# $\Sigma$-spaces

by

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**0. Introduction.** $P$-spaces due to K. Morita [8] is a basic and the most important concept in the theory of product spaces. A perfectly normal space and a countably compact space were shown to be two trivial examples of $P$-spaces by K. Morita [8], Theorem 3.2 and Corollary 3.4. Beside these spaces the following are non-trivial examples of $P$-spaces:

(a) Paracompact Hausdorff spaces which are complete in the sense of E. Čech (Z. Frolik [3]).

(b) $M$-spaces (K. Morita [8]).

(c) Paracompact $p$-spaces (A. Arhangelskii [1]).

(d) $M^*$-spaces (T. Ishii [4]).

As a matter of fact the second concept is a generalization of the first and the last three cases are the same with each other for paracompact Hausdorff spaces (cf. A. Arhangelskii [1], K. Morita [8] and [9]). The purpose of this paper is to introduce $\Sigma$-spaces, which are $P$-spaces and offer a concept of real generalization of $M$-spaces, and study several features of those. The following are some of their features.

(i) If a space $X$ is a countable sum of closed $\Sigma$-spaces $X_i$, $i=1,2,\ldots$, then $X$ is a $\Sigma$-space.

(ii) If $X_i$, $i=1,2,\ldots$, are paracompact $\Sigma$-spaces, then $\prod X_i$ is a paracompact $\Sigma$-space.

(iii) If $\{X_\alpha\}$ is a locally finite closed covering of a space $X$ and each $X_\alpha$ is a $\Sigma$-space, then $X$ is a $\Sigma$-space.

(iv) If $f\colon X\to Y$ is a quasi-perfect mapping onto, then $X$ is a $\Sigma$-space if and only if $Y$ is a $\Sigma$-space.

(v) If $X$ is a regular $\Sigma$-space and $S$ is a paracompact $G_\delta$-set of $X$, then $S$ is a $\Sigma$-space.

(vi) Every regular space with a $\sigma$-locally finite net is a $\Sigma$-space.

(vii) If $X$ is a paracompact $P$-space and $Y$ is a paracompact $\Sigma$-space, then $X\times Y$ is paracompact.

K. Morita [9] constructed a non-$M$-space $X$ which is the sum of two closed $M$-spaces. So the property (i) is remarkable and convenient to handle Σ-spaces. All spaces considered in this paper are Hausdorff spaces. All mappings are continuous. A mapping $f\colon X\to Y$ is *quasi-perfect* if $f$ is closed and $f^{-1}(y)$ is countably compact for every point $y$ in $Y$. If moreover every $f^{-1}(y)$ is compact, then $f$ is *perfect*. The index $i$ runs always through positive integers.

Section 1 gives definitions and related observation which will be needed for the next section. Section 2 illustrates a location of Σ-spaces among other classes of spaces. Further properties of Σ-spaces will be given in Section 3 and the last Section 4 offers applications of Σ-spaces to product spaces.

## 1. Preliminaries.

1.1. Definition. Let $\mathcal{F}$ be a covering of a space $X$ and $x$ a point of $X$. Then we set

$$C(x,\mathcal{F})=\bigcap\{F\colon x\in F\in\mathcal{F}\}.$$

A *Σ-net* of a space $X$ is a sequence $\{\mathcal{F}_i\}$ of locally finite closed coverings satisfying the following condition:

If $K_1\supset K_2\supset\ldots$ is a sequence of non-empty closed sets of $X$ such that

$$K_i\subset C(x,\mathcal{F}_i)$$

for some point $x$ in $X$ and for each $i$, then

$$\bigcap K_i\neq\emptyset.$$

If we set

$$C(x)=\bigcap C(x,\mathcal{F}_i),$$

then it is to be noted that every $C(x)$ is closed and countably compact. A *strong Σ-net* is a Σ-net such that each $C(x)$ is compact. A space $X$ is a *Σ-space* or a *strong Σ-space*, if $X$ has respectively a Σ-net or a strong Σ-net. Clearly every paracompact Σ-space is a strong Σ-space.

1.2. Definition [1]. A Σ-net is a *σ-net*, if $C(x)=x$ for each point $x$. A space is a *σ-space* if it has a σ-net.

1.3. Lemma. *Let $\{\mathcal{F}_i\}$ be a Σ-net of a space $X$. If for each $i$ $\mathcal{H}_i$ is a locally finite closed covering of $X$ refining $\mathcal{F}_i$, then $\{\mathcal{H}_i\}$ is a Σ-net of $X$.*

1.4. Lemma. *Let $X$ be a Σ-space. Then $X$ has a Σ-net $\{\mathcal{F}_i\}$ which satisfies the following:*

(i) *Every $\mathcal{F}_i$ is (finitely) multiplicative.*

(ii) $\mathcal{F}_i=\{F(\alpha_1\ldots\alpha_i)\colon \alpha_1,\ldots,\alpha_i\in\Omega\}$.

(iii) *Every $F(\alpha_1\ldots\alpha_i)$ is the sum of all $F(\alpha_1\ldots\alpha_i\alpha_{i+1})$, $\alpha_{i+1}\in\Omega$.*

(iv) *For every $x\in X$ there exists a sequence $\alpha_1,\alpha_2,\ldots,\in\Omega$ such that if $C(x)\subset U$ with $U$ open, then*

$$C(x)\subset F(\alpha_1\ldots\alpha_i)\subset U$$

*for some $i$.*

Proof. Let $\{\mathcal{H}_i\}$ be a Σ-net of $X$. Let $\mathcal{H}_i'$ be the collection of all finite intersections of elements of $\mathcal{H}_i$. Then $\mathcal{H}_i'$ is a locally finite multiplicative closed covering of $X$. Set

$$\mathcal{H}_i'=\{H_i(\alpha_i)\colon \alpha_i\in A_i\}.$$

Let $\Omega$ be a set containing all $A_i$ whose power $|\Omega|$ is the supremum of all $|A_i|$. If we set

$$H_i(\alpha_i)=\emptyset\quad\text{for}\quad \alpha_i\in\Omega-A_i,$$

then we can express $\mathcal{H}_i'$ as

$$\mathcal{H}_i'=\{H_i(\alpha)\colon \alpha\in\Omega\}.$$

Set

$$F(\alpha_1\ldots\alpha_i)=\bigcap_{j\leqslant i}H_j(\alpha_j).$$

$$\mathcal{F}_i=\{F(\alpha_1\ldots\alpha_i)\colon \alpha_1,\ldots,\alpha_i\in\Omega\}.$$

Since $\mathcal{F}_i<$ (refines) $\mathcal{H}_i$, $\{\mathcal{F}_i\}$ is, by Lemma 1.3, a Σ-net satisfying the conditions (i), (ii) and (iii). Let $x$ be an arbitrary point of $X$. Since $\mathcal{H}_i'$ is multiplicative, there exists an $\alpha_i\in\Omega$ such that

$$C(x,\mathcal{H}_i')=H_i(\alpha_i).$$

Then it can easily be seen that the sequence $\alpha_1,\alpha_2,\ldots$ satisfies the condition (iv).

1.5. Definition. A Σ-net $\{\mathcal{F}_i\}$ with the property in Lemma 1.4 is *spectral*. If the power of the index set $\Omega$ is $\mathfrak{m}$, then it is a *spectral $\Sigma(\mathfrak{m})$-net*. A space is a *$\Sigma(\mathfrak{m})$-space* if it has a spectral $\Sigma(\mathfrak{m})$-net. A *strong $\Sigma(\mathfrak{m})$-space* is now easy to be understood.

A space is a $\Sigma(1)$-space or a strong $\Sigma(1)$-space if and only if it is respectively countably compact or compact. It is evident from the above construction that a space $X$ is a $\Sigma(\mathfrak{m})$-space if and only if $X$ has a Σ-net $\{\mathcal{F}_i=\{F_i(\alpha)\colon \alpha\in A_i\}\}$ such that $|A_i|\leqslant\mathfrak{m}$ for each $i$.

1.6. Theorem. *If $2\leqslant\mathfrak{m}\leqslant\aleph_0$, then a space $X$ is a $\Sigma(\mathfrak{m})$-space if and only if $X$ is a $\Sigma(2)$-space.*

Proof. Since the sufficiency is evident, we merely prove the necessity. Let $\{\mathcal{F}_i\}$ be a spectral $\Sigma(\mathfrak{m})$-net. Then we can write as

$$\mathcal{F}_i=\{F_{ij}\colon j=1,2,\ldots\}.$$

[1] In our previous paper [10] we name a paracompact space having a σ-locally finite net a σ-space. It is evident that a space is a σ-space in the present definition if and only if it has a σ-locally finite net. In [10] a σ-net was defined as a σ-net in the present sense with an additional condition which we shall name a spectral σ-net. Please pardon the author for this confusion, while we have much benefit for these simple expressions.

Set

$$\mathcal{F}_{ij} = \{F_{ij}, X\}.$$

Then we obtain a $\Sigma$-net

$$\{\mathcal{F}_{ij}\colon i, j = 1, 2, \ldots\}.$$

Consider this $\{\mathcal{F}_{ij}\}$ as $\{\mathcal{K}_i\}$. Starting from $\{\mathcal{K}_i\}$ we obtain a spectral $\Sigma(2)$-net as in the proof of Lemma 1.4.

1.7. Lemma. *Let $f\colon X \to Y$ be a quasi-perfect mapping and $\mathcal{F}$ a locally finite closed collection of $X$. Then $f(\mathcal{F})$ is a locally finite closed collection of $Y$.*

This is proved by A. Okuyama [12].

