Embedding Cantor sets in a manifold

Part II: An extension theorem for homeomorphisms on Cantor sets

by

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In 1921 L. Antoine [3] gave an example of a Cantor set in E^3 whose complement was not simply connected. This was the first known example of a wild embedding of a Cantor set in E^n. Shortly thereafter (1924) J. W. Alexander [2] showed that the Cantor set of Antoine, often called Antoine's necklace, was contained in a 2-sphere in E^3 disproving the Schoenflies theorem for E^3. Concurrently, Alexander [1] gave an example of a 2-sphere in E^3 that was wild at a tame Cantor set of points. In 1951, Blankinship published a paper [4] in which he generalized the construction of Antoine's necklace to E^n for any $n \geqslant 3$, that is, he constructed Cantor sets in E^n whose complements were not simply connected. In this same paper, he showed that these generalized necklaces must lie on the boundary of a k-cell $(0 < k \leqslant n)$; thus giving a method for constructing wild k-cells in E^n.

In this paper we shall show that any Cantor set in a manifold (open or closed) is tamely embedded in the boundary of a k-cell, $0 < k \leqslant n$.

It should be remarked that each Cantor set in an n-manifold need not lie in an open n-cell. In fact, it has been shown by Hocking and Doyle in [6] that if each Cantor set in a compact 3-manifold lies in an open 3-cell then the manifold is S^3.

The following lemmas are easily proved by routine methods.

LEMMA 1. *Let $\varepsilon > 0$ be given and let A be a compact, 0-dimensional subset of E^n. Then there exists a finite collection of disjoint, open connected subsets $\{U_i\}$: $(i = 1, 2, \ldots, k)$ of E^n which cover A so that* (1) $\operatorname{diam} U_i < \varepsilon$, (2) $\mathrm{Cl}(U_i)$ *is a polyhedron, and* (3) $\mathrm{E}^n - \mathrm{Cl}(U_i)$ *is connected.*

LEMMA 2. *Let P and Q be disjoint, compact polyhedra in E^n both of which are the union of n-simplexes, $\varepsilon > 0$, and let a be a polyhedral arc with endpoints p and q such that $a \cap P = \{p\}$ and $a \cap Q = \{q\}$ and q is in the interior of an $(n-1)$ simplex δ of* $\mathrm{Bd}\,Q$. *Then a can be "blown up" into*

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a polyhedral n-cell C such that $C \cap P$ and $C \cap Q$ are $(n-1)$-cells in $\mathrm{Bd}P$ *and* $\mathrm{Bd}Q$ *respectively and $d(x, a) < \varepsilon$ for ever $x \in C$.*

The following theorem is proved by pulling arms out of an n-cell much as was originally done by Alexander [1].

THEOREM 3. *Let A be a Cantor set in E^n. Then A lies on the boundary of an n-cell $C^n \subset E^n$. Furthermore, C^n can be so chosen that A is tamely embedded in* $\mathrm{Bd}\,C^n$. (Note that C^n itself may well be wild in E^n and in fact C^n must be wild if A is wild. [7])

Proof. Let C_0 be a polyhedral n-cell in E^n whose distance from A is 1. Let $P_{1,1}, P_{1,2}, \ldots, P_{1,k_1}$ be disjoint polyhedra of diameter less than 1/2 such that $\mathrm{Int}P_{1,i}$ is connected and $A \subset \bigcup_{i=1}^{k_1} \mathrm{Int}P_{1,i}$ (Lemma 1). Let a be a polygonal arc in $\mathrm{Bd}\,C_0$, let $x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}$ be k_1 distinct points of a and let $y_{1,1}, y_{1,2}, \ldots, y_{1,k_1}$ be k_1 points such that $y_{1,i}$ lies in the interior of an $(n-1)$-simplex on the boundary of $P_{1,i}$. Choose disjoint polyhedral arcs $a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}$ so that the endpoints of $a_{1,i}$ are $x_{1,i}$ and $y_{1,i}$ and so that

$$a_{1,i} \cap C_0 = \{x_{1,i}\} \quad \text{and} \quad a_{1,i} \cap \Big(\bigcup_{j=1}^{k_1} p_{1,i}\Big) = \{y_{1,i}\}.$$

Applying Lemma 2, blow each $a_{1,i}$ up into a polyhedral n-cell $C_{1,i}$ such that $C_{1,i} \cap (\bigcup_{j\neq i} C_{1,j}) = \emptyset$, $C_{1,i} \cap C_0$ is a polyhedral $(n-1)$-cell, and $C_{1,i} \cap P_{1,i}$ is a polyhedral $(n-1)$-cell. Let $T_0 = C_0$ and let h_1 be a homeomorphism of C_0 onto $C_1 = C_0 \cup (\bigcup_{i=1}^{k_1} C_{1,i})$ such that $h_1(x_{1,i}) = y_{1,i}$. (Note: We must choose $T_0, T_1, T_2, \ldots$ in C_0 so that we do not stretch any part of C_0 too much in the construction. This will help to insure that the final composition of homeomorphism will be a homeomorphism. As we will show $\bigcap_{i=1}^{\infty} T_i$ will be the Cantor set that is mapped by the final homeomorphism onto A.)

