An extended arithmetic of ordinal numbers *

by

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Introduction

In this paper we shall define a transfinite sequence of binary operations O_γ on and to ordinals and establish the basic arithmetical properties of these operations. The operations O_γ are indexed by arbitrary ordinals. The first two, O_0 and O_1, are addition and multiplication; O_2 differs but slightly from exponentiation (and actually coincides with the latter except in the cases 1^β and α^0). The sequence of operations O_γ is defined in a uniform way by means of a simple recursive formula; the definition is a natural extension of the well-known definition of multiplication in terms of addition, or exponentiation in terms of multiplication. In most of the arithmetical theorems concerning arbitrary operations O_γ the reader will readily recognize natural generalizations of familiar results from the traditional arithmetic of ordinals referring to the three lowest operations; however, he will also find here some rather interesting exceptions to this rule.

Detailed proofs will not always be given in this paper. Many theorems whose proofs are omitted can be obtained by a straightforward application of the principle of transfinite induction. The material is arranged in such a way that the reader will easily be able to reconstruct less obvious arguments. Sometimes we indicate the principal theorems previously stated from which a given result can be derived. In a few more difficult cases we supply more complete proofs.

We shall use the customary set-theoretical notation. Lower case Greek letters $\alpha, \beta, \ldots$ will represent ordinals; in particular $\varkappa, \lambda, \ldots$ will normally be used to represent finite ordinals. Greek capitals will represent

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arbitrary classes of ordinals. In particular, ω will denote as usual the least infinite ordinal; Ω will be used to denote the class of all ordinals. As is customary in contemporary set theory, we identify an ordinal with the set of all smaller ordinals; in consequence, the relation $<$ among ordinals coincides with the membership relation ϵ and also with the relation of proper inclusion $\subset$. Some other consequences of this identification are: ω is the set of all finite ordinals, i.e., natural numbers; a class Γ of ordinals is a set if and only if there is an ordinal a such that $\Gamma \subseteq a$ or, equivalently, if the union $\bigcup \Gamma$ of Γ is an ordinal; if Γ is a set of ordinals, then its union is what is usually called the least upper bound of Γ; the intersection $\bigcap \Gamma$ of a non-empty class Γ of ordinals is the least ordinal in Γ. Expressions of the form $\bigcup_{\eta<\beta}\tau(\eta)$ and of various related forms will have the usual meaning; e.g.,

$$\bigcup_{\eta<\beta}(a+\eta) = \bigcup \{\delta\colon\ \delta = a+\eta \textit{ for some } \eta<\beta\}.$$

The union $\bigcup a$ of an arbitrary ordinal a is either the ordinal immediately preceding a or the ordinal a itself, depending on whether or not a has an immediate predecessor. Hence the formula $a = \bigcup a$ expresses the fact that a is a limit ordinal; thus, in particular, we regard 0 as a limit ordinal. For more information concerning ordinals the reader may consult Sierpiński [12].

This paper is divided into three sections. In Section 1 we formulate the definition of the operations O_γ and draw from it a number of elementary consequences. One group of these consequences are monotony laws. Another such group are various recursion formulas which in many cases simplify the general recursion schema used in the definition of O_γ and facilitate the application of transfinite induction in the development of the extended arithmetic. In Section 2 we concern ourselves with identities satisfied by the higher operations. Only few such identities have been discovered so far, and their derivations are more intricate than the arguments used in other portions of the paper. Moreover, none of the really interesting identities which are known at present are satisfied unconditionally by all ordinals upon which the operations are performed. This applies in particular to operations O_γ with finite indices $\gamma \geqslant 4$; an interesting open problem is that of the existence of nontrivial and unconditionally satisfied identities involving these operations.

Section 3 contains a detailed discussion of main numbers of operations O_γ, i.e., those ordinals which, when construed as sets of all smaller ordinals, prove to be closed under these operations. In the course of this discussion we come upon many far-reaching analogies and some rather striking differences between the traditional operations of the arithmetic of ordinals and the newly introduced higher operations. In particular, it is known from the traditional arithmetic that the main numbers of O_0 and O_1 respectively coincide with the ordinals of the form $\omega^{1+\eta}$ and ω^{ω^η}, i.e., with $\omega O_2(1+\eta)$ and $\omega O_3(1+\eta)$ in our notation; however, no analogous analytic representation can be found there for main numbers of O_2, i.e., the so-called epsilon numbers. The explanation of this phenomenon appears in Section 3. It turns out that analytic expressions for the main numbers of any operation O_γ are always provided by corresponding higher operations, in fact by $O_{\gamma+2}$ or $O_{\gamma+3}$ in case γ is even, and by $O_{\gamma+1}$ or $O_{\gamma+2}$ in case γ is odd. In particular, to get such expressions for the epsilon numbers we have to use O_4 or O_5; the main numbers of O_2 turn out to coincide with the ordinals of the form $\omega O_5(1+\eta)$.