1.8. Theorem. *Let $f\colon X \to Y$ be a quasi-perfect mapping onto. Then $X$ is a $\Sigma$-space if and only if $Y$ is a $\Sigma$-space.*

Proof. Suppose that $X$ is a $\Sigma$-space. By Lemma 1.4 there exists a spectral $\Sigma$-net $\{\mathcal{F}_i\}$ of $X$. By Lemma 1.7 every $f(\mathcal{F}_i)$ is a locally finite closed covering of $Y$. To see $\{f(\mathcal{F}_i)\}$ is a $\Sigma$-net of $Y$ let $L_1 \supset L_2 \supset \ldots$ be a sequence of non-empty closed sets of $Y$ with

$$L_i \subset C\big(y, f(\mathcal{F}_i)\big)$$

for some point $y$ in $Y$ and each $i$. Take an arbitrary point $x$ in $f^{-1}(y)$. If there would exist an $i$ with

$$f^{-1}(L_i) \cap C(x) = \emptyset,$$

then there would exist a $j$ with

$$f^{-1}(L_i) \cap C(x, \mathcal{F}_j) = \emptyset.$$

Let $k$ be the maximum of $i$ and $j$. Then

$$f^{-1}(L_k) \cap C(x, \mathcal{F}_k) = \emptyset.$$

Since $\mathcal{F}_k$ is multiplicative, there is an element $F$ of $\mathcal{F}_k$ with

$$F = C(x, \mathcal{F}_k).$$

Then $L_k \cap f(F) = \emptyset$ and hence

$$L_k \cap C\big(y, f(\mathcal{F}_k)\big) = \emptyset,$$

a contradiction. Thus $f^{-1}(L_i) \cap C(x) \neq \emptyset$ for any $i$ and

$$\bigcap f^{-1}(L_i) \neq \emptyset$$

by the countable compactness of $C(x)$. Hence $\bigcap L_i \neq \emptyset$. Therefore $\{f(\mathcal{F}_i)\}$ is a $\Sigma$-net of $Y$ and $Y$ is a $\Sigma$-space.

Conversely suppose that $Y$ is a $\Sigma$-space. Let $\{\mathcal{K}_i\}$ be a spectral $\Sigma$-net of $Y$. Then each $f^{-1}(\mathcal{K}_i)$ is a locally finite closed covering of $X$. To see



$\{f^{-1}(\mathcal{K}_i)\}$ is a $\Sigma$-net of $X$ let $K_1 \supset K_2 \supset \ldots$ be a sequence of non-empty closed sets of $X$ such that

$$K_i \subset C\big(x, f^{-1}(\mathcal{K}_i)\big)$$

for some point $x$ in $X$ and for each $i$. If there would exist an $i$ with

$$f(K_i) \cap C(y) = \emptyset,$$

then there would exist a $j$ with

$$f(K_i) \cap C(y, \mathcal{K}_j) = \emptyset.$$

Let $k$ be the maximum of $i$ and $j$. Then

$$f(K_k) \cap C(y, \mathcal{K}_k) = \emptyset.$$

Let $H$ be an element of $\mathcal{K}_k$ with $H = C(y, \mathcal{K}_k)$. Then

$$K_k \cap f^{-1}(H) = \emptyset,$$

which would imply

$$K_k \cap C\big(x, f^{-1}(\mathcal{K}_k)\big) = \emptyset,$$

a contradiction. Therefore

$$f(K_i) \cap C(y) \neq \emptyset$$

for any $i$ and $\bigcap f(K_i) \neq \emptyset$ by the countable compactness of $C(y)$. Choose a point $y_1$ in $\bigcap f(K_i)$. Then

$$K_i \bigcap f^{-1}(y_1) \neq \emptyset$$

for any $i$ and hence

$$\bigcap K_i \neq \emptyset$$

by the countable compactness of $f^{-1}(y_1)$. Thus $X$ is a $\Sigma$-space and the theorem is completely proved.

1.9. Corollary. *Let $X$ be a $\Sigma$-space and $Y$ be a compact space. Then $X \times Y$ is a $\Sigma$-space.*

Proof. Since the projection of $X \times Y$ onto $X$ is perfect, the assertion is trivially true by Theorem 1.8.

1.10. Corollary. *Let $X$ be a space and $\{X_\alpha\}$ a locally finite closed covering of $X$. If each $X_\alpha$ is a $\Sigma$-space, then $X$ is a $\Sigma$-space.*

Proof. Let $E$ be the topological disjoint sum of $X_\alpha$. Then $E$ is evidently a $\Sigma$-space. Let $f\colon E \to X$ be the natural mapping. Then $f$ is quasi-perfect. Thus $X$ is a $\Sigma$-space by Theorem 1.8.

1.11. Definition. A space $X$ is a *pre-$\sigma$-space* if there exists a quasi-perfect mapping of $X$ onto a $\sigma$-space $Y$.

Clearly every pre-$\sigma$-space is a $\Sigma$-space by Theorem 1.8. Recall that a space $X$ is an $M$-space if and only if there exists a quasi-perfect mapping of $X$ onto a metric space (cf. A. Arhangelskii [1] and K. Morita [8], Theorem 6.1). Since every metric space is a $\sigma$-space, every $M$-space is pre-$\sigma$. However there exists a pre-$\sigma$-space which is not an $M$-space by Example 2.3 below. Furthermore there exists a $\Sigma$-space which is not pre-$\sigma$ as will be shown in Example 2.4.

## 2. Location of $\Sigma$-spaces.

2.1. Example. *A paracompact $\sigma$-space which is not an $M$-space.* Consider a paracompact $\sigma$-space $X$ due to E. Michael [7], Example 12.1, which is not metric. Then $X$ is not an $M$-space, since every paracompact $\sigma$-space which is an $M$-space at the same time is always metric by A. Okuyama [11].

2.2. Example. *A paracompact $M$-space which is not a $\sigma$-space.* Let $Y$ be the product of $\mathfrak{m}$ copies of the closed unit interval, where $\mathfrak{m} > \aleph_0$. Then it is an $M$-space. Since every open set of a $\sigma$-space is an $F_\sigma$, $Y$ is not a $\sigma$-space.

2.3. Example. *A paracompact pre-$\sigma$-space which is neither a $\sigma$-space nor an $M$-space.* Let $X$ and $Y$ be the spaces given in Examples 2.1 and 2.2 respectively. Then $X\times Y$ is the desired. Let $\pi\colon X\times Y\to X$ be the projection. Since $\pi$ is perfect, $X\times Y$ is pre-$\sigma$. Since every closed subset of an $M$-space is an $M$-space, $X\times Y$ is not an $M$-space. By an analogous reason for a $\sigma$-space, $X\times Y$ is not a $\sigma$-space.

2.4. Example. *A paracompact $\Sigma$-space which is not a pre-$\sigma$-space but a countable sum of closed pre-$\sigma$-spaces.* Let $P$ be the space consisting of all ordinals less than or equal to the first uncountable ordinal $\omega_1$ with the order topology. Then $P$ is compact. Let $P_i$, $i = 1, 2, \ldots$, be copies of $P$. Let $J$ be the sum of all $P_i$ where $\omega_1$ is one and only one common point of $P_i$ and $P_j$ with $i\neq j$. When $x\in J-\{\omega_1\}$, a neighborhood base of $x$ in $J$ is a neighborhood base of $x$ in $P_i$ with $x\in P_i$. For $\alpha<\omega_1$, let $P_i(\alpha)$ be the set of all ordinals in $P_i$ greater than $\alpha$. Set

$$U(\alpha_1\alpha_2\ldots) = \bigcup P_i(\alpha_i)\,.$$

The collection of all possible $U(\alpha_1\alpha_2\ldots)$ is a neighborhood base of $\omega_1$ in $J$. Then $J$ is regular. Since $J$ is $\sigma$-compact, $J$ is a paracompact $\Sigma$-space. Since each $P_i$ is a closed pre-$\sigma$-space in $J$, $J$ is a countable sum of closed pre-$\sigma$-spaces.

To prove $J$ is not pre-$\sigma$ assume the contrary. Then there would exist a quasi-perfect mapping $f$ of $J$ onto some $\sigma$-space $X$. Since every closed set of a $\sigma$-space is a $G_\delta$, $f^{-1}(f(\omega_1))$ would be a closed $G_\delta$ containing $\omega_1$. Hence there would exist a sequence $\alpha_1, \alpha_2, \ldots$ such that

$$U(\alpha_1\alpha_2\ldots)\subset f^{-1}(f(\omega_1))\,.$$



The countably compact set $f^{-1}(f(\omega_1))$ would contain a closed subset

$$\{\alpha_1+1, \alpha_2+1, \ldots\}$$

which is not countably compact, a contradiction. Thus $J$ is not pre-$\sigma$.

2.5. Remark. K. Morita [9] constructed a space $X$ which is not an $M$-space but the sum of two closed $M$-spaces. Thus the finite sum theorem is not true for $M$-spaces. However his space is not normal. The finite sum theorem for normal $M$-spaces is always true by J. Suzuki [13], Theorem 2. It is worth enough to point out that $J$ is a paracompact non-$M$-space which is the countable sum of closed $M$-spaces $P_i$. Here is another remark: $J$ is not an $M^*$-space as can be seen by analogous argument to 2.4.

2.6. Theorem. *Every $M^*$-space is a $\Sigma$-space.*

This is evident if we recall the definition of $M^*$-spaces: A space $X$ is an $M^*$-space, if $X$ has a sequence $\{\mathcal{F}_i\}$ of locally finite closed coverings such that two propositions $K_1\supset K_2\supset\ldots$ and Star $(x, \mathcal{F}_i)\supset K_i\neq\emptyset$, $i = 1, 2, \ldots$, imply $\bigcap \overline{K}_i\neq\emptyset$. This type of sequence is itself a $\Sigma$-net of $X$.