Suppose now that the sets $P_{m,1}, P_{m,2}, \ldots, P_{m,k_m}, C_{m,1}, C_{m,2}, \ldots, C_{m,k_m}$ and C_m together with T_m and h_m have been defined. For the moment we restrict our attention to a polyhedron $P_{m,i}$. Applying Lemma 1, we get disjoint polyhedra $P^i_{m+1,1}, P^i_{m+1,2}, \ldots, P^i_{m+1,q_i}$ in $\mathrm{Int}P_{m,i}$ of diameter less than $1/2^{m+1}$ whose interiors cover $P_{m,i} \cap A$. Let $f_m = h_m \circ h_{m-1} \circ \ldots \circ h_1$ and choose distinct points $x^i_{m+1,1}, x^i_{m+1,2}, \ldots, x^i_{m+1,q_i}$ of $f_m(a) \cap \mathrm{Bd}P_{m,i}$. For each $j = 1, 2, \ldots, q_i$ let $y^i_{m+1,j}$ be a point in the interior of an $(n-1)$-simplex on the boundary of $P^i_{m+1,j}$. At this point, in order to avoid ever increasing numbers of superscripts, we shall reorder the $P^i_{m+1,j}$'s $(j = 1, 2, \ldots, q_i;\ i = 1, 2, \ldots, k_m)$ lexicographically in (i, j) (and correspondingly the $x^i_{m+1,j}$'s and $y^i_{m+1,j}$'s). We now have an ordering using two subscripts $P_{m+1,1}, P_{m+1,2}, \ldots, P_{m+1,k_{m+1}}$.

Now let $a_{m+1,1}, a_{m+1,2}, \ldots, a_{m+1,k_{m+1}}$ be disjoint polyherdal arcs such that (1) $x_{m+1,i}$ and $y_{m+1,i}$ are the endpoints of $a_{m+1,i}$, (2) $a_{m+1,i} \subset P_{m,j_i}$ for some j_i, (3) $a_{m+1,i} \cap \mathrm{Bd}P_{m,j_i} = \{x_{m+1,i}\}$, (4) $a_{m+1,i} \cap \mathrm{Bd}P_{m+1,i} = \{y_{m+1,i}\}$, and (5) $a_{m+1,i} \cap (\bigcup_{q\neq 1} P_{m+1,q}) = \emptyset$.

Next we wish to define the set T_{m+1}. Let $S(x, \varepsilon) = \{y\colon y \in C_0$ and $d(x, y) \leqslant \varepsilon\}$ and let $z_{m+1,i} = f_m^{-1}(x_{m+1,i})$. Choose $\varepsilon_{m+1} > 0$ small enough so that $S(z_{m+1,i}, \varepsilon_{m+1}) \subset \mathrm{Int}\,T_m$, $\varepsilon_{m+1} < 1/2^{m+1}$ and $S(z_{m+1,i}, \varepsilon_{m+1}) \cap S(z_{m+1,j}, \varepsilon_{m+1}) = \emptyset$ for $i \neq j$. Define $T_{m+1} = \bigcup_{i=1}^{k_{m+1}} S(z_{m+1,i}, \varepsilon_{m+1})$. Applying Lemma 2, blow each $a_{m+1,i}$ up into a polyhedral n-cell $C_{m+1,i}$ such that (1) $C_{m+1,i} \cap C_{m+1,j} = \emptyset$ for $i \neq j$, (2) $C_{m+1,i} \cap C_{m,j_i}$ is an $(n-1)$-cell in $f_m(T_{m+1})$, and (3) $C_{m+1,i} \cap P_{m+1,i}$ is an $(n-1)$-cell. Let $C_{m+1} = C_m \cup (\bigcup_{j=1}^{k_{m+1}} C_{m+1,j})$ and choose a homeomorphism $h_{m+1}\colon C_m \twoheadrightarrow C_{m+1}$ such that $h_{m+1}(x_{m+1,i}) = y_{m+1,i}$ and $h_{m+1}/C_m - f_m(T_{m+1}) = \mathrm{id}$. Finally we define $f(x) = \lim f_m(x)$. Since f is the uniform limit of a sequence of continuous functions, f is continuous. Because the domain of f is C_0, a compact set, we need only show that f is one-to-one to establish that f is a homeomorphism. Clearly $T_{m+1} \subset T_m$ and $T = \bigcap_{m=1}^{\infty} T_m$ is a Cantor set in $\mathrm{Bd}\,C_0$. For any point $x \in C_0 - T$ there exists an N such that $x \notin T_m$ for $m > N$, thus for all $m > N$ we have $f_m(x) = h_m(f_{m-1}(x)) = f_{m-1}(x)$. So $f = f_N$ in a neighborhood of x and f_N is a homeomorphism in a neighborhood of x. We see that f is one-to-one on $C_0 - T$. Since f is continuous, $f(\mathrm{Bd}\,C_0)$ is compact. Now $d(a, f_m(\mathrm{Bd}\,C_0)) < 1/2^m$ for any point $a \in A$; hence, $d(a, f(\mathrm{Bd}\,C)) = 0$ so $a \in f(\mathrm{Bd}\,C_0)$ and $A \subset f(\mathrm{Bd}\,C_0)$. Because $f_m(\mathrm{Bd}\,C_0) \cap A = \emptyset$ for each m and h_m is the identity near $z \notin T$ for sufficiently large m, it follows that $A \subset f(T)$. For each $z \in T$ there exists a sequence $\{z_m\}$ of points such that $z_m \in \{z_{m,i}\colon i = 1, 2, \ldots, k_m\}$, $d(z_m, z) < 1/2^m$ and $d(f_m(z_m), A) < 1/2^m$. Let $\varepsilon > 0$ be given. By uniform continuity of f, there is a $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon/2$ for $d(x, y) < \delta$. Choose m large enough so that $1/2^m < \delta$ and $1/2^{m-2} < \varepsilon/2$. We have

