Maximal chains and antichains

by

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An *antichain* of a partially ordered set E is a subset A of E such that any two distinct elements of A are uncomparable. Our main concern is the characterization and study of the p.o. sets in which every maximal chain intersects each maximal antichain (we call such p.o. set chain-antichain complete, or CAC), and the application to some existence results in spaces of continuous functions.

Characterization of CAC p.o. sets is achieved under strong finiteness conditions by considering four element subsets. We call a quadruple (a, b, c, d) of elements of a p.o. set E an N if $a < b$, $c < b$, $c < d$ and a and c, a and d, b and d are uncomparable; and a *proper* N if furthermore there is no $c < x < b$ such that x is uncomparable to a and to d, or an N' if furthermore b covers c (i.e. there is no $c < x < b$). If E is finite, then E is CAC if and only if it contains no proper N, if and only if it contains no N'. This holds also if E is *regular*, i.e. if every non-empty chain C of E has a l.u.b. and a g.l.b. which are in the closure of C for the interval topology. Any finite p.o. set is obviously regular.

Next we prove that any regular p.o. set can be embedded into a regular CAC p.o. set, which one can choose "minimum". We conjecture that *any* p.o. set can be so completed.

Next we prove that a finite modular lattice which is CAC has dimension at most 2 (in the sense of [2]; see also [3], [4]). This provides a very convenient criterion to recognize that the dimension is at most 2. Again we suspect that it can be extended to a much larger class of p.o. sets.

Finally we consider all continuous real-valued functions on a compact metric space satisfying simple conditions of boundedness and equicontinuity (so that the chains will have a minimum of good properties), ordered by either $f(x) \leqslant g(x)$ or $f(x) < g(x)$ for all x. We show that neither p.o. set is CAC or regular, yet can prove that any maximal chain for the first ordering intersects each maximal antichain for the second ordering. An example of existence result is immediately derived as a corollary. The technique of proof is the same as in the abstract cases.

The reader is referred to [1] for the fundamentals of p.o. sets. Throughout, E denotes a given p.o. set.

1. Maximal antichains. We shall write $a\|b$ for: a and b are uncomparable, so that a subset A of E is an antichain if and only if, for any $a, b \in A$, either $a = b$ or $a\|b$. This property is of finite character, whence the set of all antichains of E, ordered by inclusion, is inductive; therefore every antichain of E, in particular every singleton of E, is contained in a maximal antichain. The set of all maximal elements of E is an antichain (possibly empty); if E is inductive, it is a maximal antichain, by:

PROPOSITION 1. *An antichain A of E is maximal if and only if any element of E is comparable to some element of A.*

Proof. If there exists $x \in E$ which is comparable to no element of A, then $A \cup \{x\}$ is an antichain and A is not maximal. If conversely A is not maximal, then any element x of a larger antichain, such that $x \notin A$, is comparable to no element of A.

If A is an antichain, we set:

$$A^+ = \{x \in E;\ a < x \text{ for some } a \in A\},$$
$$A^- = \{x \in E;\ x < a \text{ for some } a \in A\}.$$

PROPOSITION 2. *If A is a maximal antichain, and if A^-, $A^+ \neq \emptyset$, then $\{A^-, A, A^+\}$ is a partition of E.*

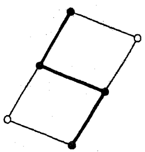
Proof. By definition of an antichain, A^-, A, A^+ are pairwise disjoint. If A is maximal, then they cover E by Proposition 1.

2. Chain-antichain-completeness. If a chain and an antichain intersect, they have no more that one common element. This observed, we look for conditions under which E is CAC.

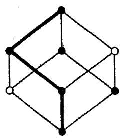
PROPOSITION 3. *If E is* CAC, *then E contains no proper N.*

Proof. Let E be CAC and (a, b, c, d) be an N of E. Then $\{b, c\}$ is contained in a maximal chain C and $\{a, d\}$ is contained into a maximal antichain A. Let $C \cap A = \{x\}$. Since $a\|c$, $x \neq a$; similarly, $x \neq d$. Therefore $x\|a$ and $x\|d$. It follows that $x \leqslant c$ and $b \leqslant x$ are impossible, whence $c < x < b$. Therefore (a, b, c, d) is not a proper N.