In the appendix following Section 3 we discuss briefly some metamathematical problems concerning the extended arithmetic of ordinals. A detailed presentation of the results mentioned in this appendix will appear in later publications.

We wish to express our gratefulness to Jean Rubin, who read an earlier draft of this paper and advised us of several defects, which we have corrected. (1)

Section 1. Definition. Monotony laws and recursion formulas

DEFINITION 1. *For every $\gamma \epsilon \Omega$, O_γ is the binary operation on $\Omega \times \Omega$ to Ω determined recursively by the following formulas which are assumed to hold for any $a, \beta \epsilon \Omega$:*

(i) $aO_\gamma\beta = a+\beta$ *in case* $\gamma = 0$;

(ii) $aO_\gamma\beta = \bigcup_{\eta<\beta,\zeta<\gamma}[(aO_\gamma\eta)O_\zeta a]$ *in case* $\gamma \geqslant 1$.

COROLLARY 2. (i) $aO_\gamma 0 = 0O_\gamma a = 0$ *for* $\gamma \geqslant 1$;

(ii) $aO_\gamma 1 = 1O_\gamma a = a$ *for all* $\gamma \geqslant 1$;

(iii) $2O_\gamma 2 = 4$ *for all* γ;

(iv) *if* $a, \beta, \gamma < \omega$, *then* $aO_\gamma\beta < \omega$. (2)

(1) The definition of the operations O_γ actually used in this paper is due to Tarski. Related ideas, which to some extent influenced our discussion, can be found in [8] and [15]. After a preliminary version of the paper received limited distribution (see footnote *), our attention was called to the fact that several years ago a sequence of operations φ_γ closely related to our O_γ was introduced and discussed in [6] and subsequently in [3] and [9]. (The essential relationship between φ_γ and O_γ is expressed by the formulas $\varphi_{\gamma+1}(\beta, a) = aO_\gamma(1+\beta)$ for $a \geqslant \omega$ and $\gamma \geqslant 3$, and $\varphi_\gamma(\beta, a) = \bigcup_{\zeta<\gamma}[aO_\zeta(1+\beta)]$ for $a \geqslant \omega$ and $\gamma = \bigcup\gamma \neq 0$. These formulas were also obtained independently by H. Levitz.) The discussion in these papers differs considerably from ours in goals and general character, and the results overlap in but few places. Thus, some of the monotony laws given here in Section 1, such as Theorem 4 and Corollary 5, can essentially be found in [6]; certain arithmetical lemmas proved in [9], when expressed in terms of the operations O_γ, yield some special cases of the "limit type identity" established in Section 2 as Theorems 27 (i) and 32 (ii).

(2) In formulating definitions and theorems we omit as a rule the initial quantifier expressions "*For all ordinals* $a, \beta, \ldots$".

Corollary 2(iv) can be considerably generalized; in particular, it remains valid if ω is replaced by an arbitrary initial ordinal ω_δ.

THEOREM 3. (i) $\alpha O_1 \beta = \alpha \cdot \beta$;

(ii) $\alpha O_2 \beta = \alpha^\beta$ *for* $\alpha \neq 1$ *and* $\beta \neq 0$;

(iii) $\alpha O_3 (1+\beta) = \alpha^{\alpha^\beta}$ *for* $\alpha \neq 1$ (*and hence* $\alpha O_3 \beta = \alpha^{\alpha^\beta}$ *for* $\alpha \neq 1$ *and* $\beta \geqslant \omega$).

In deriving 3(ii),(iii) from 1 we make use of the known fact that $\alpha+\beta \leqslant \alpha\cdot\beta \leqslant \alpha^\beta$ for $\alpha, \beta \geqslant 2$.

From 1(i) and 3(i),(ii) we see that the first three operations O_γ actually coincide, or almost coincide, with the ordinary arithmetical operations. (However, by 2(i),(ii), $\alpha O_2 \beta$ differs from α^β in case $\beta = 0$ or $\alpha = 1$.) We shall make some comments on 3(iii) in our later discussion.

