Atomic compactness and graph theory

by

Walter Taylor (Boulder, Colo.)

The aim of this paper is to answer in the negative a question of Mycielski (see [9], Problem 484): is every atomic compact algebra a retract of a compact topological algebra? This question has been answered in the affirmative by Łoś [8] in the case of Abelian groups, and by Węglorz [14] in the case of linear spaces and the case of Boolean algebras. Our example will be a graph, but we will also show how to convert it to an example of an algebra which answers Mycielski's question in the negative.

1. Preliminaries. For background in the theory of models and application of the theory of ultraproducts to the theory of models, the reader is referred to [2], [7], and [12].

Our source of definitions in topology is [1]. Notice in particular that a compact topological space is assumed to be Hausdorff, except in § 8.

Our definition of atomic compactness will be the same as in [14], to which we refer the reader for definition of satisfaction of a set of formulas with constants. Briefly, a *structure* $\mathfrak{A} = \langle A, R_i \rangle$ *is atomic compact* if and only if every set of atomic formulas with constants in A which is finitely satisfiable in $\mathfrak{A}$, is satisfiable in $\mathfrak{A}$.

A compact topological structure is a pair $\langle \mathfrak{A}, \mathcal{T} \rangle$, where $\mathfrak{A} = \langle A, R_t \rangle_{t \in T}$ is a relational structure, and $\mathcal{T}$ is a compact (Hausdorff) topology on A, such that for every $t \in T$, the $n(t)$-ary relation R_t is a closed subset of $A^{n(t)}$ in the product topology.

By a *graph* we mean a relational structure $\langle A, R \rangle$ where $R \subseteq A^2$ is antireflexive and symmetric. A circuit of length k in A is a k-tuple of distinct elements of A, each related to the next, and the last related to the first. A set $B \subseteq A$ is called independent iff $R \cap B^2 = \emptyset$. The chromatic number of A is the least cardinal n such that A is the union of n independent subsets. See [11] or [13] for further explanation of these ideas.

We will let L denote the first-order predicate language with no constants, no function symbols, and one two-place relational symbol, R. (R will also stand for the binary relation of whatever graph we are describing.) Given a graph G, we will let L_G denote the language L augmented by the addition of one constant c_g for each $g \in G$. If U is any graph with a fixed embedding i: $G \to U$, then U can be supplied with an L_G structure by letting each constant c_g stand for $i(g)$ in U. $U \equiv V$ (resp. $U \equiv_G V$) will mean that U and V are elementarily equivalent as L-structures (resp. as L_G-structures.) $U \cong_G V$ will mean that U and V are isomorphic as L_G-structures.

2. Topological compactness and chromatic number.

THEOREM 2.1. *Suppose the graph A is a compact topological structure. Then A has finite chromatic number.*

Proof. We are given that R, the binary relation of the graph A is a closed subset of A^2 in the product topology. Suppose A has infinite chromatic number. Then let $\mathfrak{I}$ denote the (proper) ideal of subsets of A of finite chromatic number. Let $\mathcal{F}$ be an ultrafilter on A extending the filter of complements of members of $\mathfrak{I}$. Thus

$$(\forall K \in \mathcal{F})(\exists a \in K)(\exists b \in K)(\langle a, b\rangle \in R) . \tag{2.1}$$

Since A is compact, $\mathcal{F}$ converges to some $h \in A$. (See [1].) If U_1 and U_2 are neighborhoods of h, then $U_1 \cap U_2 \in \mathcal{F}$, and so taking $U_1 \cap U_2$ as K in (2.1), we have $\langle a, b\rangle \in R \cap (U_1 \times U_2)$. Since R is closed, $\langle h, h\rangle \in R$, which is a contradiction, since R is antireflexive. Q.E.D.

COROLLARY 2.2. *Suppose the graph G is a retract of a compact topological structure. Then G has finite chromatic number.*

Proof. Let S denote the binary relation of the graph G. We are given a compact topological structure $\langle A, T\rangle$, and we are given homomorphisms (see [14], p. 291) p and i

$$\langle G, S\rangle \underset{p}{\overset{i}{\rightleftarrows}} \langle A, T\rangle$$

where $p \circ i$ is the identity map on G. Clearly, since p is a homomorphism, T is antireflexive. Next we let R denote the relation which is the union of T and its converse, namely:

$$(\forall a \in A)(\forall b \in A)\big(\langle a, b\rangle \in R \text{ iff } (\langle a, b\rangle \in T \text{ or } \langle b, a\rangle \in T)\big) .$$

Clearly $\langle A, R\rangle$ is a compact topological graph which has G as a subgraph. By Theorem 2.1, G is thus a subgraph of a graph of finite chromatic number, and hence G has finite chromatic number. Q.E.D.