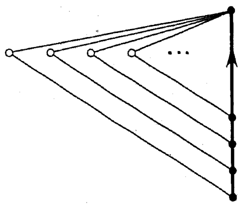
Any chain, any lattice of at most 5 elements, are CAC. The following are examples of p.o. sets, in fact lattices, which are not CAC:



Example 1



Example 2



Example 3

In each example, boldface indicates a maximal chain C, "white" elements a maximal antichain A, such that $C \cap A = \emptyset$. In both example 1 and example 2, proper N's are obvious. On the other hand, example 3 is a complete lattice satisfying the descending chain condition but contains no proper N; it suggests the following additional condition.

DEFINITION 4. We say that E is *sup-regular* if every non-empty chain C of E has a l.u.b. $\sup C$ and if $x < \sup C$ implies $x < c$ for some $c \in C$. *Inf-regular* is defined dually. Finally we say that E is *regular* if it is both sup-regular and inf-regular.

If E satisfies the ascending chain condition, then E is sup-regular, for then $\sup C \in C$ for any chain $C \neq \emptyset$. In particular, any finite p.o. set is regular.

THEOREM 5. *If E is regular, then the following conditions are equivalent:*

(i) *E is* CAC;

(ii) *E contains no proper N;*

(iii) *E contains no N'.*

Proof. By Proposition 3, (i) implies (ii); also (ii) implies (iii). Finally we show that, if E is not CAC, then E contains an N'.

LEMMA 6. *Let E be regular and C be a maximal chain, A be a maximal antichain of E such that $C \cap A = \emptyset$. Then $C \cap A^-$ has a maximum element c, $C \cap A^+$ has a minimum element b, and b covers c.*

Proof. By the hypothesis and Proposition 2, $C \cap A^-$ and $C \cap A^+$ are disjoint and cover C. First $\inf C \in C$ and $\inf C$ is a minimal element of E, since C is maximal. Therefore $\inf C \notin A^+$ and $\inf C \in C \cap A^-$. Dually $C \cap A^+ \neq \emptyset$.

Let $c = \sup C \cap A^-$. For any $x \in C$, either $y \leqslant x$ for all $y \in C \cap A^-$ and then $c \leqslant x$; or $x < y$ for some $y \in C \cap A^-$ and then $x < c$; in either case, x and c are comparable. Since C is maximal, $c \in C$. Also, $c \in A^+$ would imply $a < c$ for some $a \in A$ and $a < x$ for some $x \in C \cap A^-$ (by sup-regularity), which is impossible. Therefore $c \in C \cap A^-$ and c is maximum element of $C \cap A^-$.

Dually $C \cap A^+$ has a minimum element b. Since $b \in A^+$, $c \in A^-$, one must have $c < b$. If b does not cover c, C is not maximal. This completes the proof of the lemma.

To complete the proof of the theorem, we find, in the situation of the lemma, $b \in A^+$ and $c \in A^-$ such that $c \lessdot b = b$ covers c. Since $b \in A^+$, $a < b$ for some $a \in A$; similarly $c < d$ for some $d \in A$. Then $c < a$ contradicts $c \lessdot b$; $a \leqslant c$ contradicts $c \in A^-$; therefore $a\|c$. Dually $b\|d$. Finally $a \neq d$ since $c < d$, so that $a\|d$. Therefore (a, b, c, d) is an N', which completes the proof.

3. The embedding theorem.

THEOREM 7. *Let E be regular. There exists a p.o. set $\bar{E}$ such that:*

(i) *$E \subseteq \bar{E}$ and for any $x, y \in E$, $x < y$ in $\bar{E}$ if and only if $x < y$ in E;*

(ii) *$\bar{E}$ is regular and* CAC;

(iii) *if $E \subseteq E_1 \subset \bar{E}$, then E_1 (with the induced order) is not* CAC;

(iv) *if f is an order-preserving mapping of E into a* CAC *p.o. set E', then f can be extended to an order-preserving of $\bar{E}$ into E'.*

Remarks. The two first properties show that E has been completed to a regular CAC p.o. set. The two last properties show that this completion is minimal in two ways; observe that (iv) is not a universal property however, since the extension of f to $\bar{E}$ need not be unique.

