The range of a planar function with ambiguous points

by

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By an arc Λ at a point ζ in the plane P we shall mean a simple continuous curve $z = z(t)$ $(0 \leqslant t < 1)$ such that $z(t) \neq \zeta$ for $0 \leqslant t < 1$ and $\lim_{t\to 1} z(t) = \zeta$. Let $w = f(z)$ be an arbitrary single valued function mapping P into itself. If Λ is an arc at $\zeta \in P$ we define the cluster set of f at ζ along Λ, denoted by $C_\Lambda(f, \zeta)$, to be the set of all values a such that there exists a sequence $\{z_n\}$ on Λ with $z_n \to \zeta$ and $f(z_n) \to a$. We call ζ an *ambiguous point* of f if there exist arcs Λ and Λ' at ζ such that $C_\Lambda(f, \zeta) \cap C_{\Lambda'}(f, \zeta) = \emptyset$. If in addition Λ and Λ' are rectilinear segments having opposite directions then ζ is called a *rectilinearly oppositely ambiguous point*, and the angle θ between Λ or Λ' and a horizontal line shall be called the *corresponding direction of ambiguity*. We shall let $f(r, s)$ denote the following sentence: Given the distinct directions $\theta_1, \theta_2, \ldots, \theta_r$, there exists a single valued function f having a range of at most s values, such that every point in the plane is a rectilinearly oppositely ambiguous point of f with at least one of $\theta_1, \theta_2, \ldots, \theta_r$ as direction of ambiguity.

The intent of this paper is to answer a question which arises from the following two theorems.

THEOREM 1. *If* $2^{\aleph_0} \leqslant \aleph_1$ *then* $f(3, 4)$ [1].

THEOREM 2. *If* $2^{\aleph_0} \leqslant \aleph_n$ *then* $f(n+2, 2^{n+2}-1)$ [2].

It is evident that for $n = 1$ Theorem 2 is a weaker form of Theorem 1. We intend to show that Theorem 2 remains true for a range of 2^{n+1} values.

Let X_n be the set consisting of all $n+2$ tuples of the form $(j_1, j_2, \ldots, j_{k-1}, k, j_{k+1}, \ldots, j_{n+2})$ where $k = 1, 2, 3, \ldots, n+2$ and each j_i is zero or one. For any $n+2$ tuple of the form $(j_1, j_2, \ldots, j_{n+2})$ where each j_i is zero or one we define the subset $B(j_1, j_2, \ldots, j_{n+2})$ of X_n to consist of the elements

$$\begin{array}{l}
(1, j_2, j_3, \ldots, \qquad\qquad j_{n+2}),\\
(j_1, 2, j_3, \ldots, \qquad\qquad j_{n+2}),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(j_1, j_2, j_3, \ldots, j_{k-1}, k, j_{k+1}, \ldots, j_{n+2}),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(j_1, j_2, j_3, \ldots, \qquad j_{n+1}, n+2).
\end{array}$$

Let $B_1 = B(j_1, j_2, \ldots, j_{n+2})$ and $B_2 = B(j'_1, j'_2, \ldots, j'_{n+2})$. If $j_i \neq j'_i$ for at least two values of i then $B_1 \cap B_2 = \emptyset$.

LEMMA. *Let Y_n be the set of all $n+2$ tuples having coordinates that are zero or one. For any natural number n, there exists a subset A_n of Y_n containing exactly 2^{n+1} elements, such that any two elements of A_n differ in at least two coordinates.*

We shall actually prove the existence of two such disjoint sets, both satisfying the conditions of the lemma, and our proof is by induction.

For $n=1$ let $A_1 = \{(0,0,0), (0,1,1), (1,1,0), (1,0,1)\}$ and $B_1 = \{(1,1,1), (1,0,0), (0,0,1), (0,1,0)\}$. Then $A_1 \cap B_1 = \emptyset$ and A_1 and B_1 both satisfy the conditions of the lemma.