2.7. Theorem. *Every $\Sigma$-space is a $P$-space.*

Proof. Let $X$ be a $\Sigma$-space and $\{\mathcal{F}_i\}$ a spectral $\Sigma$-net of $X$. Let

$$\{G(\alpha_1\ldots\alpha_i)\colon\ \alpha_1, \ldots, \alpha_i\in\Omega\,,\ i = 1, 2, \ldots\}$$

be a system of open sets of $X$ such that

$$G(\alpha_1\ldots\alpha_i)\subset G(\alpha_1\ldots\alpha_i\alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \ldots$ Set

$$F(\alpha_1\ldots\alpha_i) = \bigcup\{F\in\mathcal{F}_i\colon\ F\subset G(\alpha_1\ldots\alpha_i)\}\,.$$

Then it is a closed set which is contained in $G(\alpha_1\ldots\alpha_i)$. Suppose that

$$\bigcup_i G(\alpha_1\ldots\alpha_i) = X.$$

To prove the corresponding sum

$$\bigcup_i F(\alpha_1\ldots\alpha_i)$$

covers $X$ assume the contrary. Then there would be a point $x$ of $X$ such that

$$x\in X-\bigcup_i F(\alpha_1\ldots\alpha_i)\,.$$

Since $\mathcal{F}_i$ is multiplicative, we can set

$$F_i = C(x, \mathcal{F}_i)$$

for a suitable element $F_i$ of $\mathcal{F}_i$. Since $\{\mathcal{F}_i\}$ is spectral, $F_1 \supset F_2 \supset \ldots$ Since

$$K_i = F_i - G(a_1 \ldots a_i) \neq \emptyset\,,$$

we obtain $\bigcap K_i \neq \emptyset$. On the other hand

$$\bigcap K_i \subset X - \bigcup_i G(a_1 \ldots a_i) = \emptyset\,,$$

a contradiction. The proof is completed.

## 3. Further features of $\Sigma$-spaces.

3.1. LEMMA. *Let $\{\mathcal{F}_i\}$ be a $\Sigma$-net of a space $X$. Let $\{K_i\}$ be a collection of closed sets of $X$ with the finite intersection property such that for some point $x$ and for each $i$ there exists a $j$ with $K_j \subset C(x, \mathcal{F}_i)$. Then $\bigcap K_i \neq \emptyset$.*

3.2. THEOREM. *Let $X$ be a space and $\{X_i\}$ a closed covering of $X$ such that each $X_i$ is a $\Sigma$-space. Then $X$ is a $\Sigma$-space.*

Proof. Let

$$\{\mathcal{F}_{ij}\colon\, j = 1, 2, \ldots\}$$

be a $\Sigma$-net of $X_i$. Set

$$\mathcal{F}'_{ij} = \{X\} \cup \mathcal{F}_{ij}\,.$$

Let us prove $\{\mathcal{F}'_{ij}\colon\, i, j = 1, 2, \ldots\}$ is a $\Sigma$-net of $X$. Let $\{K_{ij}\colon\, i, j = 1, 2, \ldots\}$ be a family of closed sets of $X$ with the finite intersection property such that

$$K_{ij} \subset C(x, \mathcal{F}'_{ij})$$

for some point $x$ in $X$, for each $i$ and for each $j$. Choose a $k$ with $x \in X_k$. Let $L$ be an arbitrary finite intersection of $\{K_{ij}\}$. Then for every $j$

$$L \cap K_{kj} \subset C(x, \mathcal{F}'_{kj}) = C(x, \mathcal{F}_{kj}) \subset X_k$$

and hence $L \cap X_k \neq \emptyset$. Thus $\{K_{ij}\}|X_k$ has the finite intersection property. By Lemma 3.1

$$\bigcap_{i,j=1}^{\infty} (K_{ij} \cap X_k) \neq \emptyset$$

and the theorem is proved.

It is to be noted that the sum theorem for pre-$\sigma$-spaces is not true by Example 2.4.

3.3. COROLLARY. *If a space $X$ has a closed covering $\{X_i\}$ such that each $X_i$ is countably compact, then $X$ is a $\Sigma$-space.*

This is evident from Theorem 3.2. But we can give a simple direct proof. Set

$$\mathcal{F}_i = \{X_1, \ldots, X_i, X\}\,.$$

Then $\{\mathcal{F}_i\}$ is clearly a $\Sigma$-net of $X$. $X$ is actually a $\Sigma(2)$-space.



3.4. COROLLARY. *If $X$ is a CW-complex, then $X$ is a $\Sigma$-space.*

Proof. As is well known $X$ is the sum of closed metric spaces $X_i$. Since each $X_i$ is a $\Sigma$-space, $X$ is a $\Sigma$-space by the sum theorem.

3.5. COROLLARY. *If $X$ is a totally normal $\Sigma$-space, then every open set $G$ of $X$ is a $\Sigma$-space.*

Proof. By the definition $X$ is totally normal if and only if $X$ is normal and every open set $G$ admits a locally finite (in $G$) covering $\mathcal{U}$ each element of which is an open $F_\sigma$-set of $X$. Let $\mathcal{U} = \{U_\alpha\colon\, \alpha \in A\}$ be such a covering. Since each closed set of $X$ is of course a $\Sigma$-space, each $U_\alpha$ is a $\Sigma$-space by the sum theorem. Since $G$ is normal as a relative space, there exists a covering $\{F_\alpha\colon\, \alpha \in A\}$ of $G$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$ and each $F_\alpha$ is closed in $G$. Since $U_\alpha$ is already a $\Sigma$-space, $F_\alpha$ is a $\Sigma$-space. Since $\{F_\alpha\colon\, \alpha \in A\}$ is locally finite in $G$, $G$ is a $\Sigma$-space by Corollary 1.10.

3.6. THEOREM. *Let $\{X_i\}$ be a sequence of strong $\Sigma$-spaces. Then $\prod X_i$ is a strong $\Sigma$-space.*

Proof. Let

$$\{\mathcal{F}^i_j\colon\, j = 1, 2, \ldots\}$$

be a strong $\Sigma$-net of $X_i$. Set

$$\mathcal{F}(i_1 \ldots i_j) = \mathcal{F}^1_{i_1} \times \ldots \times \mathcal{F}^j_{i_j} \times \prod_{k \geqslant j+1} X_k\,.$$

Then it is a locally finite closed covering of $\prod X_i$. Let us prove that

$$\{\mathcal{F}(i_1 \ldots i_j)\colon\, i_s = 1, 2, \ldots \text{ for } s = 1, \ldots, j\,,\; j = 1, 2, \ldots\}$$

is a $\Sigma$-net of $\prod X_i$. Let $x = (x_1, x_2, \ldots)$ be a point of $\prod X_i$ and

$$\mathcal{K} = \{K(i_1 \ldots i_j)\colon\, i_s = 1, 2, \ldots \text{ for } s = 1, \ldots, j\,,\; j = 1, 2, \ldots\}$$

a collection of closed sets of $\prod X_i$ with the finite intersection property such that

$$K(i_1 \ldots i_j) \subset C\big(x, \mathcal{F}(i_1 \ldots i_j)\big)$$

for each sequence $i_1, \ldots, i_j$. Evidently

$$C(x') = \prod C(x'_i)$$

for every point $x' = (x'_1, x'_2, \ldots)$ of $\prod X_i$. Hence $C(x')$ is compact for every point $x'$.

Assume that there exists a finite intersection $L$ of elements of $\mathcal{K}$ with

$$L \cap C(x) = \emptyset\,.$$

By the compactness of $C(x)$ there would exist an $n$ and open sets $G_i \supset C(x_i)$ for $i=1, \ldots, n$ such that

$$L \cap \Big(\prod_{i=1}^{n} G_i \times \prod_{i=n+1}^{\infty} X_i\Big) = \emptyset .$$

Choose $j_i$ for $i=1, \ldots, n$ such that

$$G_i \supset C(x_i, \mathcal{F}^i_{j_i}) \supset C(x_i) .$$

Then

$$L \cap \Big(\prod_{i=1}^{n} C(x_i, \mathcal{F}^i_{j_i}) \times \prod_{i=n+1}^{\infty} X_i\Big) = \emptyset .$$

Since

$$C\big(x, \mathcal{F}(j_1 \ldots j_n)\big) = \prod_{i=1}^{n} C(x_i, \mathcal{F}^i_{j_i}) \times \prod_{i=n+1}^{\infty} X_i ,$$

then

$$L \cap C\big(x, \mathcal{F}(j_1 \ldots j_n)\big) = \emptyset ,$$

a contradiction. Thus

$$L \cap C(x) \neq \emptyset .$$

Hence $\bigcap \{K\colon K \in \mathcal{K}\} \neq \emptyset$ and the theorem is proved.

3.7. Theorem. *Let $X$ be a regular $\Sigma$-space and $Y$ be a paracompact subset of $X$ with the expression $Y = \bigcap G_i$ with every $G_i$ open. Then $Y$ is a $\Sigma$-space.*

Proof. Let $\{\mathcal{F}_i\}$ be a $\Sigma$-net of $X$. Let $\{V\}$ be a collection of open sets of $X$ covering $Y$ such that $\{\overline{V}\}$ refines $\{G_i\}$. Let $\mathcal{H}_i$ be a locally finite (in $Y$) closed covering of $Y$ refining $\{V\}|Y$. Set

$$\mathcal{L}_i = \mathcal{H}_i \wedge (\mathcal{F}_i|Y) .$$

Then $\{\mathcal{L}_i\}$ is a $\Sigma$-net of $Y$ as follows. Let $x$ be a point of $Y$ and $\{K_i\}$ a collection of closed sets of $Y$ with the finite intersection property such that

$$K_i \subset C(x, \mathcal{L}_i)$$

for each $i$. Since $\overline{K}_i \subset C(x, \mathcal{F}_i)$,

$$\bigcap \overline{K}_i \neq \emptyset .$$

Choose a point $y$ from this intersection. Then $y \in \overline{K}_i \subset G_i$ for each $i$ and hence $y \in Y \cap \overline{K}_i$ for each $i$. Since $K_i$ is closed in $Y$, then $y \in K_i$ for each $i$, proving $\bigcap K_i \neq \emptyset$. The proof is finished.