Proof. Theorem 5 suggests that $\bar{E}$ be constructed by destroying all N' of E. This is precisely what we shall do. Let E^* be the set of all pairs (u, v) such that $u, v \in E$ and $u \lesssim v$. For any subset S of E^*, define on $\bar{E} = E \cup S$ a binary relation by:

$$x < y \text{ (in } \bar{E}) \quad \text{if and only if} \quad x < y \text{ (in } E),$$
$$x < (u, v) \quad \text{if and only if} \quad x \leqslant u,$$
$$(u, v) < x \quad \text{if and only if} \quad v \leqslant x,$$
$$(u, v) < (u', v') \quad \text{if and only if} \quad v \leqslant u',$$

for all $x, y \in E$, $(u, v), (u', v') \in S$. It is readily verified that this relation is irreflexive and transitive, so that the addition of the equality makes it a partial order relation. The p.o. set $\bar{E}$ has already property (i). We show that it is regular for any choice of S.

LEMMA 8. *If C is a chain of $\bar{E}$ and if $(u, v) \in C \cap S$, then $C \cup \{u, v\}$ is a chain of $\bar{E}$.*

Proof. If $x \in C \cap E$, then either $x < (u, v)$ and $x \leqslant u$, or $(u, v) < x$ and $v \leqslant x$. If $(u', v') \in C \cap S$, then either $(u', v') = (u, v)$ and $u < (u', v') < v$; or $(u', v') < (u, v)$ and $(u', v') < v' \leqslant u < v$; or $(u, v) < (u', v')$ and $u < v < (u', v')$. Therefore u and v are comparable to any element of C; since finally $u < v$, $C \cup \{u, v\}$ is a chain.

LEMMA 9. *If C is a non-empty chain of $\bar{E}$, then either C has a maximum element or* $\sup C$ *exists in $\bar{E}$ and coincides with the supremum in E of the chain of all $x, u, v \in E$ such that $x \in C$ or $(u, v) \in C$.*

Proof. First $C \cup \{u, v \in E\colon (u, v) \in C\}$ is a chain of $\bar{E}$ by Lemma 8 and an easy transfinite induction (adding successively all pairs $(u, v) \in C$). Therefore this intersects E along a chain, which has a l.u.b. in E, say a, by sup-regularity. We have to prove that, if C does not have a maximal element, then a is the l.u.b. of C in $\bar{E}$.

First $x \leqslant a$, $(u, v) < v \leqslant a$ for any x, $(u, v) \in C$, so that a is an upper bound of C. Let b be another upper bound of C. If $b \in E$, then $x \leqslant b$ and $u < v \leqslant b$ for any x, $(u, v) \in C$, so that $a \leqslant b$ by definition of a. If $b = (u', v') \in S$, then, for any $x \in C \cap E$, $x \leqslant (u', v')$ and $x \leqslant u'$; for any $(u, v) \in C$, either $(u, v) = (u', v')$ and C has a maximum element, or $(u, v) < (u', v')$ and $u < v \leqslant u'$; therefore $a \leqslant u' < b$, or else C has a maximum element. This completes the proof.

LEMMA 10. *$\bar{E}$ is regular.*

Proof. Let C be a non-empty chain of $\bar{E}$. By Lemma 9, C has a l.u.b. in $\bar{E}$. Let $a = \sup C$ and take $b < a$. If C has a maximum element, then $b < a \in C$. From now on, we assume that C has no maximum element. Then $a \in E$ and a is the supremum in E of the chain of all $x, u, v \in E$ such that $x \in C$, $(u, v) \in C$.

If $b \in E$, then either $b < x$ or $b < u$ or $b < v$ for some $x \in C \cap E$ or some $(u, v) \in C$, by sup-regularity of E. In the first case $b < x \in C$; in the second case $b < (u, v) \in C$. In the third case, (u, v) is not the largest element of C so that $(u, v) < c$ for some $c \in C$; then $b < c$.

If $b = (u', v')$, then $v' \leqslant a$. We show that $v' = a$ is impossible. If $v' = a$, then $u' < a$ and by the previous case $u' < c$ for some $c \in C$. Since C has no maximum element, $u' < c < a = v'$, which implies $c = (u', v') = b$, so that $b \in C$. But again C has no maximum element, whence $b < c' < a$ for some $c' \in C$. This is impossible since $(u', v') \lesssim v'$ in $\bar{E}$. This proves that $v' < a$. Then, as above, $v' < c$ for some $c \in C$, and $b < v' < c$.

