On a conjecture of Leader

by

Louis J. Nachman (Columbus, Ohio)

Suppose $X$ is a set; $(Y, A)$ is a proximity space; and $[X, Y]$ is the collection of all single valued functions from $X$ to $Y$. In [3] Leader defines two types of convergence for nets ranging in $[X, Y]$. Suppose $(f_\alpha : \alpha \in D)$ is a net of functions from $X$ into $Y$. $(f_\alpha)$ is said to converge in proximity to some $f \in [X, Y]$ iff for $A \subseteq X$ and $B \subseteq Y$, $[f(A)] \delta B$ implies the existence of an $a^* \in D$ such that $a \supseteq a^*$ implies $[f(a)] \delta B$. $(f_\alpha)$ is said to converge uniformly to $f$ iff for every pseudometric $d$ on $Y$ such that $\delta(d) \leq \delta$, $(f_\alpha)$ converges uniformly relative to $d$ to $f$. Leader shows that uniform convergence implies convergence in proximity and conjectures that the converse is not true ([3], Conjecture 2).

If $X$ is a set and $(Y, \mathcal{U})$ is a uniform space then the uniformity of uniform convergence for $[X, Y]$ determined by $\mathcal{U}$ (see [1]) will be denoted by $\mathcal{U}_{\mathfrak{u}}$, its topology by $\mathcal{W}(\mathfrak{u})$. A proximity space $(X, \delta)$ determines a class $\mathcal{H}(\delta)$ of all uniformities for $X$ whose proximity is $\delta$. The unique smallest member of $\mathcal{H}(\delta)$ will be denoted by $\mathcal{U}_s(\delta)$ and $\mathcal{U}_s(\delta)$ will represent $\sup\{\mathcal{U}_s: \mathcal{U} \in \mathcal{H}(\delta)\}$. In this paper we relate Leader's convergence in proximity and uniform convergence to convergence relative to $\mathcal{U}_s(\delta)$ and $\mathcal{U}_s(\delta)$ and then show that Leader's conjecture is correct.

**Theorem 1.** Suppose $X$ is a non-empty set and $(Y, \delta)$ is a proximity space. Suppose $(f_\alpha : \alpha \in D)$ is a net ranging in $[X, Y]$ which converges to $f \in [X, Y]$ relative to $\mathcal{U}_s(\delta)$. Then $(f_\alpha)$ converges in proximity to $f$.

**Proof.** Suppose $A \subseteq X, B \subseteq Y$, and $[f(A)] \delta B$. Then there is a $U \in \mathcal{U}_s(\delta)$ such that $U \cap [f(A)] \cap B = \emptyset$. Let $U^* \in \mathcal{U}_s(\delta)$ be symmetric such that $U^* = U^* \subseteq U$. Since $(f_\alpha)$ converges to $f$ relative to $\mathcal{U}_s(\delta)$ there is an $a^* \in D$ such that $a \supseteq a^*$ implies $[(f_\alpha(a)), f(\epsilon)] \subseteq U^*$ for all $\epsilon \in X$. Thus $f_\alpha([f(A)]) \subseteq U^*[f(A)]$ and $U^*[f(A)] \subseteq U*[f(A)]$. But then $U^*[f(A)] \cap B = \emptyset$ and therefore $f_\alpha([f(A)]) \delta B$.

The following lemma appears in various forms throughout the literature on proximity spaces. A proof of this statement of the result can be found in [3].
LEMMA 1. Suppose \((X, \delta)\) is a proximity space and \(\mathcal{U}\) is a uniformity for \(X\) such that \(\delta(\mathcal{U}) < \delta\). Then
\[
\mathcal{U} \cup \{y \in X : y \in Y, d(y, a) < \varepsilon\}.
\]

It is clear that, if \(d\) is a pseudometric for \(X\) and \((f_x) \in \mathcal{Y}\) then \((f_x)\) converges uniformly to \(f\) relative to \(d\) iff \((f_x)\) converges to \(f\) relative to \(\mathcal{U}\) where \(\mathcal{U}\) is the uniformity with base \(\{V\} : \varepsilon > 0\);
\[
V_x = \{(a, b) \in X \times X : d(a, b) < \varepsilon\}.
\]

THEOREM 2. Suppose \(X\) is a non-empty set and \((Y, \delta)\) is a proximity space. Suppose \((f_x : a \in D)\) is a net ranging in \([X, Y]\); \(f \in (X, Y)\). Then \((f_x)\) converges uniformly to \(f\) iff \((f_x)\) converges to \(f\) relative to \(\mathcal{U}(\delta)\).