3.8. Lemma. *Let $\mathcal{F} = \bigcup \mathcal{F}_i$ be a $\sigma$-locally finite closed covering of a space $X$, where each $\mathcal{F}_i$ is a locally finite closed collection. Let $x$ be a point of $X$ and $\{F_i(x)\}$ be the subcollection of $\mathcal{F}$ which consists of all elements of $\mathcal{F}$ containing $x$. Let the following condition be satisfied:*



*If $\{K_i\}$ is a collection of closed sets of $X$ with the finite intersection property such that $K_i \subset F_i(x)$ for some point $x$ and for each $i$, then $\bigcap K_i \neq \emptyset$.*

*Then $X$ has a $\Sigma$-net.*

Proof. Set

$$\mathcal{H}_i = \{X\} \cup \mathcal{F}_i .$$

Then $\{\mathcal{H}_i\}$ is a $\Sigma$-net.

3.9. Theorem. *Let $A$ be an $F_{\sigma\delta}$-set of a $\Sigma$-space $X$. Then $A$ is a $\Sigma$-space.*

Proof. Set

$$A = \bigcap_i A_i , \quad A_i = \bigcup_j A_{ij} ,$$

where each $A_{ij}$ is a closed set of $X$ such that $A_{ij} \subset A_{i,j+1}$ for each $j$. Let $\{\mathcal{F}_i\}$ be a $\Sigma$-net of $X$. Set

$$\mathcal{F}_{ijk} = \mathcal{F}_i|A_{jk} ,$$

$$\mathcal{H} = \bigcup \{\mathcal{F}_{ijk}|A\colon i, j, k = 1, 2, \ldots\} .$$

Then $\mathcal{H}$ is a $\sigma$-locally finite closed covering of $A$. Let $x$ be a point of $A$. Set

$$\{H \in \mathcal{H}\colon x \in H\} = \{H_1, H_2, \ldots\} .$$

Let $\{K_i\}$ be the collection of closed sets of $A$ with the finite intersection property such that $K_i \subset H_i$ for each $i$. Let $j_i$ be the smallest integer of $j$ such that $x \in A_{ij}$. Let $\{F_i\}$ be the collection of all elements of $\bigcup \mathcal{F}_i$ containing $x$. Set

$$\mathcal{L} = \{F_k \cap A_{ij}\colon k = 1, 2, \ldots, j \geqslant j_i, i = 1, 2, \ldots\} = \{L_1, L_2, \ldots\} .$$

Then

$$\{H_i\} = \mathcal{L}|A .$$

Since $C(x, \mathcal{F}_i)$ contains a finite intersection of elements of $\mathcal{L}$,

$$\bigcap \overline{K}_i \neq \emptyset .$$

Choose a point $y$ from this intersection. Since

$$y \in \cap \overline{K}_i \subset \cap \overline{H}_i \subset \cap \overline{L}_i ,$$

then

$$y \in A_{ij} \quad \text{for} \quad j \geq j_i \text{ and } i = 1, 2, \ldots$$

Thus $y \in A_i$ for each $i$ and $y$ is a point of $A$. Hence

$$y \in \bigcap K_i$$

and the theorem is proved by Lemma 3.8. The proof is completed.

3.10. Lemma. *A normal space $X$ is strongly normal (i.e. countably paracompact and collectionwise normal) if and only if the following condition in $X$ is satisfied:*

*For each locally finite closed collection $\{F_\alpha\colon \alpha\in A\}$ there exists a locally finite open collection $\{G_\alpha\colon \alpha\in A\}$ such that $F_\alpha\subset G_\alpha$ for each $\alpha\in A$.*

This was proved by M. Katětov [5].

3.11. LEMMA. *Let $X$ be a regular strong $\Sigma$-space and*

$$\{\mathcal{F}_i=\{F_{i\alpha}\colon \alpha\in A_i\}\}$$

*a strong $\Sigma$-net of $X$. If there exists, for each $i$, a locally finite open covering*

$$\mathcal{U}_i=\{U_{i\alpha}\colon \alpha\in A_i\}$$

*of $X$ with $F_{i\alpha}\subset U_{i\alpha}$, $\alpha\in A_i$, then $X$ is paracompact.*

Proof. First we assume that each $\mathcal{F}_i$ is multiplicative without loss of generality. Let $\mathcal{G}$ be an arbitrary open covering of $X$. Let $\mathcal{G}(x)$ be a finite subcollection of $\mathcal{G}$ covering $C(x)$. Set

$$G(x)=\bigcup\{G\colon G\in\mathcal{G}(x)\},$$
$$\mathcal{H}=\{G(x)\colon x\in X\},$$
$$\mathcal{F}_i'=\{F\in\mathcal{F}_i\colon F<\mathcal{H}\}=\{F_{i\alpha}\colon \alpha\in B_i\}.$$

Since $C(x,\mathcal{F}_i)\subset G(x)$ for some $i$ and $\mathcal{F}_i$ is multiplicative, $\bigcup\mathcal{F}_i'$ covers $X$. For each $F_{i\alpha}\in\mathcal{F}_i'$ let $G(x_{i\alpha})$ be an element of $\mathcal{H}$ with $F_{i\alpha}\subset G(x_{i\alpha})$. Set

$$V_{i\alpha}=U_{i\alpha}\cap G(x_{i\alpha}),\qquad \alpha\in B_i,$$
$$\mathcal{L}=\bigcup\{\mathcal{G}(x_{i\alpha})|V_{i\alpha}\colon \alpha\in B_i,\ i=1,2,\ldots\}.$$

Then $\mathcal{L}$ is a $\sigma$-locally finite open covering of $X$ refining $\mathcal{G}$. Thus $X$ is paracompact by E. Michael [6].

3.12. THEOREM. *If $X$ is a collectionwise normal, strong $\Sigma$-space, then $X$ is paracompact.*

Proof. Since $X$ is a $P$-space by Theorem 2.6, $X$ is countably paracompact by K. Morita [8], Theorem 3.10. Thus by Lemmas 3.10 and 3.11, $X$ is paracompact.

3.13. THEOREM. *If $\{X_i\}$ is a sequence of paracompact $\Sigma$-spaces then $\prod X_i$ is a paracompact $\Sigma$-space.*

Proof. Let $\{\mathcal{F}_j^i\colon j=1,2,\ldots\}$ be a $\Sigma$-net of $X_i$ and $\{\mathcal{U}_j^i\colon j=1,2,\ldots\}$ a sequence of locally finite open covering of $X$ such that every $\mathcal{U}_j^i$ is in one-one correspondence with $\mathcal{F}_j^i$ as stated in the condition of Lemma 3.11. Then

$$\Big\{\mathcal{F}_{i_1}^1\times\ldots\times\mathcal{F}_{i_j}^j\times\prod_{k\geq j+1}X_k\Big\}$$

is a strong $\Sigma$-net of $\prod X_i$ as was shown in the proof of Theorem 3.6. Moreover every

$$\mathcal{F}_{i_1}^1\times\ldots\times\mathcal{F}_{i_j}^j\times\prod_{k\geq j+1}X_k$$



is in 'good' one-one correspondence with a locally finite open covering

$$\mathcal{U}_{i_1}^1\times\ldots\times\mathcal{U}_{i_j}^j\times\prod_{k\geq j+1}X_k.$$

Hence by Lemma 3.11 $\prod X_i$ is paracompact and the proof is completed.

3.14. COROLLARY. *If $\{X_i\}$ is a sequence of paracompact pre-$\sigma$-spaces, then $\prod X_i$ is a paracompact pre-$\sigma$-space.*

Proof. $\prod X_i$ is paracompact by Theorem 3.13. Let $f_i$ be a perfect mapping of $X_i$ onto a $\sigma$-space $Y_i$. Then $\prod f_i$ is a perfect mapping of $\prod X_i$ onto $\prod Y_i$. Since $\prod Y_i$ is a $\sigma$-space by Theorem 3.6, $\prod X_i$ is a pre-$\sigma$-space.

3.15. THEOREM. *Let $X$ be a paracompact $\Sigma$-space. Then $X$ is a $\sigma$-space if and only if the diagonal $\Delta$ in $X\times X$ is a $G_\delta$-set.*

Proof. Since the sufficiency is evident, we prove the necessity. Let $\{\mathcal{F}_i\}$ be a $\Sigma$-net of $X$. Let $\{G_i\}$ be a sequence of open sets of $X\times X$ with

$$\Delta=\bigcap G_i.$$

For each point $x$ in $X$ choose an open neighborhood $U_i(x)$ with $U_i(x)\times U_i(x)\subset G_i$. Let $\mathcal{H}_i$ be a locally finite closed covering of $X$ which refines $\{U_i(x)\colon x\in X\}$. Set

$$\mathcal{L}_i=\mathcal{F}_i\wedge\mathcal{H}_i.$$

Then $\{\mathcal{L}_i\}$ is a $\Sigma$-net of $X$ by Lemma 1.3. If $C(x,\bigcup\mathcal{L}_i)$ would contain a point $x'$ different from $x$, then there would exist an $n$ with $(x,x')\notin G_n$. Let $L$ be an element of $\mathcal{L}_n$ with $\{x,x'\}\subset L$. Choose $U_n(x'')$ with $L\subset U_n(x'')$. Then $(x,x')\in U_n(x'')\times U_n(x'')$. On the other hand $U_n(x'')\times U_n(x'')\subset G_n$, a contradiction. Hence $C(x,\bigcup\mathcal{L}_i)=x$ for each point $x$ in $X$ and the proof is finished.