Therefore $\bar{E}$ is sup-regular. Dually $\bar{E}$ is inf-regular, so that it is regular.

Next we show that a suitable choice of S gives $\bar{E}$ the properties (ii) and (iii).

LEMMA 11. *$\bar{E}$ contains no N' if and only if:*

1) *S contains all pairs $(u, v) \in E^*$ such that (x, v, u, y) is an N' of E for some $x, y \in E$;*

2) *whenever $x, u, v, y \in E$ are such that $x < v$, $u < y$, $x < y$, $u \lesssim v$, $x \| u$, $y \| v$ and either $(x, v) \in S$ or $(u, y) \in S$, then $(u, v) \in S$.*

Proof. Let (a, b, c, d) be an N' of $\bar{E}$. We show first that $b, c \in E$. If $b, c \in S$, then $c = (u, v)$, $b = (u', v')$ and

$$c = (u, v) < v \leqslant u' < (u', v') = b$$

so that b does not cover c in $\bar{E}$. If $c = (u, v) \in S$, $b \in E$, then $c < v \leqslant b$ and $b = v$ since $c \lesssim b$; but then $b = v \leqslant d$ which is impossible. It is dually impossible that $c \in E$, $b \in S$. Since $c \lesssim b$ in $\bar{E}$, also $c \lesssim b$ in E and furthermore $(c, b) \notin S$. Now we proceed to show that either 1) or 2) does not hold.

Case A: $a, d \in E$. Then (a, b, c, d) is an N' of E and $(c, b) \notin S$ so that 1) does not holds.

Case B: $a=(u,v)\in S$, $d\in E$, $v\neq b$. In this case: $v<b$ since $a<b$; $c<v$ contradicts $c\lessdot b$ and $v\leqslant c$ contradicts $a\| c$, so that $v\| c$; $d<v$ contradicts $b\| d$ and $v\leqslant d$ contradicts $a\| d$. Therefore (v,b,c,d) is an N' of E and $(c,b)\notin S$ so that 1) does not hold.

Case C: $a=(u,b)\in S$, $d\in E$, $u\| d$. In this case: $c\leqslant u$ contradicts $a\| c$ and $u\leqslant c$ implies $u=c$ (since $u\lessdot b$) and contradicts $a\| c$, so that $u\| c$. Then (u,b,c,d) is an N' of E and 1) does not hold.

Case D: $a=(u,b)\in S$, $d\in E$, u and d are comparable. Then $u\| c$ as in case C; also $d\leqslant u$ contradicts $a\| d$, so that $u<d$. In this case 2) does not hold.

The cases when $a\in E$, $d\in S$ are treated dually. If finally $a,d\in S$, there is as above three cases to consider for each of a and d. We leave these cases to the reader since they are quite similar to cases B, C, D; in each case again, either 1) or 2) does not hold.

If conversely 1) does not hold, then (x,v,u,y) is an N' of E where $(u,v)\notin S$; (x,v,u,y) is then also an N' of $\bar{E}$. If 2) does not hold, and if $x,y,u,v\in E$ are such that $x<v$, $u<y$, $x<y$, $u\lessdot v$, $x\| u$, $v\| y$ and $(x,v)\in S$ for instance, with $(u,v)\notin S$, then (a,v,u,y), where $a=(x,v)\in S$, is an N' of $\bar{E}$. Indeed, $u\lessdot v$ in $\bar{E}$ since $(u,v)\notin S$; $a<v$; $u<y$; $v\| y$; if a and u were comparable, then either $v\leqslant u$ or $u\leqslant x$, so that $a\| u$; if a and y were comparable, then either $v\leqslant y$ or $y\leqslant x$, so that $a\| y$. This completes the proof of the lemma.

