Then

$$\mathcal{M}_i = \bigcup \{\mathcal{K}_j(\alpha_1 \dots \alpha_i)\colon\ j = 1, 2, \dots, \alpha_1, \dots, \alpha_i \in \Omega\}$$

is a locally finite closed collection of $X \times Y$. Thus $\bigcup \mathcal{M}_i$ is a σ-locally finite closed covering of $X \times Y$ refining $\mathcal{G}$. By Lemma 4.9 $X \times Y$ is countably paracompact and the proof is completed.

4.11. **Remark.** Almost all propositions about Σ-spaces are also true if we replace Σ-spaces with $\Sigma(\mathfrak{m})$-spaces. The following are such ones: Theorems 1.8, 3.2, 3.6, 3.9, 3.13 and Corollaries 1.8, 1.19.

References

[1] A. Arhangelskii, *On a class of spaces containing all metric and all locally bicompact spaces*, Dokl. Akad. Nauk SSSR 151 (1963), pp. 751-754; Soviet Math. Dokl. 4 (1963), pp. 1051-1055.

[2] C. Borges, *Stratifiable spaces*, Pacific J. Math. 17 (1966), pp. 1-16.

[3] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. 8 (1960), pp. 747-750.

[4] T. Ishii, *On closed mappings and M-spaces* I, Proc. Japan Acad. 43 (1967), pp. 752-756.

[5] M. Katětov, *On extensions of locally finite coverings*, Colloq. Math. 6 (1958), pp. 145-151.

[6] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. 4 (1953), pp. 831-838.

[7] — *$\aleph_0$-spaces*, J. Math. and Mech. 15 (1966), pp. 983-1002.

[8] K. Morita, *Products of normal spaces with metric spaces*, Math. Annalen 154 (1964), pp. 365-382.

[9] — *Some properties of M-spaces*, Proc. Japan Acad. 43 (1967), pp. 869-872.

[10] K. Nagami, *σ-spaces and product spaces*, forthcoming.

[11] A. Okuyama, *On metrizability of M-spaces*, Proc. Japan Acad. 40 (1964), pp. 176-179.

[12] — *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku sect. A 9 (1967), pp. 236-254.

[13] J. Suzuki, *On a theorem for M-spaces*, Proc. Japan Acad. 43 (1967), pp. 610-614.

Reçu par la Rédaction le 20. 3. 1968

A generalized contraction principle

by

R. E. Chandler (Raleigh, N. C.)

Various versions and generalizations of the Banach contraction mapping theorem ([1], p. 160) have been given. For only two of many examples see [4], p. 43, 50 (where an application is given by solving the Volterra type integral equation) and [2] (where an application is given to analytic mappings of a compact connected set in the complex plane into itself.) We discuss a general definition of contraction mapping here for which we can prove the necessary result that a contraction mapping of a complete metric space into itself has a unique fixed point. In order to make this definition it is convenient to work with uniform spaces having a countable symmetric base rather than metric spaces although, of course, the two are equivalent.

See Kelley ([3], Chapter 6) for the necessary terminology and results. In what follows Z will denote the integers and Δ the diagonal of $X \times X$ $(\Delta = \{(x, x) | x \in X\})$.

DEFINITION. Let $(X, \mathcal{U})$ be a uniform space. A mapping $f\colon X \to X$ is *u-contracting* provided there is a collection of symmetric sets $\{V_n\}_{n \in Z}$, cofinal in $\mathcal{U}$ (with respect to the ordering $U_1 \geqslant U_2$ if and only if $U_1 \subseteq U_2$) which satisfy

(i) $V_i \subseteq V_j$ if $i \leqslant j$, $\bigcap_{n \in Z} V_n = \Delta$, $\bigcup_{n \in Z} V_n = X \times X$,

(ii) for each $n \in Z$ there is an integer $p(n) > 0$ such that $\{p(n) | n \in Z\}$ is bounded and $V_{n-p(n)} \circ V_{n-p(n)} \subseteq V_n$,

(iii) if $(x, y) \in V_n$ then $\big(f(x), f(y)\big) \in V_{n-1}$.