This theorem is to be compared to a metrization theorem due to A. Okuyama [11] and to C. Borges [2], Theorem 8.1: A paracompact $M$-space is metrizable if and only if the diagonal is a $G_\delta$-set. There may not be an elegant metrization theorem for $\Sigma$-spaces, because of the character of $\Sigma$-spaces itself such that they generalize $M$-spaces and $\sigma$-spaces at the same time. For the convenience of the reader let us give the following which is not elegant at all: A $\Sigma$-space $X$ with a $\Sigma$-net $\{\mathcal{F}_i\}$ is metrizable if and only if $X$ is an $M$-space and $\bigcup\mathcal{F}_i$ has a subcovering each element of which is metrizable. This is a direct consequence of Okuyama–Borges' theorem. The condition for $X$ to be an $M$-space cannot be dropped, since any CW-complex is a $\Sigma$-space satisfying the last condition.

3.16. THEOREM. *Let $X$ be a space and $X_1, X_2, \ldots$ a sequence of subsets of $X$. If each $X_i$ is a strong $\Sigma$-space, then $\bigcap X_i$ is a strong $\Sigma$-space.*

Proof. (i) Let us prove first that $X_1 \cap X_2$ is a strong $\Sigma$-space. Let $\{\mathcal{F}_{ki}: i=1,2,\ldots\}$ be a spectral strong $\Sigma$-net of $X_k$ for $k=1,2$. Set

$$\mathcal{F}_i = (\bigwedge_{k=1,2} \mathcal{F}_{ki})|X_1 \cap X_2 .$$

To see $\{\mathcal{F}_i\}$ forms a strong $\Sigma$-net of $X_1 \cap X_2$ let $K_1 \supset K_2 \supset \ldots$ be a sequence of non-empty closed sets of $X_1 \cap X_2$ such that

$$K_i \subset C(x, \mathcal{F}_i)$$

for some point $x$ in $X_1 \cap X_2$ and for each $i$. Set

$$C_k(x) = \bigcap_i C(x, \mathcal{F}_{ki}) , \qquad k=1,2 ,$$

$$C(x) = \bigcap_i C(x, \mathcal{F}_i) .$$

Since

$$C_1(x) \cap C_2(x) \subset C(x, \mathcal{F}_{1i}) \cap C(x, \mathcal{F}_{2i}) = C(x, \mathcal{F}_i)$$

for each $i$, $C_1(x) \cap C_2(x) \subset C(x)$. Since it is evident that $C(x) \subset C_1(x) \cap C_2(x)$, we obtain

$$C(x) = C_1(x) \cap C_2(x) .$$

Thus $C(x)$ is compact.

Let $K_i'$ be the closure of $K_i$ in $X_1$. Set

$$K = \bigcap K_i' .$$

Since $K_i' \subset C(x, \mathcal{F}_{1i})$ for each $i$, $K$ is not empty. To prove $K \cap C(x) \neq \emptyset$ assume the contrary. Since $K \subset C_1(x)$, then $K \cap C_2(x) = \emptyset$. Since $K$ and $C_2(x)$ are compact and all spaces considered in this paper have been assumed to be Hausdorff, there exists an open set $U$ in $X$ such that

$$C_2(x) \subset U \subset \bar{U} \subset X - K .$$

Since $\{\mathcal{F}_{2i}\}$ is a spectral $\Sigma$-net of $X_2$, there exists a $j$ with

$$C(x, \mathcal{F}_{2j}) \subset U .$$

Since $K_j \subset C(x, \mathcal{F}_{2j})$, $\bar{K}_j \subset \bar{U}$ and hence $K_j' \subset \bar{U}$. Therefore $K \subset \bar{U} \subset X-K$, a contradiction. Thus we obtain $K \cap C(x) \neq \emptyset$. Since $C(x) \subset X_1 \cap X_2$, then $K_i' \cap C(x) = K_i \cap C(x)$. Therefore

$$(\bigcap K_i) \cap C(x) = K \cap C(x) \neq \emptyset .$$

Thus we know that $X_1 \cap X_2$ is a strong $\Sigma$-space.

(ii) By the above observation every finite intersection of $X_i$'s is a strong $\Sigma$-space. Thus we assume without loss of generality that $X_1 \supset X_2 \supset \ldots$ Set

$$Y = \bigcap X_i .$$



Let $\{\mathcal{F}_{ij}: j=1,2,\ldots\}$ be a spectral strong $\Sigma$-net of $X_i$ such that

$$\mathcal{F}_{i1} > \mathcal{F}_{i2} > \ldots ,$$

$$\mathcal{F}_{1j} > \mathcal{F}_{2j} > \ldots$$

Set

$$\mathcal{F}_i = \mathcal{F}_{ii}|Y .$$

Let us prove $\{\mathcal{F}_i\}$ is a strong $\Sigma$-net of $Y$. Let $K_1 \supset K_2 \supset \ldots$ be a sequence of non-empty closed sets of $Y$ such that

$$K_i \subset C(y, \mathcal{F}_i)$$

for some point $y$ in $Y$ and for each $i$.

Now $C(y)$ and $C_i(y)$ can be defined naturally. Since $\bigcap C(y, \mathcal{F}_{ii}) \subset Y$,

$$\bigcap C_i(y) \subset \bigcap C(y, \mathcal{F}_{ii}) = \bigcap C(y, \mathcal{F}_{ii}|Y) = C(y) .$$

Since it is evident that $C(y) \subset \bigcap C_i(y)$, we obtain

$$C(y) = \bigcap C_i(y) .$$

Thus $C(y)$ is compact and hence closed. Let $K_i^j$ be the closure of $K_i$ with respect to $X_j$ and set

$$K^j = \bigcap_i K_i^j .$$

Then $K^j$ is not empty for any $j$ and $K^1 \supset K^2 \supset \ldots$ Since $K^j$ is a closed subset of $C_j(y)$ and $C_j(y)$ is closed in $X$, $K^j$ is closed in $X$. To prove $K^j \cap C(y) \neq \emptyset$ for any $j$, assume that there would exist an $m$ with $K^m \cap C(y) = \emptyset$. Then there would exist an $n$ with $K^m \cap C_n(y) = \emptyset$. Choose an $s$ with $s \geqslant m$, $s \geqslant n$ and with

$$K^m \cap C(y, \mathcal{F}_{ns}) = \emptyset .$$

Then

$$K^s \cap C(y, \mathcal{F}_{ss}) = \emptyset ,$$

yielding

$$K^s \subset K_s^s \subset C(y, \mathcal{F}_{ss}) \subset X - K^s ,$$

a contradiction.

Set

$$K = \bigcap K^j .$$

Since $K^j \cap C(y) \neq \emptyset$ for any $j$, then

$$K \cap C(y) \neq \emptyset .$$

Since

$$K_i^j \cap C(y) \subset \bar{K}_i \cap C(y) = K_i \cap C(y) \subset K_i^j \cap C(y) ,$$

then

$$K \cap C(y) = (\bigcap_{i,j} K_i^j) \cap C(y) = (\bigcap_i K_i) \cap C(y) .$$

Thus $\bigcap K_i \neq \emptyset$ and the proof is completed.

3.17. Remark. In view of Theorems 3.2 and 3.16 one may expect that if $X_1, X_2, \ldots$ are strong $\Sigma$-spaces contained in a space $X$, then $\bigcup X_i$

may be a $\Sigma$-space. But it is not the case by E. Michael's celebrated example. Let $I$ be the unit interval, $J$ the rationals in $I$ and $K$ the irrationals in $I$. Let $L$ be the space obtained by retopologizing $I$ in such a way that every set of type $U \cup V$, with $U$ open in $I$ and with $V \subset K$, is a basic open set in $L$. Then $L$ is a hereditarily paracompact space, $J$ is an $F_\sigma$-set of $L$ and $K$ is a $G_\delta$-set of $L$. $J$ and $K$ are strong $\Sigma$-spaces, while $J \cup K$ is not a $P$-space as is well known and hence not a $\Sigma$-space.

3.18. LEMMA. *Let $X$ be a regular space, $X_1$ a subset of $X$ having a strong $\Sigma$-net $\{\mathcal{F}_i\}$ and $x$ a point of $X_1$. Set*

$$\mathcal{H}_i = \overline{\mathcal{F}}_i \cup \{X\},$$
$$C_1(x) = \bigcap C(x, \mathcal{F}_i),$$
$$C(x) = \bigcap C(x, \mathcal{H}_i).$$

*Then $C_1(x) = C(x)$. If $K_1 \supset K_2 \supset \ldots$ is a sequence of non-empty closed sets of $X$ with*

$$K_i \subset C(x, \mathcal{H}_i)$$

*for each $i$, then $\bigcap K_i \neq \emptyset$.*

Proof. Let $y$ be an arbitrary point of $X - C_1(x)$. Choose an open set $U$ of $X$ with

$$C_1(x) \subset U \subset \overline{U} \subset X - \{y\}$$

and a $j$ with

$$C(x, \mathcal{F}_j) \subset U.$$

Since $C(x) \subset C(x, \mathcal{H}_j) \subset \overline{U}$, $C(x)$ does not contain $y$, proving $C(x) \subset C_1(x)$. Since it is evident that $C_1(x) \subset C(x)$, we obtain $C_1(x) = C(x)$.