Call now S_0 the set of all $(u,v)\in E^*$ such that (x,v,u,y) is an N' of E for some $x,y\in E$. Observe that any intersection of subsets of E^* satisfying 2) again satisfies 2); since E^* itself satisfies 2) and contains S_0, there exists a smallest subset S of E^* verifying 1) and 2). The corresponding p.o. set $\bar{E}$ satisfies (iii) by Proposition 3 and Lemma 11; it is regular by Lemma 10 and, since it contains no N' by Lemma 11, it is CAC by Theorem 5.

Finally we show that $\bar{E}$ has property (iv). In fact this does not depend of the choice of S. Let f be an order-preserving mapping of E into some CAC p.o. set E'. If $(u,v)\in S$, we extend f to (u,v) in the following manner. If $(u,v)\notin S_0$, define $\bar{f}(u,v)=f(v)$. If $(u,v)\in S_0$, then (x,v,u,y) is an N' of E for some $x,y\in E$; if $\big(f(x),f(v),f(u),f(y)\big)$ is not an N of E', define $\bar{f}(u,v)=f(v)$; if $\big(f(x),f(v),f(u),f(y)\big)$ is an N of E', then by Proposition 3 it is not a proper N and there exists $z'\in E'$ such that $f(u)<z'<f(v)$, $f(x)\| z'\| f(y)$; define $\bar{f}(u,v)=z'$. It is immediate by the definition of the partial order on $\bar{E}$ that $\bar{f}$ is order-preserving. This completes the proof of the theorem.

4. The dimension theorem.

Recall that E is said *of dimension* n if n is the smallest cardinal number such that the partial order of E is intersection of n total orders (cf. [2]). We shall prove the following

THEOREM 12. *If E is a finite modular lattice and is* CAC, *then the dimension of E is at most* 2.

Example 1 of Section 2 shows that the converse does not hold. Consider also the direct product $\bar{R}\times\bar{R}\times\bar{R}$, where R is the chain of all real numbers and $\bar{R}=R\cup\{+\infty\}\cup\{-\infty\}$; it can be shown that this is a distributive lattice of dimension 3 and is CAC; this shows that the finiteness assumption cannot be removed in Theorem 12.

The proof can be sketched as follows. We shall assume that E has dimension at least 3. We consider all order relations R on E such that $x\leqslant y$ implies xRy and that there does not exist $x,y,z\in E$ such that $x\lessdot z$, $x\| y\| z$, $xRyRz$; a total order with these properties can be found if and only if E has dimension at most 2 (Lemmas 14, 15). Under our assumption, an order relation R maximal with the two properties above is not total, so that the addition to R of a pair of uncomparable elements will make R lose the second property. This yields a pattern of elements of E from which we manage to extract an N'.

DEFINITION 13. We say that the antichains of E can be *coherently oriented* if there exists a total order T on E (= a *coherent orientation* of the antichains) such that $x\leqslant y$ implies xTy and that $x\| y\| z$, $xTyTz$ implies $x\| z$.

(It can be shown that this is equivalent to the existence for each antichain A of E a total order T_A on A such that, for any antichains A,B: 1) if $x,y\in A\cap B$, then xT_Ay if and only if xT_By; 2) if $x,y\in A$, $y,z\in B$, xT_AyT_Bz implies $x\| z$ or $x=y=z$.)

LEMMA 14. *The antichains of E can be coherently oriented if and only if the dimension of E is at most* 2.

Proof. Assume that the order relation on E is intersection of two total orders T, T'. Then $x\leqslant y$ implies xTy. Also $x\| y\| z$, $xTyTz$ implies xTz and also $zT'yT'x$ (since T' is total and $xT'y$ for instance would imply $x\leqslant y$). Therefore xTz, $zT'x$, so that $x\| z$. Hence T is a coherent orientation of the antichains of E.

Assume conversely that there exists a coherent orientation T of the antichains of E. Define a binary relation T' on E by: $xT'y$ if and only if either $x\leqslant y$ or $x\| y$, yTx. Then T' is reflexive and antisymmetric. Transitivity follows from the condition on T; if for instance $x\leqslant y$, $y\| z$, zTy, then $z\leqslant x$ is impossible; also $x\| z$, xTz is impossible; therefore $xT'z$. It follows that T' is an order relation, clearly total. Finally it is immediate that xTy and $xT'y$ implies $x\leqslant y$ and conversely. Hence the dimension of E is at most 2.

