous, $f(\mathcal{V}_q^\alpha(x)) \geqslant \mathcal{V}_p^\alpha(f(x))$ is valid for all ordinal numbers α by Theorem 1. Let $V \in f(\mathcal{V}_q^\alpha(x))$; then there is $U \in \mathcal{V}_q^\alpha(x)$ such that $f(U) \subset V$. Thus $x \in I_q^\alpha(U) = I_q(I_q^{\alpha-1}(U))$ implies $I_q^{\alpha-1}(U) \in \mathcal{V}_q(x)$, and hence $f(I_q^{\alpha-1}(U))$ $f(\mathcal{V}_q(x)) = \mathcal{V}_p(f(x))$. But then, by the induction hypothesis and Proposition 7, $f(I_q^{\alpha-1}(U)) \subset I_p^{\alpha-1}(f(U))$. Thus $I_p^{\alpha-1}(f(U)) \in \mathcal{V}_p(f(x))$, and $f(x)$ $\in I_p^\alpha(f(U))$, which further implies that $f(U) \in \mathcal{V}_p^\alpha(f(x))$. It remains to consider the case where α is a limit ordinal; in this case $f(\mathcal{V}_q^\sigma(x)) = \mathcal{V}_p^\sigma(f(x))$ is assumed valid for all $\sigma < \alpha$ and all $x \in S$. Let $V \in f(\mathcal{V}_q^\alpha(x))$, where $f(U) \subset V$ and $U \in \mathcal{V}_q^\alpha(x)$. Then, by the induction hypothesis, $f(U) \in \mathcal{V}_p^\sigma(x)$, all $\sigma < \alpha$. Thus

$$f(U) \in \bigcap \{\mathcal{V}_p^\sigma(f, x))\colon \sigma < \alpha\} = \mathcal{V}_p^\alpha(f(x))\,,$$

and the proof of (1) is complete.

(2) Let $I_p(A)$ be an open base set for $\varphi(p)$. Then, by Propositions 2 and 7, $f^{-1}(I_p(A)) = I_q(f^{-1}(A))$ is $\varphi(q)$-open.

(3) Let $f\colon (S, \pi^\alpha(q)) \to (T, \pi^\alpha(p))$ and $f\colon (S, \pi^\beta(q)) \to (T, \pi^\beta(p))$ be two distinct components of the translation of the decomposition series for (S, q) into that of (T, p). If $\pi^\alpha(q) = \pi^\beta(q)$, then it is clear that $\pi^\alpha(p)$ $= \pi^\beta(p)$, and this observation establishes the desired inequality.

It is interesting to note that in Example 1, where all three parts of Theorem 3 break down, the given function f is neighborhood-preserving except at the single point z. This suggests no significant weakening of the conditions imposed in Theorem 3 can be found in the general case.

Let $C(T)$ denote the complete lattice of all convergence structures on T, and assume that p is the infimum of some set $\{p_i\}$ of convergence structures on T. If (S, q) is the direct union of the $\{(T, p_i)\}$ and f the natural projection of S onto T, then one can show without difficulty that f is a convergence quotient map. Since a direct union of topologies is a topology, it follows from Theorem 1, [5], that every convergence space is a convergence quotient image of some topological space.

4. Topological implications. "Careful analysis and broad classification of entities that at first glance appear to be dissimilar are the essence of point-set topology and form its main task. It is clear that if one is to take this path, then it immediately becomes necessary to invent new topological ideas and objects". This quotation is from a paper by A. V. Arhangelskii, [1], and the purpose of this section is to classify some of the various types of topological quotient maps which have been studied by Arhangelskii and others as special cases of convergence quotient maps.

In this section, unless otherwise indicated, we will consider f to be a continuous mapping of a topological space (S, q) onto another topological space (T, p). The term "quotient map" will mean "topological quotient map". The four definitions that follow describe concepts which are intermediate in generality between "quotient map" and "open map"; they are taken from references [1], [2], and [7] and listed in order of decreasing generality.

Definition 5. f is *hereditarily quotient* if, for each $A \subset T$, $f|f^{-1}(A)$ is a quotient map onto A.

Definition 6. f is *pseudo-open* if, for each point y in T and each neighborhood U of $f^{-1}(y)$ in S, y is in the interior of $f(U)$.

Definition 7. f is *bi-quotient* if, whenever $y \in T$ and $\mathfrak{U}$ is a covering of $f^{-1}(y)$ be open subsets of S, then finitely many $f(U)$, with $U \in \mathfrak{U}$, cover some neighborhood of y in T.

Definition 8. f is *almost open* if, for each y in T, there is x in $f^{-1}(y)$ with a base of open sets each element of which is mapped by f onto an open set in Y.

Definitions 5 and 6 are known to be equivalent (see [1]).

Proposition 8. *The following statements about f are equivalent.*

(a) *f is bi-quotient.*

(b) *If $\mathfrak{B}$ is a filter base in T and $y \in T$ adheres to $\mathfrak{B}$, then some $x \in f^{-1}(y)$ adheres to $f^{-1}(\mathfrak{B})$.*

(c) *If $\mathcal{F}$ is an ultrafilter on T which p-converges to y, then there is an ultrafilter $\mathfrak{I}$ on S which maps on $\mathcal{F}$ and q-converges to some $x \in f^{-1}(y)$.*

Michael [7] has shown that (a) and (b) are equivalent; the equivalence of (b) and (c) is easy to verify.

When the spaces under consideration are topological spaces, we see immediately that the characterization of pretopological convergence maps given in Proposition 6 coincides with Definition 6, and that the characterization of convergence quotient maps given in Proposition 5 is equivalent to Definition 8. Thus we have proved

Theorem 4. (a) *f is a pseudo-open map if and only if f is a pretopological quotient map.*

(b) *f is almost open if and only if f is a convergence quotient map.*

For the sake of completeness, it might be mentioned that f is open if and only if f is neighborhood-preserving.

To obtain a characterization of bi-quotient maps, we turn to a type of convergence structure which (like the notion of pretopology) was defined by Choquet [3] in 1948.

Definition 8. A convergence structure r on a set R is a *pseudo-topology* if $\mathcal{F}$ r-converges to x whenever each ultrafilter $\mathfrak{I}$ finer than r-converges to x.

Proposition 9. *Let f map the pseudo-topology (S, q) onto the pseudo-topology (T, p). Then p is the finest pseudo-topology on T relative to which f is continuous if and only if f satisfies condition* (c) *of Proposition* 8.

The proof of Proposition 9 is similar to that of Proposition 3 and will be omitted. By analogy with Definitions 2 and 3, we will call any function f which satisfies condition (c) of Proposition 8 a *pseudo-topological quotient map*.

Theorem 5. *A mapping f from a topological space (S, q) to a topological space (T, p) is bi-quotient if and only if f is a pseudo-topological quotient map.*

References

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