To prove the rest assume that $K_i \cap C(x, \mathcal{F}_i) = \emptyset$ for some $i$. Since $X$ is regular, there is an open set $V$ of $X$ with

$$C_1(x) \subset V \subset \overline{V} \subset X - K_i.$$

Choose a $j$ with $j \geqslant i$ and with $C(x, \mathcal{F}_j) \subset V$. Then $C(x, \mathcal{H}_j) \subset \overline{V}$, yielding

$$K_j \subset C(x, \mathcal{H}_j) \subset X - K_i,$$

a contradiction. Thus $K_i \cap C(x, \mathcal{F}_i) \neq \emptyset$ for each $i$ and $\bigcap K_i \neq \emptyset$. The proof is finished.

3.19. LEMMA. *Let $X$ be a regular space. Let $X_1$ or $X_2$ be a subset of $X$ having respectively a strong $\Sigma$-net $\{\mathcal{F}_{1i}\}$ or $\{\mathcal{F}_{2i}\}$ such that each $\mathcal{F}_{ji}$ is locally finite in $X$. We name a $\Sigma$-net with this additional condition a special $\Sigma$-net. Then $X_1 \cup X_2$ has a special strong $\Sigma$-net $\{\mathcal{F}_i\}$.*

Proof. Set

$$\mathcal{H}_{ij} = \overline{\mathcal{F}}_{ji} \cup \{X\},$$
$$\mathcal{H}_i = \mathcal{H}_{1i} \wedge \mathcal{H}_{2i},$$
$$\mathcal{F}_i = \mathcal{H}_i | X_1 \cup X$$



Then $\mathcal{H}_i$ is a locally finite closed covering of $X$. Let $K_1 \supset K_2 \supset \ldots$ be a sequence of non-empty closed sets of $X_1 \cup X_2$ such that

$$K_i \subset C(x, \mathcal{F}_i)$$

for some point $x$ in $X_1 \cup X_2$ and for each $i$. When $x \in X_1$,

$$C(x, \mathcal{F}_i) \subset C(x, \mathcal{H}_i) \subset C(x, \mathcal{H}_{1i}).$$

Thus

$$K_i \subset C(x, \mathcal{H}_{1i} | X_1 \cup X_2)$$

for each $i$ and hence $\bigcap K_i \neq \emptyset$ by Lemma 3.18. When $x \in X_2$, we obtain $\bigcap K_i \neq \emptyset$ too.

To see $\{\mathcal{F}_i\}$ is strong let $x$ be an arbitrary point of $X_1$. Then

$$\bigcap C(x, \mathcal{F}_i) \subset \bigcap C(x, \mathcal{H}_i) \subset \bigcap C(x, \mathcal{H}_{1i}).$$

Since

$$\bigcap C(x, \mathcal{H}_{1i}) = \bigcap C(x, \mathcal{F}_{1i}) \subset X_1$$

by the preceding lemma, $\bigcap C(x, \mathcal{F}_i)$ is compact and the proof is finished.

3.20. THEOREM. *Let $X$ be a perfectly normal strong $\Sigma$-space. Then each Borelian set of $X$ is also a strong $\Sigma$-space.*

Proof (by transfinite induction). (i) Let $\alpha$ be an arbitrary ordinal less than the first uncountable ordinal $\omega_1$. Let us define the families $\mathcal{B}_{\alpha\sigma}$ and $\mathcal{B}_{\alpha\delta}$ inductively. Let $\mathcal{B}_{0\sigma}$ be the family of all open sets of $X$ and $\mathcal{B}_{0\delta}$ the family of all closed sets of $X$. If $\alpha > 0$, let $\mathcal{B}_{\alpha\sigma}$ or $\mathcal{B}_{\alpha\delta}$ be respectively the family of all sets $B$ of type:

$$B = \bigcup_{i=1}^{\infty} B_i \quad \text{or} \quad B = \bigcap_{i=1}^{\infty} B_i,$$

where

$$B_i \in \mathcal{B}_{\beta_i\sigma} \cup \mathcal{B}_{\beta_i\delta},$$
$$\beta_i < \alpha, \quad i = 1, 2, \ldots$$

Set

$$\mathcal{B}_\alpha = \mathcal{B}_{\alpha\sigma} \cup \mathcal{B}_{\alpha\delta},$$
$$\mathcal{B} = \cup \{\mathcal{B}_\alpha \colon \alpha < \omega_1\}.$$

Then $\mathcal{B}$ is the family of all Borelian sets.

(ii) Let $P(\alpha)$ be the following proposition:

Each set in $\mathcal{B}_\alpha$ has a special strong $\Sigma$-net.

Clearly every closed set has a special strong $\Sigma$-net. Since every open set $G$ is an $F_\sigma$-set, the method in the proof of Theorem 3.2 can be applied and $G$ has a special strong $\Sigma$-net. Thus $P(0)$ is true. Let $\alpha$ be an ordinal with $0 < \alpha < \omega_1$ and put the transfinite induction assumption that $P(\beta)$ is true for each $\beta$ less than $\alpha$.

(iii) Let $D$ be an arbitrary element in $\mathcal{B}_{\alpha\delta}$. Then in the same fashion as in the proof of Theorem 3.16, $D$ can be proved to have a special strong $\Sigma$-net.

Let $E$ be an arbitrary element of $\mathcal{B}_{a\sigma}$. Then $E$ can be expressed as:

$$E = \bigcup E_i\,, \quad \text{where} \quad E_i \in \mathcal{B}_{\beta_i}\,,\ \beta_i < \alpha\,.$$

By Lemma 3.19 we can assume without loss of generality that

$$E_1 \subset E_2 \subset \ldots\,,$$
$$\beta_1 \leqslant \beta_2 \leqslant \ldots$$

Let

$$\{\mathcal{F}_{ij}\colon j = 1, 2, \ldots\}$$

be a special strong $\Sigma$-net of $E_i$. We assume here without loss of generality that $\{\mathcal{F}_{ij}\colon j = 1, 2, \ldots\}$ is spectral. This assumption is possible if we construct a spectral $\Sigma$-net from a special $\Sigma$-net by the standard way as in the proof of Lemma 1.4. Set

$$\mathcal{H}_{ij} = \overline{\mathcal{F}}_{ij} \cup \{X\}\,,$$
$$\mathcal{H}_i = \bigwedge \{\mathcal{H}_{st}\colon s \leqslant i\,,\ t \leqslant i\}\,,$$
$$\mathfrak{L}_i = \mathcal{H}_i | E\,.$$

Then $\mathcal{H}_i$ is a locally finite closed covering of $X$. Let us prove that $\{\mathfrak{L}_i\}$ is a special strong $\Sigma$-net of $E$.

(iv) Let $K_1 \supset K_2 \supset \ldots$ be a sequence of non-empty closed sets in $E$ such that

$$K_i \subset C(x, \mathfrak{L}_i)$$

for some point $x$ in $E$ and for each $i$. Choose a $k$ with $x \in E_k$. Then for each $i$ greater than or equal to $k$

$$K_i \subset C(x, \mathfrak{L}_i) \subset C(x, \mathcal{H}_i) \subset C(x, \mathcal{H}_{ki})\,.$$

Thus

$$K_i \subset C(x, \mathcal{H}_{ki}|E)\,, \quad i \geqslant k\,.$$

Notice that

$$\{\mathcal{F}_{ki}\colon i = k, k+1, \ldots\}$$

is a strong $\Sigma$-net of $E_k$, since $\{\mathcal{F}_{ki}\colon i = 1, 2, \ldots\}$ was taken to be spectral. If we apply Lemma 3.18, we obtain:

$$\bigcap_{i=k}^{\infty} K_i \neq \emptyset\,.$$

Therefore

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset\,.$$

Set

$$C(x) = \bigcap_i C(x, \mathfrak{L}_i)\,,$$
$$C'_i(x) = \bigcap_j C(x, \mathcal{H}_{ij})\,,$$
$$C_i(x) = \bigcap_j C(x, \mathcal{F}_{ij})\,.$$

Then by Lemma 3.18

$$C_i(x) = C'_i(x)\,, \quad i = k, k+1, \ldots$$

Let $p$ be an arbitrary non-negative integer. Then

$$C(x, \mathcal{H}_{k+p}) \subset C(x, \mathcal{H}_{k+p,q})\,, \quad q = 1, \ldots, k+p\,,$$
$$C(x, \mathcal{H}_{k+p+r}) \subset C(x, \mathcal{H}_{k+p,k+p+r})\,, \quad r = 1, 2, \ldots$$

Thus

$$\bigcap_{r=0}^{\infty} C(x, \mathcal{H}_{k+p+r}) \subset \bigcap_{q=1}^{\infty} C(x, \mathcal{H}_{k+p,q}) = C'_{k+p}(x)\,.$$

Therefore

$$\bigcap_{i=k}^{\infty} C(x, \mathcal{H}_i) \subset \bigcap_{i=k}^{\infty} C'_i(x) = \bigcap_{i=k}^{\infty} C_i(x)\,.$$

Since

$$C(x) \subset \bigcap_{i=k}^{\infty} C(x, \mathcal{H}_i)\,,$$

then

$$C(x) \subset \bigcap_{i=k}^{\infty} C_i(x)\,.$$

Since every $C_i(x)$ is compact, $C(x)$ is compact. Thus $E$ has a special strong $\Sigma$-net and $P(\alpha)$ is true. The induction is now completed and the theorem is proved in a slightly strengthened form.

