On a conjecture of Leader

by

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Suppose X is a set: (Y, δ) is a proximity space; and $[X, Y]$ is the collection of all single valued functions from X to Y. In [2] Leader defines two types of convergence for nets ranging in $[X, Y]$. Suppose $\{f_\alpha\colon \alpha \in D\}$ is a net of functions from X into Y. $\{f_\alpha\}$ is said to *converge in proximity* to some $f \in [X, Y]$ iff for $A \subseteq X$ and $B \subseteq Y$, $f[A]\bar{\delta}B$ implies the existence of an $\alpha^* \in D$ such that $\alpha \geqslant \alpha^*$ implies $f_\alpha[A]\bar{\delta}B$, $\{f_\alpha\}$ is said to *converge uniformly to* f iff for every pseudometric d on Y such that $\delta(d) \leqslant \delta$, $\{f_\alpha\}$ converges uniformly relative to d to f. Leader shows that uniform convergence implies convergence in proximity and conjectures that the converse is not true ([2], Conjecture 2).

If X is a set and $(Y, \mathfrak{U})$ is a uniform space then the uniformity of uniform convergence for $[X, Y]$ determined by $\mathfrak{U}$ (see [1]) will be denoted by $\mathfrak{U}$, its topology by $\mathfrak{T}(\mathfrak{U})$. A proximity space (X, δ) determines a class $\Pi(\delta)$ of all uniformities for X whose proximity is δ. The unique smallest member of $\Pi(\delta)$ will be denoted by $\mathfrak{U}(\delta)$ and $\mathfrak{U}'(\delta)$ will represent $\sup\{\mathfrak{U}\colon \mathfrak{U} \in \Pi(\delta)\}$. In this paper we relate Leader's convergence in proximity and uniform convergence to convergence relative to $\mathfrak{U}(\delta)$ and $\mathfrak{U}'(\delta)$ and then show that Leader's conjecture is correct.

THEOREM 1. *Suppose X is a non-empty set and (Y, δ) is a proximity space. Suppose $\{f_\alpha\colon \alpha \in D\}$ is a net ranging in $[X, Y]$ which converges to $f \in [X, Y]$ relative to $\mathfrak{U}(\delta)$. Then $\{f_\alpha\}$ converges in proximity to f.*

Proof. Suppose $A \subseteq X$, $B \subseteq Y$, and $f[A]\bar{\delta}B$. Then there is a $U \in \mathfrak{U}(\delta)$ such that $U[f[A]] \cap B = \emptyset$. Let $U^* \in \mathfrak{U}(\delta)$ be symmetric such that $U^* \circ U^* \subseteq U$. Since $\{f_\alpha\}$ converges to f relative to $\mathfrak{U}(\delta)$ there is an $\alpha^* \in D$ such that $\alpha \geqslant \alpha^*$ implies $\big(f(x), f_\alpha(x)\big) \in U^*$ for all $x \in X$. Thus $f_\alpha[A] \subseteq U^*[f[A]]$ and $U^*[f_\alpha[A]] \subseteq U[f[A]]$. But then $U^*[f_\alpha[A]] \cap B = \emptyset$ and therefore $f_\alpha[A]\bar{\delta}B$.

The following lemma appears in various forms throughout the literature on proximity spaces. A proof of this statement of the result can be found in [3].

LEMMA 1. *Suppose* (X, δ) *is a proximity space and* $\mathfrak{U}$ *is a uniformity for* X *such that* $\delta(\mathfrak{U}) \leqslant \delta$. *Then*

$$\mathfrak{U} \vee \mathfrak{U}(\delta) \in \Pi(\delta).$$

It is clear that, if d is a pseudometric for Y and $\{f_a\}$ a net in $[X, Y]$ then $\{f_a\}$ converges uniformly to f relative to d iff $\{f_a\}$ converges to f relative to $\mathfrak{V}_d$ where $\mathfrak{V}_d$ is the uniformity with base $\{V_\varepsilon\colon \varepsilon > 0\}$;

$$V_\varepsilon = \{(a, b) \in Y \times Y\colon d(a, b) < \varepsilon\}.$$

THEOREM 2. *Suppose* X *is a non-empty set and* (Y, δ) *is a proximity space. Suppose* $\{f_a\colon a \in D\}$ *is a net ranging in* $[X, Y]$; $f \in [X, Y]$. *Then* $\{f_a\}$ *converges uniformly to* f *iff* $\{f_a\}$ *converges to* f *relative to* $\mathfrak{U}'(\delta)$.

Proof. Suppose $\{f_a\}$ converges to f relative to $\mathfrak{U}'(\delta)$. Let d be a pseudometric on Y such that $\delta(d) \leqslant \delta$. If $\mathfrak{V}_d$ is the uniformity generated by d, then by Lemma 1, $\mathfrak{V}_d \vee \mathfrak{U}(\delta) \in \Pi(\delta)$. Since $\{f_a\}$ converges to f relative to $\mathfrak{U}'(\delta)$, $\{f_a\}$ converges to f relative to $\mathfrak{V}_d \vee \mathfrak{U}(\delta)$ and hence relative to $\mathfrak{V}_d$. By the above remark we infer that $\{f_a\}$ converges uniformly to f relative to d.