Let $n \geqslant 1$ and suppose A_n and B_n are disjoint sets, both satisfying the conditions of the lemma. Let

$$C = \{(j_1, j_2, \ldots, j_{n+2}, 0) | (j_1, j_2, \ldots, j_{n+2}) \in A_n\},$$
$$D = \{(j_1, j_2, \ldots, j_{n+2}, 1) | (j_1, j_2, \ldots, j_{n+2}) \in B_n\},$$
$$E = \{(j_1, j_2, \ldots, j_{n+2}, 0) | (j_1, j_2, \ldots, j_{n+2}) \in B_n\},$$
$$F = \{(j_1, j_2, \ldots, j_{n+2}, 1) | (j_1, j_2, \ldots, j_{n+2}) \in A_n\}$$

and set $A_{n+1} = C \cup D$ and $B_{n+1} = E \cup F$. Then $A_{n+1} \cap B_{n+1} = \emptyset$ and both satisfy the conditions of the lemma, which completes the proof.

Now associated with the 2^{n+1} elements of A_n there are 2^{n+1} sets of the form $B(j_1, j_2, \ldots, j_{n+2})$ where $(j_1, j_2, \ldots, j_{n+2}) \in A_n$ which for convenience we label arbitrarily as $B_1, B_2, \ldots, B_{2^{n+1}}$. By the above comment the B_i's are mutually disjoint since elements of A_n differ in at least two coordinates, and so $\bigcup_{i=1}^{2^{n+1}} B_i$ has $(n+2)2^{n+1}$ elements. Since X_n has $(n+2)2^{n+1}$ elements, it follows that $X_n = \bigcup_{i=1}^{2^{n+1}} B_i$.

THEOREM 3. *If $2^{\aleph_0} \leqslant \aleph_n$, then $f(n+2, 2^{n+1})$.*

Proof. With the above work behind us our proof essentially parallels the proof of Theorem 1. By a theorem of Davies [3] the hypothesis implies that $P = \bigcup_{j=1}^{n+2} E_j$ where the E_j's are mutually disjoint and every line having direction θ_j intersects E_j in at most finitely many points. Let L be any line in the plane. We impose an ordering on L as follows. For $z, z' \in L$ $z < z'$ if either $\mathrm{Im}(z) < \mathrm{Im}(z')$ or if $\mathrm{Im}(z) = \mathrm{Im}(z')$ and $\mathrm{Re}(z) < \mathrm{Re}(z')$. Let $L(\theta_j)$ be any line in the plane having direction θ_j. Then $L(\theta_j)$ intersects E_j in only finitely many points which we label as $e_1 < e_2 < e_3 < \ldots < e_m$. Let $z \in L(\theta_j)$. We call z a *class zero point with respect to* E_j if either $z < e_1$, $e_2 < z < e_3$, $e_4 < z < e_5, \ldots, e_p < z < e_{p+1}$ ($p \leqslant m$ and p even) and we call z a *class one point with respect to* E_j if either $e_1 < z < e_2$, $e_3 < z < e_4$, $e_5 < z < e_6, \ldots, e_p < z < e_{p+1}$ ($p \leqslant m$ and p odd). For $p = m$ we define $e_p < z < e_{p+1}$ to mean $e_p < z$. If $L(\theta_j) \cap E_j$ is empty then we call $z \in L(\theta_j)$ a class zero point with respect to E_j. We have thus partitioned every line in the plane having direction $\theta_1, \theta_2, \ldots, \theta_{n+2}$ into alternating intervals of class zero or class one points. Now let $z \in P$. Then $z \in E_k$ for some unique k. Through z construct the $(n+1)$ lines having directions θ_j with $j \neq k$, say $L(\theta_j)$ $(j \neq k)$. For each $j \neq k$ since $z \notin E_j$, z is a class zero or a class one point with respect to E_j. We define the correspondence φ mapping the plane into X_n by $\varphi(z) = (j_1, j_2, \ldots, j_{k-1}, k, j_{k+1}, \ldots, j_{n+2})$ where $z \in E_k$ and z is a class j_i point with respect to E_i $(i \neq k)$.

Using the B_j's defined previously and the correspondence φ we are now ready to define our function f. For $z \in P$ define $f(z) = j$ where $\varphi(z) \in B_j$. The function f is clearly single valued and has a range of at most 2^{n+1} values.