LEMMA 15. *If E is finite, a total order T on E is a coherent orientation of the antichains of E if and only if*: 1) $x\leqslant y$ *implies* xTy; 2) $x\| y\| z$, $xTyTz$, $x\lessdot z$ *is impossible.*

Proof. These condition are obviously necessary. Conversely, suppose that T satisfies them but that $x\|y\|z$, $xTyTz$ for some $x, y, z \in E$ such that x and z are comparable. To obtain a contradiction, observe first that xTz, so that $x < z$. Choose a maximal chain from x to z, with n elements. If $n = 2$, then $x \prec z$, contradicting the assumptions on T. If there is contradiction whenever $n = p$, and if $n = p+1$, let t be in the chain and such that $x \prec t < z$. Then $y \leqslant t$ contradicts $y\|z$; $t \leqslant y$ contradicts $x\|y$; therefore $y\|t$. Finally T is total so that either yTt, contradicting 2), or tTy, contradicting the inductive hypothesis. This completes the proof.

Lemma 16. *Let R be a partial order relation on E and $a, b \in E$ be uncomparable for R. Then*

$$R' = R \cup \{(x, y);\ xRa,\ bRy\}$$

is an order relation containing R and such that $aR'b$.

Proof. This is straightforward. See also [2] for a proof.

Lemma 17. *Let E be a modular lattice. If (a, b, c, d) is an N of E such that $a \prec b$ or $c \prec d$, then E contains a proper N.*

Proof. If N is proper, the proof is over. If not, there exists $x \in E$ such that $c < x < b$, $a\|x\|d$. If $a \prec b$, then $a < a \vee x \leqslant b$ makes $a \vee x = b$; hence $a \wedge x \prec x$. Also $a \wedge x < a$, $c < x$, $x\|a\|c$. Finally $c \leqslant a \wedge x$ contradicts $a\|c$, $a \wedge x < c$ contradicts $a \wedge x \prec x$, so that $a \wedge x\|c$. Therefore $(c, x, a \wedge x, a)$ is an N' of E. The case when $c \prec d$ is dual.

Lemma 18. *Let E be a modular lattice. If E contains elements x, y, z such that $x\|y\|z$, $x \prec z$, then E contains a proper N.*

Proof. Observe that

$$x \leqslant x \vee (y \wedge z) = (x \vee y) \wedge z \leqslant z$$

so that either $x \vee (y \wedge z) = x$ or $x \vee (y \wedge z) = z$. In the first case $x \vee y \leqslant z$ contradicts $y\|z$ and $z \leqslant x \vee y$ contradicts $(x \vee y) \vee z = x$, so that $x \vee y\|z$. Therefore $(y, x \vee y, x, z)$ is an N of E, where $x \prec z$, and the conclusion follows from Lemma 17. The other case is dual.

Proof of Theorem 12. Assume that E is a finite modular lattice of dimension at least 3. Let R be an order relation on E which satisfies 1) and 2) of Lemma 15, and is maximal with these properties: such R exists since 2) is of finite character. If R were total, then it would be a coherent orientation of the antichains of E by Lemma 15, and E would have dimension at most 2 by Lemma 14. Therefore R is not total. By Lemma 16, we can find an order relation R' such that $R \subset R'$. Since R' then satisfies condition 1) of Lemma 15, it cannot satisfy 2), or else R is not maximal. Therefore there exists $x, y, z \in E$ such that $x\|y\|z$, $xR'yR'z$ and $x \prec z$. Then E contains a proper N by Lemma 18 and is therefore not CAC.

5. Sets of functions. Let R (R^+) be the set of all real numbers (all positive real numbers) and K be any compact metric space. For any mapping φ of R^+ into R^+ and any $M \in R^+$, let $C(M, \varphi)$ be the set of all (continuous) real-valued functions f on K such that

(i) $|f(x)| \leqslant M$ for all $x \in K$;

(ii) $|f(x) - f(y)| \leqslant \varepsilon$ whenever $\varepsilon > 0$, $x, y \in K$ and $d(x, y) \leqslant \varphi(\varepsilon)$.