LEMMA 1. *If $f\colon X \to X$ is u-contracting then f has at most one fixed point.*

Proof. Suppose $f(x) = x$ and $y \neq x$. Let n be the least integer for which $(x, y) \in V_n$. (n exists since $\bigcap V_n = \Delta$ and $\bigcup V_n = X \times X$.) Then $(x, y) \in V_n$ so $\big(f(x), f(y)\big) \in V_{n-1}$. If $y = f(y)$ we would have $(x, y) \in V_{n-1}$, a contradiction.

LEMMA 2. *If $f\colon X \to X$ is u-contracting then so is any iterate, f^p, of f.*

Proof. The sequence of V_n which demonstrates that f is u-contracting will suffice.

Lemma 3. *If $f\colon X\to X$ is u-contracting then f is uniformly continuous.*

Proof. Define $f_2\colon X\times X\to X\times X$ by $f_2(x,y)=\big(f(x),f(y)\big)$.

$$f_2^{-1}(V_n)\supseteq V_n \quad \text{so that} \quad f_2^{-1}(V_n)\in\mathfrak{U}.$$

Theorem. *Let $f\colon X\to X$ be u-contracting where $(X,\mathfrak{U})$ is a complete uniform space. Then there is exactly one $x_0\in X$ for which $f(x_0)=x_0$.*

Proof. Let $p=\max\{p(n)|\ n\in Z\}$ and let x be an arbitrary point of X. Let g denote the pth iterate of f. Rename, if necessary, the V_n so that $\big(x,g(x)\big)\in V_0$. Then

$$\big(g(x),g^2(x)\big)\in V_{-p},\ \big(g^2(x),g^3(x)\big)\in V_{-2p},\ \ldots,\ \big(g^n(x),g^{n+1}(x)\big)\in V_{-np},\ \ldots,$$
$$\ldots,\ \big(g^{n+q}(x),g^{n+q+1}(x)\big)\in V_{-(n+q)p}.$$

Thus

$$\big(g^n(x),g^{n+q+1}(x)\big)\in V_{-np}\circ V_{-(n+1)p}\circ\ldots\circ V_{-(n+q-1)p}\circ V_{-(n+q)p}.$$

Now $V_{-(n+q)p}\subseteq V_{-(n+q-1)p}$ so that

$$V_{-(n+q-1)p}\circ V_{-(n+q)p}\subseteq V_{-(n+q-1)p}\circ V_{-(n+q-1)p}\subseteq V_{-(n+q-2)p}.$$

Consequently, we see that

$$V_{-np}\circ V_{-(n+1)p}\circ\ldots\circ V_{-(n+q-1)p}\circ V_{-(n+q)p}\subseteq V_{-np}\circ V_{-np}\subseteq V_{-(n-1)p}.$$

For each $U\in\mathfrak{U}$ there is an N such that if $(n-1)p>N$ then $V_{-(n-1)p}\subseteq U$ since $\{V_n\}_{n\in Z}$ is cofinal in $\mathfrak{U}$. Thus, if $n>N/p+1$ and $q\geqslant 0$, we have $\big(g^n(x),g^{n+q+1}(x)\big)\in V_{-(n-1)p}\subseteq U$. Therefore, $\{g^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X,\mathfrak{U})$. Let $x_0=\lim g^n(x)$. Since g is uniformly continuous we have $g(x_0)=g\big(\lim g^n(x)\big)=\lim g^{n+1}(x)=x_0$ and so x_0 is a fixed point of g. However,

$$g\big(f(x_0)\big)=f\big(g(x_0)\big)=f(x_0).$$

Thus, $f(x_0)$ is also a fixed point of g. We conclude that $f(x_0)=x_0$.

