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Reçu par la Rédaction le 15. 9. 1967

On countable multiple point compactifications*

by

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Throughout this paper a space X always denotes a Tychonoff space and $\hat{X}$ a Hausdorff compactification of it. The Stone–Čech compactification will be denoted by βX. Countable will mean countably infinite.

In [4], Magill characterizes those spaces X which possess finite compactifications (i.e. $|\hat{X}-X|$ is finite) and in [5], those which possess countable compactifications (i.e. $|\hat{X}-X|$ is countable). A much wider class of compactifications consists of those having a finite or countable number of multiple points as defined by Njåstad, [6].

In Section 1 of this paper we show that every non pseudocompact space has countable multiple point compactifications. It then follows that although the Euclidean n-spaces do not admit countable compactifications (cf. [5]) they do have countable multiple point compactifications. Section 1 will also provide examples of spaces X such that $|\beta X-X|$ is infinite yet X does not possess a countable multiple point compactification. These examples are obtained by using the fact that for any space Y, there exists a space X such that $\beta X-X = Y$ (cf. [2], p. 133).

Recent work (cf. [1], [3], [6], [7], [8]) has shown that many compactifications are of the Wallman-type (henceforth called Wallman compactifications) as defined by Frink [1]. In [6], Njåstad shows that if the set of multiple points in $\hat{X}$ is contained in some subset of $\hat{X}-X$ which is normally and zero-dimensionally embedded in $\hat{X}$, then $\hat{X}$ is a Wallman compactification. It follows that all finite multiple point compactifications are Wallman. The authors, in a previous work [8], have shown that every countable compactification is Wallman. The purpose of Section 2 is to generalize this by showing that all countable multiple point compactifications are Wallman. Since there are examples where the countable set of multiple points is not contained in any subset of $\hat{X}-X$ which is normally and zero-dimensionally embedded in $\hat{X}$, this does not follow from the theorem of Njåstad.

* Research supported by the National Science Foundation, Grant GP 6529.

1. Let f be a continuous mapping of βX onto the compactification $\hat{X}$ which is the identity on X. If $f^{-1}(y)$ consists of more than one point, y is called a *multiple point* of $\hat{X}$.

A space is *pseudocompact* if every continuous real-valued function on it is bounded. It is not difficult to see that X is pseudocompact if and only if every nonempty zero-set in βX meets X.

THEOREM 1. *Every space which is not pseudocompact has a countable multiple point compactification.*

Proof. Suppose X is not pseudocompact. Then there is a nonempty zero-set $Z \subset \beta X$ which misses X and $|Z| \geqslant 2^c$ ([2], p. 132).

Since Z is an infinite Hausdorff space, there is a countable discrete subset $A = \{x_1, x_2, \ldots\} \subset Z$.

Let $\mathfrak{D}$ be a decomposition of βX whose members are $\bar{A} - A$, the pairs $\{x_{2n-1}, x_{2n}\}$, $n = 1, 2, \ldots$, and all sets $\{x\}$ for $x \in \beta X - \bar{A}$.

We will show that $\mathfrak{D}$ is an upper semi-continuous decomposition, of βX. Let O be an open set containing $\{x_{2n-1}, x_{2n}\}$. Since A is discrete, there is an open set U_n, $n = 1, 2, \ldots$, such that $\{x_n\} = U_n \cap A$. Thus, letting

$$V = (U_{2n-1} \cup U_{2n}) \cap \big(\beta X - (\bar{A} - A)\big) \cap O\,,$$

we see that V is a saturated open set and $\{x_{2n-1}, x_{2n}\} \subset V \subset O$.

If $\bar{A} - A \subset O$, where O is open, then O contains all but a finite number of elements of A. If infinitely many points of A are in $\beta X - O$, a limit point of A would also be in $\beta X - O$ since $\beta X - O$ is compact. Letting

$$V = O - (\{x_{2n}|\ x_{2n-1} \notin O\} \cup \{x_{2n-1}|\ x_{2n} \notin O\})$$

we see that V is a saturated open set and $\bar{A} - A \subset V \subset O$.