Let $z \in P$. Then $z \in E_k$ for some unique k. Let $L(\theta_k)$ be a line segment with direction θ_k containing z in its interior and small enough so that it intersects E_k in no other point. Let $z_1, z_2 \in L(\theta_k)$ with $z_1 < z < z_2$. Then $\varphi(z_1)$ and $\varphi(z_2)$ differ in their kth coordinates. We assert that $f(z_1) \neq f(z_2)$, for if not $f(z_1) = f(z_2) = j$ implies that $\varphi(z_1), \varphi(z_2) \in B_j$ and since $\varphi(z_1)$ and $\varphi(z_2)$ differ in their kth coordinates it follows from the definition of B_j and φ that one of the elements z_1 or z_2 must be in E_k, which contradicts the definition of $L(\theta_k)$. Thus if we let Λ and Λ' be the arcs at z determined by the two sides of $L(\theta_k)$ we see that $C_\Lambda(f, z) \cap C_{\Lambda'}(f, z) = \emptyset$ and hence z is a rectilinearly oppositely ambiguous point of f with θ_k as direction of ambiguity.

It remains an open question whether the range can be reduced further. The function we have defined in Theorem 3 depends upon the partition of X_n into the 2^{n+1} disjoint sets B_j and the question immediately arises as to whether or not we can find a partition of X_n into say m disjoint sets, with $m < 2^{n+1}$, that would yield a function satisfying Theorem 3 but with a range of at most m values. It can be shown however, that such an accomplishment would require additional, as yet unknown, information about Davies' decomposition of the plane mentioned at the beginning of the proof of Theorem 3.

There is in fact reason to suspect that the range cannot be further reduced. We define the minimal range as the smallest number ϱ_n such that $2^{\aleph_0} = \aleph_n$ implies that $f(n+2, \varrho_n)$. It has been shown in [1] that there does not exist a function having a range of three values or less such that every point in the plane is a rectilinearly oppositely ambiguous point. Note that we make no restriction here on directions of ambiguity. Thus $\varrho_1 = 4$ and so Theorem 3 is minimal for $n = 1$.

The reader is referred to [1] and [2] for other interesting theorems on the subject. In particular $f(n+2, \aleph_0)$ implies that $2^{\aleph_0} \leqslant \aleph_n$, so that the converses to all the above results hold.

References

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On countable multiple point compactifications*

by

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Throughout this paper a space X always denotes a Tychonoff space and $\hat{X}$ a Hausdorff compactification of it. The Stone–Čech compactification will be denoted by βX. Countable will mean countably infinite.

In [4], Magill characterizes those spaces X which possess finite compactifications (i.e. $|\hat{X}-X|$ is finite) and in [5], those which possess countable compactifications (i.e. $|\hat{X}-X|$ is countable). A much wider class of compactifications consists of those having a finite or countable number of multiple points as defined by Njåstad, [6].

In Section 1 of this paper we show that every non pseudocompact space has countable multiple point compactifications. It then follows that although the Euclidean n-spaces do not admit countable compactifications (cf. [5]) they do have countable multiple point compactifications. Section 1 will also provide examples of spaces X such that $|\beta X-X|$ is infinite yet X does not possess a countable multiple point compactification. These examples are obtained by using the fact that for any space Y, there exists a space X such that $\beta X-X = Y$ (cf. [2], p. 133).

Recent work (cf. [1], [3], [6], [7], [8]) has shown that many compactifications are of the Wallman-type (henceforth called Wallman compactifications) as defined by Frink [1]. In [6], Njåstad shows that if the set of multiple points in $\hat{X}$ is contained in some subset of $\hat{X}-X$ which is normally and zero-dimensionally embedded in $\hat{X}$, then $\hat{X}$ is a Wallman compactification. It follows that all finite multiple point compactifications are Wallman. The authors, in a previous work [8], have shown that every countable compactification is Wallman. The purpose of Section 2 is to generalize this by showing that all countable multiple point compactifications are Wallman. Since there are examples where the countable set of multiple points is not contained in any subset of $\hat{X}-X$ which is normally and zero-dimensionally embedded in $\hat{X}$, this does not follow from the theorem of Njåstad.

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