3.21. Remark. If $X$ in the above theorem is not perfectly normal but hereditarily normal, the theorem is not true. Let $X$ be the set of all ordinals less than or equal to $\omega_1$. This $X$ with the interval topology is a strong $\Sigma$-space (actually a compact space) and is hereditarily normal. Let $G$ be the set of all countable ordinals. $G$ is of course a $\Sigma$-space. Let us prove $G$ cannot have a strong $\Sigma$-net. Let $\{\mathcal{F}_i\}$ be a spectral $\Sigma$-net of $G$. Let $\mathcal{F}'_i$ be the subcollection of $\mathcal{F}_i$ consisting of all cofinal (in $G$) elements $\mathcal{F}_i$. Set

$$K_i = \bigcap \{F\colon F \in \mathcal{F}'_i\}\,,$$
$$K = \bigcap K_i\,.$$

Then each $K_i$ is cofinal and $K_1 \supset K_2 \supset \ldots$ Hence $K$ is also cofinal. Set

$$S = \bigcup \{F\colon F \in \mathcal{F}_i - \mathcal{F}'_i\,,\ i = 1, 2, \ldots\}\,.$$

Then $S$ is not cofinal. Hence $K - S$ is not empty. Let $x$ be a point of $K - S$. Then $C(x)$ is cofinal and hence not compact. Thus $\{\mathcal{F}_i\}$ is not strong.

## 4. Product spaces.

4.1. Theorem. *Let $X$ be a paracompact $P(\mathfrak{m})$-space and $Y$ a paracompact $\Sigma(\mathfrak{m})$-space. Then $X \times Y$ is paracompact.*

Proof. Let $\mathcal{G}$ be an arbitrary open covering of $X \times Y$. Let

$$\{\mathcal{F}_i = \{F(\alpha_1 \ldots \alpha_i)\colon \alpha_1, \ldots, \alpha_i \in \Omega\}\}$$

be a spectral $\Sigma$-net of $Y$ with $|\Omega| \leqslant \mathfrak{m}$. Let

$$\mathcal{H}_i = \{H(\alpha_1 \ldots \alpha_i)\colon \alpha_1, \ldots, \alpha_i \in \Omega\}$$

be a locally finite open covering of $Y$ such that

$$F(\alpha_1 \ldots \alpha_i) \subset H(\alpha_1 \ldots \alpha_i)$$

for each $\alpha_1, \ldots, \alpha_i$. Let

$$\mathfrak{W}(\alpha_1 \ldots \alpha_i) = \{U_\lambda \times V_\lambda\colon \lambda \in \Lambda(\alpha_1 \ldots \alpha_i)\}$$

be the maximal collection satisfying the following three conditions:

(i) Each $U_\lambda$ is an open set of $X$.

(ii) Each $V_\lambda$ is an open set of $Y$ such that

$$F(\alpha_1 \ldots \alpha_i) \subset V_\lambda \subset H(\alpha_1 \ldots \alpha_i)\,.$$

(iii) Each $V_\lambda$ is a finite union of open sets $V_{\lambda 1}, \ldots, V_{\lambda n(\lambda)}$ such that

$$\mathcal{G}_\lambda = \{U_\lambda \times V_{\lambda i}\colon i = 1, \ldots, n(\lambda)\} < \mathcal{G}\,.$$

Set

$$\mathfrak{W} = \cup \{\mathfrak{W}(\alpha_1 \ldots \alpha_i)\colon \alpha_1, \ldots, \alpha_i \in \Omega,\ i = 1, 2, \ldots\}\,.$$

By an analogous way to that in K. Nagami [10], Theorem 3, we can see that $\mathfrak{W}$ is a normal open covering of $X \times Y$. Hence there exists a locally finite open covering

$$\mathfrak{W}_0 = \{W_\lambda\colon \lambda \in \Lambda(\alpha_1 \ldots \alpha_i),\ \alpha_1, \ldots, \alpha_i \in \Omega,\ i = 1, 2, \ldots\} = \{W_\lambda\colon \lambda \in \Lambda\}$$

of $X \times Y$ such that

$$W_\lambda \subset U_\lambda \times V_\lambda$$

for each $\lambda$. Now

$$\bigcup \{\mathcal{G}_\lambda | W_\lambda\colon \lambda \in \Lambda\}$$

is a locally finite open covering of $X \times Y$ refining $\mathcal{G}$ and the theorem is proved.

4.2. Corollary. *Let $X$ be a paracompact $P$-space and $Y$ a paracompact $\Sigma$-space. Then $X \times Y$ is paracompact.*

This generalizes the essential part of K. Morita [8], Theorem 6.5, and K. Nagami [10], Theorem 3, at the same time.

4.3. Definition. Let $\mathfrak{m}$ be a power. A space $X$ has the *property* $L(\mathfrak{m})$ if every open covering of $X$ has a subcovering consisting of at most $\mathfrak{m}$ elements.

When $\mathfrak{m}$ is finite, the property $L(\mathfrak{m})$ implies that $X$ consists of at most $\mathfrak{m}$ points. A space $X$ has the property $L(\aleph_0)$ if and only if $X$ is a Lindelöf space. It is to be noted that the property $L(\mathfrak{m})$ is not always hereditary. If $X$ has the property $L(\mathfrak{m})$, then each $F_\sigma$-set of $X$ has the property $L(\mathfrak{m})$.



4.4. Lemma. *If $X$ is a space with the property $L(\mathfrak{m})$, then each locally finite collection $\mathcal{F}$ of subsets of $X$ consists of at most $\mathfrak{m}$ elements.*

Proof. Since the proposition is trivially true for a finite $\mathfrak{m}$, we prove it for an infinite $\mathfrak{m}$. Suppose there would exist a locally finite collection $\mathcal{F}$ with $|\mathcal{F}| > \mathfrak{m}$. Pick a point from each element of $\mathcal{F}$. Let $S$ be the sum of all these points. Since the collection of such points is locally finite, $S$ is a closed discrete subset of $X$ with $|S| > \mathfrak{m}$. Hence $X$ has not the property $L(\mathfrak{m})$, a contradiction.

4.5. Lemma. *Let $\mathfrak{m}$ be an infinite power. If a space $X$ is a strong $\Sigma(\mathfrak{m})$-space, then $X$ has the property $L(\mathfrak{m})$.*

Proof. Let $\mathcal{G}$ be an arbitrary open covering of $X$ and $\mathcal{H}$ the collection of all finite sum of elements of $\mathcal{G}$. Let

$$\{\mathcal{F}_i = \{F_{i\alpha}\colon \alpha \in A_i\}\}$$

be a strong spectral $\Sigma$-net of $X$ such that $|A_i| \leqslant \mathfrak{m}$ for each $i$. Set

$$\mathcal{F}'_i = \{F_{i\alpha}\colon \alpha \in B_i\} = \{F \in \mathcal{F}_i\colon F \subset \text{some } H \in \mathcal{H}\}\,.$$

To prove that $\bigcup \mathcal{F}'_i$ covers $X$ let $x$ be an arbitrary point of $X$. Since $C(x)$ is compact, there is an element $H$ of $\mathcal{H}$ with $C(x) \subset H$. Then there exists an $i$ such that $C(x, \mathcal{F}_i) \subset H$. Since $C(x, \mathcal{F}_i) = F$ for some element $F$ of $\mathcal{F}_i$, $C(x, \mathcal{F}_i) \in \mathcal{F}'_i$. Thus $\bigcup \mathcal{F}'_i$ covers $X$.

For each element $F_{i\alpha}$ in $\mathcal{F}'_i$ let $\mathcal{G}_{i\alpha}$ be a finite subcollection of $\mathcal{G}$ covering $F_{i\alpha}$. Then

$$\bigcup \{\mathcal{G}_{i\alpha}\colon \alpha \in B_i,\ i = 1, 2, \ldots\}$$

is a subcovering of $\mathcal{G}$ consisting of at most $\mathfrak{m}$ elements.

4.6. Theorem. *Let $\mathfrak{m}$ be an infinite power. A paracompact $\Sigma$-space $X$ is a $\Sigma(\mathfrak{m})$-space if and only if $X$ has the property $L(\mathfrak{m})$.*

Proof. The necessity is evident by Lemma 4.5. Assume that $X$ has the property $L(\mathfrak{m})$. Let

$$\{\mathcal{F}_i = \{F_{i\alpha} \neq \emptyset\colon \alpha \in A_i\}\}$$

be a $\Sigma$-net of $X$. Then by Lemma 4.4 $|A_i| \leqslant \mathfrak{m}$. Hence $X$ is a $\Sigma(\mathfrak{m})$-space by the remark at the end of Definition 1.5.

4.7. Theorem. *Let $\mathfrak{m}$ be an infinite power. Let $X$ be a paracompact $P(\mathfrak{m})$-space with the property $L(\mathfrak{m})$ and $Y$ a paracompact $\Sigma(\mathfrak{m})$-space. Then $X \times Y$ is a paracompact space with the property $L(\mathfrak{m})$.*

Proof. The paracompactness of $X \times Y$ is assured by Theorem 4.1. To prove $X \times Y$ has the property $L(\mathfrak{m})$ let $\mathcal{G}$ be an arbitrary open covering of $X \times Y$. Let

$$\{\mathcal{F}_i = \{F(\alpha_1 \ldots \alpha_i)\colon \alpha_1, \ldots, \alpha_i \in \Omega\}\}$$

be a spectral $\Sigma(\mathfrak{m})$-net of $Y$. Let

$$\mathcal{G}(\alpha_1\ldots\alpha_i) = \{G_\lambda(\alpha_1\ldots\alpha_i):\ \lambda\in\Lambda(\alpha_1\ldots\alpha_i)\}$$

be the collection of all possible open sets of $X$ such that

$$G_\lambda(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i)$$

is a sum of a finite collection $\mathcal{K}_\lambda(\alpha_1\ldots\alpha_i)$ refining $\mathcal{G}$. Then

$$\bigcup\{\mathcal{K}_\lambda:\ \lambda\in\Lambda(\alpha_1\ldots\alpha_i),\ \alpha_1,\ldots,\alpha_i\in\Omega,\ i=1,2,\ldots\}$$

covers $X\times Y$. Set

$$G(\alpha_1\ldots\alpha_i) = \bigcup\{G_\lambda(\alpha_1\ldots\alpha_i):\ \lambda\in\Lambda(\alpha_1\ldots\alpha_i)\}.$$

Then

$$G(\alpha_1\ldots\alpha_i)\subset G_1(\alpha_1\ldots\alpha_i\alpha_{i+1})$$

for each sequence $\alpha_1,\alpha_2,\ldots$ Let $\{H(\alpha_1\ldots\alpha_i)\}$ be a collection of closed sets of $X$ such that

(i) $H(\alpha_1\ldots\alpha_i)\subset G(\alpha_1\ldots\alpha_i)$,

(ii) $\bigcup_i G(\alpha_1\ldots\alpha_i)=X$ implies $\bigcup_i H(\alpha_1\ldots\alpha_i)=X$.