Finally, if $x \in \beta X - \bar{A}$ and $\{x\} \subset O$, O open, there is an open set V containing x such that $V \cap \bar{A} = \emptyset$. Thus $\{x\} \subset V \cap O \subset O$ and $V \cap O$ is a saturated open set.

Thus the quotient space of the decomposition $\mathfrak{D}$ is a Hausdorff compactification of X which possesses a countable number of multiple points.

Thus we see that the Euclidean n-spaces and many other common spaces possess countable multiple point compactifications. We will now show that this is not always the case, no matter how large $|\beta X - X|$ may be.

LEMMA 1. *Every space X such that $|\beta X - X|$ is infinite has a countable multiple point compactification if and only if every compactification $\hat{Y}$ of every infinite space Y allows a decomposition $\mathfrak{D}$ satisfying:*

(i) *$\mathfrak{D}$ is upper semi-continuous,*

(ii) *if $y \in \hat{Y} - Y$ then $\{y\} \in \mathfrak{D}$,*

(iii) *the number of sets in $\mathfrak{D}$ having more than one element is countable.*

Proof. First we notice that any decomposition of βX whose quotient is a compactification of X must be an upper semi-continuous decomposition, in which the single points of X are elements. Hence the decomposition is determined by its restriction to $\beta X - X$. It is upper semi-continuous if it is upper semi-continuous on $\mathrm{cl}_{\beta X}(\beta X - X)$.

Thus, any decomposition $\mathfrak{D}$ of $\mathrm{cl}_{\beta X}(\beta X - X)$ satisfying (i), (ii), and (iii), where $Y = \beta X - X$, determines a decomposition of βX whose quotient is a countable multiple point compactification of X.

Conversely, let $\hat{Y}$ be any compactification of Y, and let $W(\omega_\alpha)$ be the set of all ordinals less than ω_α, where ω_α is the smallest ordinal of cardinal $\aleph_\alpha$. Suppose $|\hat{Y}| < \aleph_\beta$ where β is a nonlimit ordinal > 0.

If $Z = \hat{Y} \times W(\omega_\beta)$, then $\beta Z = \hat{Y} \times W(\omega_\beta + 1)$ ([2], p. 138). Let $X = \beta Z - (Y \times \{\omega_\beta\})$. Since $Z \subset X \subset \beta Z$, we have that $\beta X = \beta Z$ and $\beta X - X = Y \times \{\omega_\beta\}$. The closure of $Y \times \{\omega_\beta\}$ in βX is $\hat{Y} \times \{\omega_\beta\}$. If X has a countable multiple point compactification, it is determined by an upper semi-continuous decomposition of βX which induces on $\hat{Y} \times \{\omega_\beta\}$ (consequently, on $\hat{Y}$) a decomposition satisfying conditions (i), (ii), and (iii).

We remark that this lemma allows us to consider all compactifications when looking for examples instead of restricting our attention to Stone–Čech compactifications. But more important, the non-singleton sets of the decomposition considered occur in Y rather than in $\hat{Y} - Y$.

LEMMA 2. *If X is an infinite discrete space, βX does not allow a decomposition satisfying* (i), (ii) *and* (iii) *of Lemma* 1.

Proof. Suppose that $\mathfrak{D}$ is a decomposition of βX satisfying conditions (ii) and (iii). By the axiom of choice, there is a set $U \subset X$ which contains exactly one element from each non-singleton set in $\mathfrak{D}$. Since U is open and closed in X, it follows that $\mathrm{cl}_{\beta X} U$ is open. From (iii), U is infinite, hence not compact, so there is a $y \in \mathrm{cl}_{\beta X} U \cap (\beta X - X)$. From (ii), $\{y\} \in \mathfrak{D}$. Clearly, no open subset of $\mathrm{cl}_{\beta X} U$ is saturated; thus $\mathfrak{D}$ fails to be upper semi-continuous.