3. Elementary extensions of graphs.

Throughout this section G and H will stand for graphs.

DEFINITION 3.1. Let $g \in G$. The *valence* of g is the cardinality of the set $\{u: \langle g, u\rangle \in R\}$.

DEFINITION 3.2. If $A \subseteq G$, and $B \subseteq G$, then A and B will be said to be *unrelated* iff no element of A is related to any element of B.

THEOREM 3.3. *Suppose $G \equiv H$. If G has a finite number of circuits of length k, then H has the same number of circuits of length k.*

Proof. Obvious.

THEOREM 3.4. *Suppose each element of G has finite valence. Then if E is an elementary extension of G, $E \cong_G G \cup B$, where G and B are unrelated.*

Proof. Obvious.

THEOREM 3.5. *Suppose each element of G has finite valence, and G has only finitely many circuits of length k for each $k = 3, 4 \ldots$ Then if E is an elementary extension of G, $E \cong_G G \cup T$, where G and T are unrelated, and T is a tree, i.e. T has no circuits.*

Proof. By Theorems 3.3 and 3.4.

THEOREM 3.6. *If G is as in Theorem 3.5, then G is a retract of each of its elementary extensions.*

Proof. If E is an elementary extension, then by Theorem 3.5, $E \cong_G G \cup T$, where G and T are unrelated, and so we need only define the retraction on T. But since T is a tree, T has chromatic number 2 (see [11]), and so T may be mapped homomorphically onto any two related elements of G. Q.E.D.

4. The main result.

We will use the following result of Węglorz ([14], Theorem 2.3):

THEOREM 4.1. *A relational structure $\mathfrak{A}$ is atomic compact iff $\mathfrak{A}$ is a retract of every elementary extension of $\mathfrak{A}$.*

COROLLARY 4.2. *Suppose each element of the graph G has finite valence and G has only finitely many circuits of length k for each $k = 3, 4 \ldots$ Then G is atomic compact.*

Proof. By Theorems 3.6 and 4.1.

In order to construct our example, we will use the following theorem of Erdös. (See [4].)

THEOREM 4.3. *Let k and m be natural numbers. Then there is a finite graph with chromatic number $\geqslant m$, and with no circuits of length $\leqslant k$.*

Remark 4.4. The proof of Theorem 4.3 in [4] does not involve an explicit construction, but rather is probabilistic in nature. Graphs of high chromatic number and no circuits of length $\leqslant 6$ are explicitly constructed in [3] and [6]. Also see § 7 of this paper.

DEFINITION 4.5. For $n \geqslant 3$, G_n denotes an arbitrary (but fixed) *finite graph* of chromatic number $\geqslant n$ having no circuits of length $\leqslant n$.

DEFINITION 4.6. Let G be the graph $G_3 \cup G_4 \cup G_5 \cup \ldots$, where each two distinct G_i's are unrelated.

THEOREM 4.7. *G is atomic compact but not a retract of a compact topological structure. An algebra with such properties also exists.*

Proof. Clearly G has infinite chromatic number, and hence by Corollary 2.2, G is not a retract of a compact topological structure. But clearly G satisfies the hypothesis of Corollary 4.2, and so G is atomic compact. If we wish an algebra, we define G^* to be G with two new elements, 0 and 1, adjoined, and we define the binary operation $F\colon G^*\times G^*\to G^*$ as

$$F(g,h)=\begin{cases}0 & \text{if} \quad \langle g,h\rangle\notin R\,,\\ 1 & \text{if} \quad \langle g,h\rangle\in R\,.\end{cases}$$

Then the algebra $\langle G^*, F\rangle$ is atomic compact but not a retract of any compact topological algebra (or structure). Q.E.D.

PROBLEM 4.8. *Does there exist an atomic compact semigroup which is not a retract of a compact topological semigroup (or structure)?*

5. Elementary extensions of G.

THEOREM 5.1. *Let H be a graph of chromatic number k. Then H has a subgraph of chromatic number k each of whose elements has valence $\geqslant k-1$.*

Proof. Obvious. (See Theorem 14.3.1 of [11]).

DEFINITION 5.2. The *graph* G is defined as in Definition 4.6, with the further stipulation that each element of G_n have valence $\geqslant n-1$.