Let

$$\mathcal{H}(\alpha_1\ldots\alpha_i) = \{H_\mu(\alpha_1\ldots\alpha_i):\ \mu\in M(\alpha_1\ldots\alpha_i)\}$$

be a covering of $H(\alpha_1\ldots\alpha_i)$ refining $G(\alpha_1\ldots\alpha_i)$ such that

$$|M(\alpha_1\ldots\alpha_i)|\leqslant\mathfrak{m}.$$

To each $\mu\in M(\alpha_1\ldots\alpha_i)$ there corresponds a $\lambda(\mu)\in\Lambda(\alpha_1\ldots\alpha_i)$ such that

$$H_\mu(\alpha_1\ldots\alpha_i)\subset G_{\lambda(\mu)}(\alpha_1\ldots\alpha_i).$$

Set

$$\mathcal{W}_\mu(\alpha_1\ldots\alpha_i) = \mathcal{K}_{\lambda(\mu)}(\alpha_1\ldots\alpha_i)|H_\mu(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i),$$

$$\mathcal{W} = \bigcup\{\mathcal{W}_\mu(\alpha_1\ldots\alpha_i):\ \mu\in M(\alpha_1\ldots\alpha_i),\ \alpha_1,\ldots,\alpha_i\in\Omega,\ i=1,2,\ldots\}.$$

Then $\mathcal{W}$ is a covering of $X\times Y$ refining $\mathcal{G}$. Since $\mathcal{W}$ consists of at most $\mathfrak{m}$ elements, $\mathcal{G}$ has a subcovering consisting of at most $\mathfrak{m}$ elements. Thus $X$ has the property $L(\mathfrak{m})$ and the theorem is proved.

4.8. Corollary. *Let $X$ be a regular Lindelöf $P(2)$-space and $Y$ a regular Lindelöf $\Sigma$-space. Then $X\times Y$ is a Lindelöf space.*

Proof. By the condition $X$ is a paracompact $P(\aleph_0)$-space with the property $L(\aleph_0)$. By Theorem 4.6 $Y$ is a paracompact $\Sigma(\aleph_0)$-space. Thus the present corollary is an immediate consequence of Theorem 4.7.

This generalizes the essential part of K. Morita [8], Corollary 6.6, and K. Nagami [10], Theorem 4, at the same time.

4.9. Lemma. *A normal space is countably paracompact if and only if each countable open covering can be refined by a $\sigma$-locally finite closed covering.*



This is an easy exercise.

4.10. Theorem. *Let $X$ be a $P$-space and $Y$ a strong $\Sigma$-space. If $X\times Y$ is normal, then $X\times Y$ is countably paracompact.*

Proof. Let

$$\{\mathcal{F}_i = \{F(\alpha_1\ldots\alpha_i):\ \alpha_1,\ldots,\alpha_i\in\Omega\}\}$$

be a spectral $\Sigma$-net of $Y$. Let $\mathcal{G}=\{G_i\}$ be an arbitrary countable open covering of $X\times Y$. Let $\mathcal{G}_1,\mathcal{G}_2,\ldots$ be the sequence of all finite subcollection of $\mathcal{G}$. Set

$$H_i = \bigcup\{G:\ G\in\mathcal{G}_i\}.$$

Let $G_j(\alpha_1\ldots\alpha_i)$ be the maximal open set of $X$ such that

$$G_j(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i)\subset H_j.$$

Set

$$G(\alpha_1\ldots\alpha_i) = \bigcup_j G_j(\alpha_1\ldots\alpha_i).$$

Then

$$G(\alpha_1\ldots\alpha_i)\subset G(\alpha_1\ldots\alpha_i\alpha_{i+1})$$

for each sequence $\alpha_1,\alpha_2,\ldots\in\Omega$. Moreover we can verify that

$$\{G(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i):\ \alpha_1,\ldots,\alpha_i\in\Omega,\ i=1,2,\ldots\}$$

covers $X\times Y$ by the standard argument with the aid of compactness of $C(y)$, $y\in Y$. Let $K(\alpha_1\ldots\alpha_i)$ be closed sets of $X$ such that

(i) $K(\alpha_1\ldots\alpha_i)\subset G(\alpha_1\ldots\alpha_i)$,

(ii) $\bigcup_i K(\alpha_1\ldots\alpha_i)=X$ whenever $\bigcup_i G(\alpha_1\ldots\alpha_i)=X$.

Then

$$\{K(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i):\ \alpha_1,\ldots,\alpha_i\in\Omega,\ i=1,2,\ldots\}$$

covers $X\times Y$.

Since $X$ is a normal $P$-space, $X$ is countably paracompact by K. Morita [8], Theorem 3.10. Since $K(\alpha_1\ldots\alpha_i)$ is a countably paracompact normal space as a closed subset of $X$, there exist closed sets $K_j(\alpha_1\ldots\alpha_i)$, $j=1,2,\ldots$, such that

(i) $K(\alpha_1\ldots\alpha_i)=\bigcup_j K_j(\alpha_1\ldots\alpha_i)$,

(ii) $K_j(\alpha_1\ldots\alpha_i)\subset G_j(\alpha_1\ldots\alpha_i)$,

(iii) $\{K_j(\alpha_1\ldots\alpha_i):\ j=1,2,\ldots\}$ is locally finite in $X$.

Set

$$\mathcal{L}_i = \{K_j(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i):\ j=1,2,\ldots,\ \alpha_1,\ldots,\alpha_i\in\Omega\}.$$

Then $\mathcal{L}_i$ is locally finite in $X\times Y$ and $\bigcup\mathcal{L}_i$ covers $X\times Y$. Since $\mathcal{G}_j$ is a finite open covering of $K_j(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i)$, there exists a finite closed covering $\mathcal{K}_j(\alpha_1\ldots\alpha_i)$ of $K_j(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i)$ which refines

$$\mathcal{G}_j|K_j(\alpha_1\ldots\alpha_i)\times F(\alpha_1\ldots\alpha_i).$$

Then

$$\mathcal{M}_i = \bigcup \{\mathcal{K}_j(a_1 \ldots a_i) \colon j = 1, 2, \ldots, a_1, \ldots, a_i \in \Omega\}$$

is a locally finite closed collection of $X \times Y$. Thus $\bigcup \mathcal{M}_i$ is a $\sigma$-locally finite closed covering of $X \times Y$ refining $\mathcal{G}$. By Lemma 4.9 $X \times Y$ is countably paracompact and the proof is completed.

4.11. **Remark.** Almost all propositions about $\Sigma$-spaces are also true if we replace $\Sigma$-spaces with $\Sigma(\mathfrak{m})$-spaces. The following are such ones: Theorems 1.8, 3.2, 3.6, 3.9, 3.13 and Corollaries 1.8, 1.19.

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# A generalized contraction principle

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Various versions and generalizations of the Banach contraction mapping theorem ([1], p. 160) have been given. For only two of many examples see [4], p. 43, 50 (where an application is given by solving the Volterra type integral equation) and [2] (where an application is given to analytic mappings of a compact connected set in the complex plane into itself.) We discuss a general definition of contraction mapping here for which we can prove the necessary result that a contraction mapping of a complete metric space into itself has a unique fixed point. In order to make this definition it is convenient to work with uniform spaces having a countable symmetric base rather than metric spaces although, of course, the two are equivalent.

See Kelley ([3], Chapter 6) for the necessary terminology and results. In what follows $Z$ will denote the integers and $\Delta$ the diagonal of $X \times X$ $(\Delta = \{(x, x) | x \in X\})$.

Definition. Let $(X, \mathcal{U})$ be a uniform space. A mapping $f\colon X \to X$ is *u-contracting* provided there is a collection of symmetric sets $\{V_n\}_{n \in Z}$, cofinal in $\mathcal{U}$ (with respect to the ordering $U_1 \geqslant U_2$ if and only if $U_1 \subseteq U_2$) which satisfy

(i) $V_i \subseteq V_j$ if $i \leqslant j$, $\bigcap_{n \in Z} V_n = \Delta$, $\bigcup_{n \in Z} V_n = X \times X$,

(ii) for each $n \in Z$ there is an integer $p(n) > 0$ such that $\{p(n) | n \in Z\}$ is bounded and $V_{n-p(n)} \circ V_{n-p(n)} \subseteq V_n$,

(iii) if $(x, y) \in V_n$ then $\big(f(x), f(y)\big) \in V_{n-1}$.

Lemma 1. *If $f\colon X \to X$ is u-contracting then $f$ has at most one fixed point.*

Proof. Suppose $f(x) = x$ and $y \neq x$. Let $n$ be the least integer for which $(x, y) \in V_n$. ($n$ exists since $\bigcap V_n = \Delta$ and $\bigcup V_n = X \times X$.) Then $(x, y) \in V_n$ so $\big(f(x), f(y)\big) \in V_{n-1}$. If $y = f(y)$ we would have $(x, y) \in V_{n-1}$, a contradiction.

Lemma 2. *If $f\colon X \to X$ is u-contracting then so is any iterate, $f^p$, of $f$.*

Proof. The sequence of $V_n$ which demonstrates that $f$ is u-contracting will suffice.