It is well known that $C(M, \varphi)$ can be partially ordered in two fashions: first, by $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in K$; next, by $f \lll g$ if and only if $f(x) < g(x)$ for all $x \in K$. We shall refer to the latter as *strict* ordering, and expressions such as strictly comparable, strict antichain, etc. shall be understood as relative to that ordering. If for instance K is connected, then it is immediate that two functions $f, g \in C(M, \varphi)$ are strictly uncomparable (which shall be denoted by $f|||g$) if and only if $f \neq g$ and $f(x) = g(x)$ for some $x \in K$.

Obviously none of these p.o. sets contains an N'. However it is not difficult to verify that none of them is sup-regular, unless K has only one element, and the following examples will prove that they need not be CAC.

Example 19. Take $K = [0, 1] \subset R$, $M = 2$ and let φ be the identity mapping of R^+ and the ordering be the strict ordering. The set A of all functions of $C(2, \varphi)$ such that $f(0) = 0$ is clearly a strict antichain; it is maximal since, for any $f \in C(2, \varphi)$, either $f(0) = 0$ and $f \in A$, or $f(0) > 0$ and then $h \lll f$, where $h(x) = -x$, $h \in A$, or finally $f(0) < 0$ and then $f \lll -h \in A$.

On the other hand, let f_a be defined by $f_a(x) = 1 + a - x$ if $0 \leqslant a \leqslant 1$, $f_a(x) = x - 1 + a$ if $-1 \leqslant a < 0$, and let C be the set of all f_a for $-1 \leqslant a \leqslant 1$. Clearly C is a strict chain of $C(2, \varphi)$. To prove that it is maximal, take $g \in C(2, \varphi)$. If $g(1) \geqslant 0$, then $g \lll f_{g(1)}$; if $g \neq f_{g(1)}$, then $g(0) < f_{g(1)}(0) = 1 + g(1)$, so that $g ||| f_a$ whenever $g(0) - 1 < a < g(1)$. It is shown similarly that, if $g(1) < 0$, then g is either in C or is strictly uncomparable to some element of C. Therefore C is maximal.

Clearly $A \cap C = \emptyset$, so that $C(2, \varphi)$ is not CAC for the strict ordering.

Example 20. This time we use the other ordering with $K = [-1, +1] \subset R$, $M = 1$ and φ the identity mapping of R^+. For maximal chain we take the set of all constant functions of $C(1, \varphi)$. Constructing a suitable maximal antichain takes some more doing.

For each $a \in K$, let f_a be defined by: $f_a(x) = x + 1 + a$ when $-1 \leqslant x \leqslant -a$ and $f_a(x) = 1 - a - x$ when $-a \leqslant x \leqslant 1$; let g_a be defined by: $g_a(x) = a - 1 - x$ when $-1 \leqslant x \leqslant a$, $g_a(x) = x - 1 - a$ when $a \leqslant x \leqslant 1$. Observe that $f_a(-1) = g_a(-1) = a$, $f_a(1) = g_a(1) = -a$, so that $g_a \leqslant f_a$ and $g_a\|f_\beta$, $f_a\|f_\beta$, $g_a\|g_\beta$ whenever $a \neq \beta$. The set A of all f_a such that a is rational and all g_a such that a is irrational is, therefore, an antichain of $C(1, \varphi)$.

To prove that it is maximal, take $h \in C(1, \varphi)$ and set $h(-1) = a$, $h(1) = -\beta$. If $a = \beta$, then $h \leqslant f_a \in A$ if a is rational, $h \geqslant g_a \in A$ if a is irrational. If $a < \beta$, then observe that $h(x) \leqslant f_a(x)$ for $x \leqslant -a$, $h(x) \leqslant f_\beta(x)$ for $x \geqslant -\beta$; therefore $h(x) \leqslant f_\gamma(x)$ for all $x \in K$ if $a \leqslant \gamma \leqslant \beta$; choosing γ in that fashion and rational, we obtain $h \leqslant f_\gamma \in A$. We proceed similarly in case $a > \beta$, using the g_γ's. In all cases, h is comparable to some element of A, so that A is a maximal antichain.

Clearly A contains no constant, so that our p.o. set is again not CAC.

In view of these examples, the following theorem is the best that can be obtained in that situation:

THEOREM 21. *For any K, every maximal chain and every maximal strict antichain of $C(M, \varphi)$ intersect.*

The proof will use the techniques of Section 2, due to the fact that some sort of regularity appears when the two ordering are used simultaneously, as shown by the two following lemmas.