DEFINITION 5.3. A graph (or tree) is said to be *infinitely branching* iff each of its elements has infinite valence.

COROLLARY 5.4. *If E is an elementary extension of G, then $E\cong_G G\cup T$, where G and T are unrelated, and T is an infinitely branching tree.*

Proof. Theorem 3.5 and Definition 5.2.

The aim of this section is to prove the converse of Corollary 5.4 (see Theorem 5.8).

LEMMA 5.5. *Any non-principal ultrapower of a countably infinite set over a countably infinite index set has cardinality $2^{\aleph_0}$. A non-principal ultraproduct of finite sets over a countably infinite index set has cardinality $2^{\aleph_0}$ or is finite.*

Proof. Theorem 6.5 and Corollary 6.6 of [7].

LEMMA 5.6. *Let S and T be trees with the same number of components and the same valence at each element of each. Then $S\cong T$.*

THEOREM 5.7. *Let $E=G\cup T_0$, where G and T_0 are unrelated, and T_0 is a countable, infinitely branching tree. Then E is an elementary extension of G.*

Proof. Let D be a non-principal ultrafilter on a countably infinite set I. By Corollary 5.4, $G^I/D\cong_G G\cup S$, where S is an infinitely branching tree. Clearly, by Lemma 5.5, S has valence $2^{\aleph_0}$ at each element, and S has $2^{\aleph_0}$ components. A similar calculation shows that

$$(G\cup T_0)^I/D\cong_G G\cup S'\,,$$

where each element of S' has valence $2^{\aleph_0}$ and S' has $2^{\aleph_0}$ components. Thus by Lemma 5.6,

$$G^I/D\cong_G (G\cup T_0)^I/D\,,$$

and so $G\equiv_G G\cup T_0$. Q.E.D.

THEOREM 5.8. (Converse to Corollary 5.4.) *Suppose $E=G\cup T$, where G and T are unrelated, and T is an infinitely branching tree. Then E is an elementary extension of G.*

Proof. Since the language L_G is countable, we may apply the Skolem–Löwenheim Theorem (see e.g. [2] or [12]) to yield a countable L_G-structure A such that

$$A\equiv_G G\cup T\,.$$

Reasoning similar to that of Corollary 5.4 shows that A must be of the form $G\cup T_0$, where T_0 is as in Theorem 5.7. Thus,

$$G\equiv_G (G\cup T_0)\cong_G A\,.$$

Thus $G\equiv_G (G\cup T)$. Q.E.D.

6. Further applications of the graphs G_n.

Considering an ultraproduct of the graphs G_n, $n=1,2,\ldots$, we can easily see the following two theorems:

THEOREM 6.1. *The theory of graphs of chromatic number 2 is not finitely axiomatizable.*

THEOREM 6.2. *The theory of infinitely branching trees is not finitely axiomatizable.*

Finally we make a somewhat different application of Theorem 4.3 (of Erdös.) We will let H_n be a graph of chromatic number exactly k and having no circuits of length $\leqslant n$. An ultraproduct of the H_n's has no circuits, and hence

THEOREM 6.3. *The class of graphs of chromatic number k $(3\leqslant k\leqslant\aleph_0)$ is not elementary.*

7. Alternate construction.

Theorem 4.7 remains true if the graph G is replaced by a graph $H=H_3\cup H_5\cup H_7\cup\ldots$, where each H_n $(n=3,5,7,\ldots)$ has chromatic number $\geqslant n$, and has no odd circuits of length $\leqslant n$. Such graphs H_n are constructed in a manner similar to that given by Erdös in [5]. The graph H_n is a finite subset of the unit ball in k-dimensional Euclidean space. Two elements of H_n are related iff their distance exceeds $2-\varepsilon$. It is easy to check that one obtains the desired properties of H_n if ε is small and k is large. It is clear that no odd circuits are added to any elementary extension of H, and so any elementary extension of H is of the form $H \cup K$, where H and K are unrelated, and K contains no odd circuits. Thus Theorems 3.6 and 4.7 are true for the graph H. I do not know how to extend the ideas of § 5 to the graph H.

8. Atomic compactness defined topologically. Although we see that it is not possible to characterize atomic compact structures as retracts of compact topological structures, it is possible to characterize atomic compactness in terms of a certain (non-Hausdorff) topology. In this section we will be given a relational structure $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$, and we will let L be the first-order language corresponding to $\mathfrak{A}$.