From Lemmas 1 and 2 we obtain:

THEOREM 2. *There are spaces X, with $|\beta X - X|$ infinite, which do not admit countable multiple point compactifications.*

In particular, if N is the set of integers with the discrete topology, the pseudocompact space

$$X = \beta N \times W(\omega_1 + 1) - (N \times \{\omega_1\})$$

has no countable multiple point compactification.

2. In [1], Frink generalized Wallman's method of compactification by using a normal base of closed sets instead of the family of all closed sets. If the normal base can be chosen from the zero-sets of continuous real-valued functions, the compactification is called a Z-compactification (cf. [8]).

We will now show that every countable multiple point compactification is Wallman. A family $\mathcal{F}$ of sets in $\hat{X}$ has the *trace property* with respect to X if for $F_i \in \mathcal{F}$,

$$\bigcap \{F_i|\ i = i, \ldots, n\} \neq \emptyset$$

implies that

$$\bigcap \{F_i|\ i = 1, \ldots, n\} \cap X \neq \emptyset .$$

A simple criterion for $\hat{X}$ to be a Wallman compactification is that there exist a base for closed sets in $\hat{X}$ which has the trace property with respect to X. This is shown in [7] and can be deduced from a theorem of Njåstad ([6], p. 271).

THEOREM 3. *If $\hat{X}$ is a countable multiple point compactification of X, then it is Wallman.*

Proof. Let M denote the countable set of multiple points. Let $Z(X)$ be the family of all zero-sets in X and let f be the quotient map from βX onto $\hat{X}$. The family

$$\mathfrak{Z} = \{\bar{Z} \subset \hat{X}|\ Z \in Z(X) \text{ and } f^{-1}[\bar{Z}] = \mathrm{cl}_{\beta X} Z\}$$

is closed under finite unions and finite intersections, since

$$\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \mathrm{cl}_{\beta X}(Z_1 \cap Z_2) .$$

It has the trace property with respect to X since the closures in βX of zero-sets in X have the trace property with respect to X.

We will now show that $\mathfrak{Z}$ is a base for the closed sets in $\hat{X}$. Let H be a closed subset of $\hat{X}$ not containing x. There is a continuous real-valued function h on $\hat{X}$ vanishing on H, with $h(x) = 1$. Let r be a real number, $0 < r < 1$, which is not in the at most countable set $h[M]$. The set $T = \{y \in \hat{X}|\ h(y) \leqslant r\}$ is a zero-set in $\hat{X}$ so $Z = T \cap X$ is in $Z(X)$. If $V = \{y \in \hat{X}|\ h(y) < r\}$, then $V \cap X \subset Z$, so $V \subset \bar{V} \subset \bar{Z} \subset T$. Also, if $y \in \partial\bar{Z}$, then $h(y) = r$ so $y \notin M$. Thus $\partial\bar{Z} \cap M = \emptyset$, $H \subset V \subset \bar{Z}$ and $x \notin \bar{Z}$.

The only remaining thing to show is that $\bar{Z} \in \mathfrak{Z}$. Suppose $p \in f^{-1}(\bar{Z}) - \mathrm{cl}_{\beta X} Z$. Then $f(p) \in M$ and $f(p) \in \bar{Z}$. Thus $f(p) \in V$ and $p \in f^{-1}[V]$, which is open. But since $V \cap X \subset Z$ and $V \cap X = f^{-1}[V] \cap X$, we see that $p \in \mathrm{cl}_{\beta X} f^{-1}[V] \subset \mathrm{cl}_{\beta X} Z$. This contradiction proves that $\bar{Z} \in \mathfrak{Z}$.

In [8], the authors have shown that if a countable multiple point compactification is Wallman, then it is a Z-compactification. Thus we have

COROLLARY. *Every countable multiple point compactification is a Z-compactification.*

Actually this is also obvious from the proof of the theorem, since the traces of the members of $\mathfrak{Z}$ on X are zero-sets in X.

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Reçu par la Rédaction le 3. 11. 1967