$$d(f(z), A) \leqslant d(f(z), f(z_m)) + d(f(z_m), f_m(z_m)) + d(f_m(z_m), A)$$
$$\leqslant \varepsilon/2 + 1/2^{m-1} + 1/2^m < \varepsilon.$$

It follows that $f(z) \in A$; hence $f(T) \subset A$ so $f(T) = A$. Now let $y \neq z$ be another point of T and let $\{y_m\colon m = 1, 2, \ldots\}$ be a sequence of points such that $y_m \in \{z_{m,i}\colon i = 1, 2, \ldots, k_m\}$ and $d(y_m, y) < 1/2^m$. There exists N such that for $m > N$ we have $d(z_m, y_m) > \gamma > 0$. Now since $\lim_{m\to\infty} f_m(z_m) = f(z)$ and $\lim_{m\to\infty} f_m(y_m) = f(y)$ and since $f_m(y_m)$ and $f_m(z_m)$ are eventually in distinct, disjoint, polyhedral neighborhoods, it follows that $f(y) \neq f(z)$. Finally, we want to show that $A \subset f(a)$. This follows from the fact that

$$d(f_m(a), a) < 1/2^m \quad \text{for each } a \in A .$$

Note that $f(C)$ is an n-cell which is locally polyhedral except at the points of A.

At first glance, the above theorem may not so appear, but it is an extension theorem which may be stated thus:

COROLLARY 4. *Let f' be a homeomorphism mapping the Cantor ternary set on the x_1-axis in E^n into E^n. f' can be extended to a homeomorphism f of the unit cube C^n into E^n.*

If f' can be extended to C^n, it can surely be extended to any face of C^n, thus:

COROLLARY 5. *Each Cantor set in E^n is tamely embedded in the boundary of a k-cell in E^n for $0 < k \leqslant n$.*

Note that if f' could be extended further to a neighborhood of the unit interval on the x_1-axis, then A would be tamely embedded in E^n.

We are ready for the statement and proof of the main theorem.

THEOREM 6. *Let M^n be an n-manifold, $n \geqslant 2$, open or closed and let A be a Cantor set in M^n. Then A is tamely embedded in the boundary of a k-cell in M^n for $0 < k \leqslant n$.*

Proof. Let C^n be a collared n-cell in M^n which does not intersect A. There exist open n-cells $E_1, E_2, \ldots, E_k$ such that $C^n \subset E_i$ and $\bigcup_{i=1}^{k} E_i$ covers A. By the Lebesgue Covering Theorem, there exists $\eta > 0$ such that any neighborhood of a point of A of diameter less than η lies wholly in E_i for some i. Let $\{U_{i,j}\}$ be a collection of connected, disjoint, closed neighborhoods of diameter less than η, whose interiors cover A, whose boundaries do not intersect A, that do not separate M^n and that are indexed so that $U_{i,j} \subset E_i$ for each j and i. Since C^n is collared in E_i, it follows that C^n is tamely embedded in E_i [5] and there is a polyhedral structure for E_i in which C^n is a polyhedral n-cell. In each set $U_{i,j}$ let $P_{i,j}$ be a polyhedral (in E_i) neighborhood of $A \cap U_{i,j}$ satisfying the conditions of Lemma 1. Now for each i and j we connect C^n with $P_{i,j}$ by disjoint polyhedral arcs $a_{i,j}$ so that the conditions of Lemma 2 are satisfied. Using Lemma 2, we "blow each polyhedral arc up" into a tame n-cell $C_{i,j}$ such that $C_{i,j} \cap P_{i,j}$ and $C_{i,j} \cap C^n$ are $(n-1)$-cells in $\mathrm{Bd}\, C_{i,j}$. We can now proceed as in the proof of Theorem 3 to get an n-cell in M^n which has A tamely embedded in its boundary. An application of the arguments of Corollaries 4 and 5 completes the proof.

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