DEFINITION 8.1. Let E denote the set of all formulas in the language L with one free variable and of the form

$$(\exists x_1)(\exists x_2)\ldots(\exists x_m)(a_1 \wedge a_2 \wedge \ldots \wedge a_m)$$

where m and n are natural numbers and the a_i are atomic formulas with constants in A.

DEFINITION 8.2. Let P denote the set of all positive formulas in the language L, with one free variable, and with constants in A.

DEFINITION 8.3. Let $\mathcal{C}$ be the topology on A which has the subbase of closed sets:

$$\{\{x \in A\colon \mathfrak{A} \models \Phi(x)\}\colon \Phi \in E\}.$$

The following theorem is a corollary of a theorem of Mycielski and Ryll-Nardzewski ([10], Theorem 3):

THEOREM 8.4. *The following are equivalent*:

(i) $\mathfrak{A}$ *is atomic compact.*

(ii) *Let Σ be any set of formulas from E, all having the same free variable x_0. If every finite subset of Σ is satisfiable in $\mathfrak{A}$, then Σ is satisfiable in $\mathfrak{A}$.*

COROLLARY 8.5. $\mathfrak{A}$ *is atomic compact iff the topological space $\langle A, \mathcal{C}\rangle$ is compact.*

Proof. Theorem 8.4 and the Alexander Subbase Theorem.

THEOREM 8.6. *If $\mathfrak{A}$ is atomic compact and $\Psi \in P$, then the set $\{x \in A\colon \mathfrak{A} \models \Psi(x)\}$ is closed in the topology $\mathcal{C}$.*

Proof. We may take Ψ to be of the form

$$(Q_1 y_1)\ldots(Q_n y_n)(c_1 \vee \ldots \vee c_m),$$

where each Q_i is $\exists$ or $\forall$, and where each c_j is a conjunction of atomic formulas. The proof is by induction on the number of appearances of $\forall$ in the formula Ψ. If $\forall$ does not appear, our set is a finite union of subbasic closed sets and hence closed. Now suppose there is at least one appearance of $\forall$, i.e. $\Psi(x)$ has the form

$$(\exists y_1)(\exists y_2)\ldots(\exists y_s)(\forall z)\Phi(x, z, y_1, \ldots, y_s),$$

where Φ has fewer appearances of $\forall$ than does Ψ. Now we let F stand for an arbitrary finite subset of A. Since $\mathfrak{A}$ is atomic compact, it is clear that $\Psi(a)$ is true in $\mathfrak{A}$ iff every possible sentence of the form

$$(\exists y_1)\ldots(\exists y_s)\bigwedge_{b\in F}\Phi(a, b, y_1, \ldots, y_s)$$

is true in $\mathfrak{A}$. Clearly then, by induction, our given set is an intersection of closed sets, and hence closed. Q.E.D.

References

[1] N. Bourbaki, *Éléments de Mathématique*, Livre III, *Topologie Générale*, Chap. I, Sections 7-9.

[2] P. M. Cohn, *Universal Algebra*, New York 1965.

[3] B. Descartes, *Solution to advanced problem* #4526, A. Math. Monthly 61 (1954), p. 352.

[4] P. Erdös, *Graph theory and probability*, Can. J. Math. 11 (1959), pp. 34-38.

[5] — *Remarks on a theorem of Ramsay*, Bull. of the Research Council of Israel (F) 7 (1957), pp. 21-24.

[6] J. B. Kelly and L. M. Kelly, *Paths and circuits in critical graphs*, Am. J. Math. 76 (1954), pp. 786-792.

[7] S. Kochen, *Ultraproducts in the theory of models*, Annals of Math. (2) 74 (1961), pp. 221-261.

[8] J. Łoś, *Abelian groups that are direct summands of every Abelian group which contains them as pure subgroups*, Fund. Math. 44 (1957), pp. 84-90.

[9] J. Mycielski, *Some compactifications of general algebras*, Coll. Math. 13 (1964), pp. 1-9.

[10] — and C. Ryll-Nardzewski, *Equationally compact algebras* (II), Fund. Math. 61 (1968), pp. 271-281.

[11] O. Ore, *Theory of Graphs*, AMS Colloquium Publications v. 38, Providence 1962.

[12] A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra*, Amsterdam 1965.

[13] *Symposium on the Theory of Graphs and its Applications*, Smolenice, June 1963, Academic Press, 1964.

[14] B. Węglorz, *Equationally compact algebras* (I), Fund. Math. 59 (1966), pp. 289-298.

Reçu par la Rédaction le 27. 1. 1968