Corollary 1. [Banach.] *If $f\colon X\to X$, where X is a complete metric space (metric d) and $d\big(f(x),f(y)\big)\leqslant a d(x,y)$ for some $a\in[0,1)$ and all $x,y\in X$, then f has a unique fixed point.*

Proof. If $a=0$ then f is a constant mapping and so has a unique fixed point. If $a\neq 0$ then in $X\times X$ define $V_n=\{(x,y)|\ d(x,y)<a^{-n}\}$, $n\in Z$. Then $\{V_n\}_{n\in Z}$ shows that f is u-contracting.

Corollary 2. [Kolmogoroff–Fomin.] *Suppose $f\colon X\to X$, $(X,\mathfrak{U})$ a complete uniform space, and suppose some iterate of f, say f^q, is u-contracting. Then f has a unique fixed point.*

Proof. By the theorem f^q has a unique fixed point, say x_0. Then

$$f^q\big(f(x_0)\big)=f\big(f^q(x_0)\big)=f(x_0)$$

and so $f(x_0)$ is a fixed point of f^q. Thus, $f(x_0)=x_0$. If $f(y)=y$ then we would have $f^q(y)=y$ and again, $y=x_0$.

Corollary 3. *Suppose $f,g\colon X\to X$, $(X,\mathfrak{U})$ a complete uniform space and suppose $f\big(g(x)\big)=g\big(f(x)\big)$ for all $x\in X$. If either f or g is u-contracting, then f and g have a common fixed point.*

Proof. Suppose f is u-contracting. Then f has a unique fixed point, say x_0. Then $f\big(g(x_0)\big)=g\big(f(x_0)\big)=g(x_0)$ whence $g(x_0)=x_0$.

From the proof of the theorem it is clear that the definition of u-contracting is slightly more stringent than actually necessary. In particular, the requirement that for each $n\in Z$ there is a $p(n)>0$ such that $V_{n-p(n)}\circ V_{n-p(n)}\subseteq V_n$ can be relaxed to state that for each n less than some integer N there is a $p(n)>0$ such that $V_{n-p(n)}\circ V_{n-p(n)}\subseteq V_n$. Also, for a given $f\colon X\to X$ we do not need that $\bigcup_{n\in Z}V_n=X\times X$. Rather, we need that for some $x\in X$ there is an $n\in Z$ for which $\big(x,f(x)\big)\in V_n$.

Corollary 4. [Edelstein.] *If $f\colon X\to X$ is (ε,a)-uniformly locally contractive ($d\big(f(x),f(y)\big)\leqslant a d(x,y)$ when $d(x,y)<\varepsilon$, $a\in[0,1)$, and $\varepsilon>0$) where (X,d) is a complete metric space and if for each $(x,y)\in X\times X$ there is an integer $n>0$ such that $d\big(f^n(x),f^n(y)\big)<\varepsilon$, then f has a unique fixed point.*

Proof. Define

$$V_{-n}=\{(x,y)|\ d(x,y)<a^n\varepsilon\},\quad n=0,1,2,\ldots$$

and

$$V_n=\{(x,y)|\ \big(f^n(x),f^n(y)\big)\in V_0\},\quad n=1,2,\ldots$$

(If $a=0$ define

$$V_0=\{(x,y)|\ d(x,y)<\varepsilon\}$$

and

$$V_{-n}=\{(x,y)|\ d(x,y)<\varepsilon 2^{-n}\},\quad n=1,2,\ldots).$$

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922), pp. 133-181.

[2] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. 12 (1961), pp. 7-10.

[3] J. L. Kelley, *General topology*, D. Van Nostrand, New York 1955.

[4] A. N. Kolmogoroff and S. V. Fomin, *Elements of the theory of functions and functional analysis*, Vol. I, *Metric and normed spaces*, Graylock, New York 1957.

NORTH CAROLINA STATE UNIVERSITY
Raleigh, N. C.

Reçu par la Rédaction le 20. 3. 1968