LEMMA 22. *Every non-empty chain C of $C(M, \varphi)$ has a* l.u.b. $\sup C$ *in $C(M, \varphi)$. If furthermore C is a strict chain, then $f \ll \sup C$ for all $f \in C$, $f \neq \sup C$.*

Proof. If $g = \sup C$ exists, it must be defined by: $g(x) = \operatorname{l.u.b.}_{f \in C} f(x)$ for all $x \in K$. All we have to show is that the function g defined by this equation is in $C(M, \varphi)$. Clearly $|g| \leqslant M$. Let ε be in R^+ and $x, y \in K$ be such that $d(x, y) \leqslant \varphi(\varepsilon)$. First

$$g(y) = \operatorname{l.u.b.}_{f \in C} f(y) \leqslant \operatorname{l.u.b.}_{f \in C} f(x) + \varepsilon = g(x) + \varepsilon .$$

Also, for each $a > 0$, there exists $f \in C$ such that $f(x) \geqslant g(x) - a$; then

$$g(y) \geqslant f(y) \geqslant f(x) - \varepsilon \geqslant g(x) - \varepsilon - a ,$$

whence $g(y) \geqslant g(x) - \varepsilon$ since a was arbitrary. Therefore $g \in C(M, \varphi)$.

If finally C is a strict chain, and if $f \in C$, $f \neq g$, then $f \ll h$ for some $h \in C$, whence $f \ll h \leqslant g$ and $f \ll g$.

Observe that $\sup C$ is usually not a strict l.u.b. of C.

LEMMA 23. *Let C be a non-empty chain of $C(M, \varphi)$ and $h \in C(M, \varphi)$ be such that $h \ll \sup C$. Then $h \ll f$ for some $f \in C$.*

Proof. Since $g = \sup C$ and h are continuous and K is compact, $h \ll g$ implies $a = \operatorname{g.l.b.}_{x \in K} \big(g(x) - h(x)\big) > 0$. Then $\operatorname{l.u.b.}_{x \in K} \big(g(x) - f(x)\big) < a$ for some $f \in C$ (or else, $f(x) \leqslant g(x) - a$ for all $f \in C$, $x \in K$, which is impossible). Choosing f thus:

$$h(x) \leqslant g(x) - a < f(x)$$

for all $x \in K$, whence $h \ll f \in C$.

Proof of Theorem 21. Assume that a maximal chain C and a maximal strict antichain A of $C(M, \varphi)$ do not intersect. First the constant function $+M$ is in C, since $C \cup \{+M\}$ is a chain and C is maximal; this function cannot be in A nor in A^-, so it is in A^+ and $C \cap A^+ \neq \emptyset$. Similarly $C \cap A^- \neq \emptyset$. Let $g = \sup C \cap A^-$, $h = \inf C \cap A^+$. Note that $g \leqslant h$; also, every element of C is in A^+ or in A^-, hence is comparable to g, so that $g \in C$; similarly $h \in C$.

Since g is not in A, g is either in A^+ or in A^-. But $g \in A^+$ implies $a \ll g$ for some $a \in A$ and $a \ll f \in C \cap A^-$ by Lemma 23, which is absurd. Therefore $g \in C \cap A^-$ and g is the largest element of $C \cap A^-$. Similarly h is the smallest element of $C \cap A^+$. Since C is maximal it follows that h covers g; but this is not possible in $C(M, \varphi)$, which completes the proof.

From Theorem 21 can be derived existence theorems such as the following:

COROLLARY 24. *Let K be a compact connected metric space and let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_p\}$ be two finite disjoint sets of continuous real-valued functions on K such that the f_i are pairwise comparable and that the graphs of any two g_j intersect. Then there exists a continuous real-valued function f on K which is comparable to all the f_i and whose graph intersects the graph of every g_j.*

Proof. The set of all f_i and g_j is a finite set of continuous functions on K and is therefore contained in some $C(M, \varphi)$. Then the f_i form a chain and the g_j a strict antichain of this $C(M, \varphi)$. These will be contained in a maximal chain and a maximal strict antichain, which by Theorem 21 have in common some function f having all the required properties.

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