If $\mathfrak{U} \in \Pi(\delta)$ and $U \in \mathfrak{U}$ let $W(U) = \{(f, g)\colon \big(f(x), g(x)\big) \in U$, for all $x \in X\}$. Then $\{W(U)\colon U \in \mathfrak{U}$ for some $\mathfrak{U} \in \Pi(\delta)\}$ is a subbase for $\mathfrak{U}'(\delta)$. Suppose $\{f_a\}$ converges uniformly to f. If for each $U \in \mathfrak{U}$, $\mathfrak{U} \in \Pi(\delta)$, there is an $a^* \in D$ such that $a \geqslant a^*$ implies that $(f, f_a) \in W(U)$, then by the above remark we can conclude that $\{f_a\}$ converges to f relative to $\mathfrak{U}'(\delta)$. It is will known that, given such a U, there is a pseudometric d for Y such that $d(a, b) < 1$ implies $(a, b) \in U$ and $\delta(d) \leqslant \delta$. Then since $\{f_a\}$ converges uniformly to f there is an $a^* \in D$ such that $a \geqslant a^*$ implies $d\big(f(x), f_a(x)\big) < 1$ for all $x \in X$ and hence $(f, f_a) \in W(U)$.

The following lemma shows that distinct uniformities for Y determine distinct topologies of uniform convergence for $[X, Y]$.

LEMMA 2. *Suppose* X *and* Y *are non-empty sets and that the cardinality of* X *is at least as large as the cardinality of* Y. *Suppose* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are uniformities for* Y *such that* $\mathfrak{U} \subseteq \mathfrak{V}$ *and* $\mathfrak{U} \neq \mathfrak{V}$. *Then* $\mathfrak{T}(\mathfrak{U}) \subseteq \mathfrak{T}(\mathfrak{V})$ *and* $\mathfrak{T}(\mathfrak{U}) \neq \mathfrak{T}(\mathfrak{V})$.

Proof. That $\mathfrak{T}(\mathfrak{U}) \subseteq \mathfrak{T}(\mathfrak{V})$ is well known. To prove $\mathfrak{T}(\mathfrak{U}) \neq \mathfrak{T}(\mathfrak{V})$ we first note that the cardinality hypothesis insures the existence of a function $g\colon X \to Y$ which is onto. Since $\mathfrak{U} \subseteq \mathfrak{V}$ and $\mathfrak{U} \neq \mathfrak{V}$ there is a $V^* \in \mathfrak{V}$ such that for any $U \in \mathfrak{U}$ there is a pair $\big(x(U), y(U)\big) \in U$ which is not in V^*. We define an indexed set of functions $\{f_U\colon Y \to Y\colon U \in \mathfrak{U}\}$ as follows:

$$f_U(y) = y \quad \text{if} \quad y \neq y(U) \quad \text{and} \quad f_U\big(y(U)\big) = x(U).$$

The collection $\{f_U \circ g\colon U \in \mathfrak{U}\}$ is a net if $\mathfrak{U}$ is directed by reverse inclusion.

It is a routine matter to check that $\{f_U \circ g\}$ converges to g in $\mathfrak{T}(\mathfrak{U})$. But, for $V^* \in \mathfrak{V}$ there is always an $x_0 \in g^{-1}\big(y(U)\big)$ for any $U \in \mathfrak{U}$ and hence $(f_U \circ g, g) \notin W(V^*)$ for any $U \in \mathfrak{U}$. Thus $\{f_U \circ g\}$ does not converge to g in $\mathfrak{T}(\mathfrak{V})$, completing the proof of the lemma.

We are now able to show easily that convergence in proximity need not imply uniform convergence.

EXAMPLE. Let Z be the integers and (Z, δ) be the integers with the discrete proximity. Then $\mathfrak{U}'(\delta)$ will be the discrete uniformity for Z. Then:

(1) $\mathfrak{U}(\delta) \subseteq \mathfrak{U}'(\delta)$,

(2) $\mathfrak{U}(\delta) \neq \mathfrak{U}'(\delta)$.

Then by Lemma 2, $\mathfrak{T}\big(\mathfrak{U}(\delta)\big) \subseteq \mathfrak{T}\big(\mathfrak{U}'(\delta)\big)$ and $\mathfrak{T}\big(\mathfrak{U}(\delta)\big) \neq \mathfrak{T}\big(\mathfrak{U}'(\delta)\big)$. Thus there is a function $f \in [Z, Z]$ and a net $\{f_a\}$ ranging in $[Z, Z]$ which converges to f relative to $\mathfrak{U}(\delta)$ (and hence by Theorem 1, converges to f in proximity) and which does not converge to f relative to $\mathfrak{U}'(\delta)$ (and thus by Theorem 2 does not converge to f uniformly); thus proving Leader's conjecture.

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References

[1] J. L. Kelley, *General Topology*, Van Nostrand, 1955.

[2] S. Leader, *On Completion of Proximity Spaces by Local Clusters*, Fund. Math. 48. (1959), pp. 201-216.

[3] W. J. Thron, *Topological Structures*, Holt, Rinehart, and Winston, 1966.

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