Proof. Suppose \((f_x)\) converges to \(f\) relative to \(\mathcal{U}(\delta)\). Let \(d\) be a pseudometric on \(X\) such that \(d(\delta) < \delta\). If \(\mathcal{U}_d\) is the uniformity generated by \(d\), then by Lemma 1, \(\mathcal{U}_d \supset \mathcal{U}(\delta)\). Since \((f_x)\) converges to \(f\) relative to \(\mathcal{U}(\delta)\), \((f_x)\) converges to \(f\) relative to \(\mathcal{U}_d \supset \mathcal{U}(\delta)\) and hence relative to \(\mathcal{U}_d\).

By the above remark we infer that \((f_x)\) converges uniformly to \(f\) relative to \(d\).

If \(\mathcal{U} \subset \mathcal{U}(\delta)\) and \(U \in \mathcal{U}\) let \(W(U) = \{(f, g) : (f(x), g(x)) \in U, \text{for all } x \in X\}\). Then \(W(U) \subset \mathcal{U}\) for some \(\mathcal{U} = \mathcal{U}(\delta)\) is a subbase for \(\mathcal{U}(\delta)\).

Suppose \((f_x)\) converges uniformly to \(f\). If for each \(U \in \mathcal{U}\), \(f \in \mathcal{U}(\delta)\), there is an \(a^* \in D\) such that \(a \supset a^*\) implies that \((f_x, f) \in W(U)\), then by the above remark we can conclude that \((f_x)\) converges to \(f\) relative to \(\mathcal{U}(\delta)\).

It is well known that, given such a \(U\), there is a pseudometric \(d\) for \(X\) such that \(d(a, b) < 1\) implies \(a, b \in U\) and \(\delta(\delta) < \delta\). Then since \((f_x)\) converges uniformly to \(f\) there is an \(a^* \in D\) such that \(a \supset a^*\) implies \(d[f(x), f(x)] < 1\) for all \(x \in X\) and hence \((f_x, f) \in W(U)\).

The following lemma shows that distinct uniformities for \(X\) determine distinct uniformities of uniform convergence for \([X, Y]\).

LEMMA 2. Suppose \(X\) and \(Y\) are non-empty sets and that the cardinality of \(X\) is at least as large as the cardinality of \(X\). Suppose \(\mathcal{U}\) and \(\mathcal{V}\) are uniformities for \(Y\) such that \(\mathcal{U} \subset \mathcal{V}\) and \(\mathcal{U} \neq \mathcal{V}\). Then \(\mathcal{U}(\mathcal{U}) \subset \mathcal{V}(\mathcal{U})\) and \(\mathcal{V}(\mathcal{U}) \neq \mathcal{V}(\mathcal{U})\).

Proof. That \(\mathcal{V}(\mathcal{U}) \subset \mathcal{V}(\mathcal{U})\) is well known. To prove \(\mathcal{U}(\mathcal{U}) \neq \mathcal{V}(\mathcal{U})\) we first note that the cardinality hypothesis insures the existence of a function \(g : X \to Y\) which is onto. Since \(\mathcal{U} \subset \mathcal{V}\) and \(\mathcal{U} \neq \mathcal{V}\) there is a \(V^* \in \mathcal{U}\) such that for any \(U \in \mathcal{U}\) there is a \(\pi(U), y \in U\) which is not in \(V^*\). We define an indexed set of functions \(f_x : X \to Y : U \in \mathcal{U}\) as follows:
\[
(f_x)_y = y \quad \text{if} \quad y \neq y(U) \quad \text{and} \quad (f_x)_y(U) = \pi(U).
\]

The collection \(\{f_x : U \in \mathcal{U}\}\) is a net if \(\mathcal{U}\) is directed by reverse inclusion.