We wish to give some idea of the values of $\alpha O_\gamma \beta$ for $\gamma \geqslant 4$. These values increase indeed very rapidly. An easy computation gives:

$2 O_4 2 = 4$ $\qquad$ $2 O_5 2 = 4$ $\qquad$ $2 O_6 2 = 4$

$2 O_4 3 = 256$ $\qquad$ $2 O_5 3 = 2^{128}$ $\qquad$ $2^{2^{2^{392}}} < 2 O_6 3 < 2^{2^{2^{393}}}$

$2 O_4 4 = 2^{2048}$ $\qquad$ $2^{2^{2^{134}}} < 2 O_5 4 < 2^{2^{2^{135}}}$

$2 O_4 5 = 2^{2^{2059}}$

$2^{2^{2^{2059}}} < 2 O_4 6 < 2^{2^{2^{2060}}}$

$3 O_4 2 = 3^9$ $\qquad$ $3 O_5 2 = 3^{3^{20}}$

$3 O_4 3 = 3^{3^{20}}$

$3^{3^{3^{20}}} < 3 O_4 4 < 3^{3^{3^{21}}}$

$4 O_4 2 = 4^{64}$ $\qquad$ $4^{4^{4^{195}}} < 4 O_5 2 < 4^{4^{4^{196}}}$

$4 O_4 3 = 4^{4^{195}}$

$4^{4^{4^{195}}} < 4 O_4 4 < 4^{4^{4^{196}}}$

Note that $4 O_4 2 = 2 O_5 3$ and $4 O_4 4 = 4 O_5 2 = 2 O_6 3$.

On the basis of Definition 1, assuming the theory of ordinal addition, we could develop in a uniform way the general theory of the operations O_γ, from which we could derive as particular cases the basic results concerning multiplication and exponentiation. Actually, however, it proves more convenient for our purposes to apply a somewhat different procedure: in developing the extended arithmetic of ordinals we shall freely use, in an explicit or implicit way, various known results from the traditional arithmetic, i.e., the theory of addition, multiplication, and exponentiation.

A number of *monotony laws* hold for the operations O_γ; these are Theorems 4(i), 6, and 8. The corresponding *strict monotony laws* (more limited in scope) are Theorems 4(ii), 11, and 13.

THEOREM 4. (i) *If* $\beta \geqslant \beta'$, *then* $\alpha O_\gamma \beta \geqslant \alpha O_\gamma \beta'$.

(ii) *If* $\alpha \geqslant 1$ *and* $\beta > \beta'$, *then* $\alpha O_\gamma \beta > \alpha O_\gamma \beta'$.

Proof: directly from 1; no use of induction is required.

Obviously equivalent formulations of 4(i),(ii) are, respectively,

(i') *If* $\alpha O_\gamma \beta < \alpha O_\gamma \beta'$, *then* $\beta < \beta'$.

(ii') *If* $\alpha \geqslant 1$ *and* $\alpha O_\gamma \beta \leqslant \alpha O_\gamma \beta'$, *then* $\beta \leqslant \beta'$.

Analogous remarks apply to other monotony laws.

COROLLARY 5. (i) *If* $\beta \geqslant 1$, *then* $\alpha O_\gamma \beta \geqslant \alpha$.

(ii) *If* $\alpha \geqslant 1$ *and* $\beta \geqslant 2$, *then* $\alpha O_\gamma \beta > \alpha$.

THEOREM 6. *If* $\alpha \geqslant \alpha'$, *then* $\alpha O_\gamma \beta \geqslant \alpha' O_\gamma \beta$.

Proof: by 1 and an elementary induction on both β and γ.

COROLLARY 7. *If* $\alpha \geqslant 1$, *then* $\alpha O_\gamma \beta \geqslant \beta$.

THEOREM 8. *If* $\gamma \geqslant \gamma'$ *and either* $\alpha, \beta \geqslant 2$ *or* $\gamma' \geqslant 1$, *then* $\alpha O_\gamma \beta \geqslant \alpha O_{\gamma'} \beta$.

Proof: by applying 1 directly in case $\gamma' \geqslant 1$, and using the formula $\alpha+\beta \leqslant \alpha\cdot\beta$ to extend the result to the case of $\alpha, \beta \geqslant 2$ and $\gamma' = 0$.

The following consequence of 4(i), 6, and 8 is very useful in further deductions:

THEOREM 9. *If* $\alpha \geqslant 2$ *and* $\beta \geqslant 1$, *then* $\alpha O_{\gamma+1} (\beta+1) = (\alpha O_{\gamma+1} \beta) O_\gamma \alpha$.

COROLLARY 10. *If* $\alpha \geqslant 2$, *then* $\alpha O_{\gamma+1} 2 = \alpha O_\gamma \alpha$.

THEOREM 11. *If* $\alpha > \alpha'$, *then* $\alpha O_{\gamma+1} (\beta+1) > \alpha' O_{\gamma+1} (\beta+1)$.

Proof: by 4, 5(i), and 9.

Theorem 11 fails if either $\gamma+1$ or $\beta+1$ is replaced by a limit ordinal; see Theorems 26 and 29 below.

THEOREM 12. *If* $\alpha \geqslant 3$ *and* $\beta \geqslant 2$, *or* $\alpha \geqslant 2$ *and* $\beta \geqslant 4$, *then* $\alpha O_\gamma \beta \geqslant \gamma$.

Proof. With the help of 2(ii) and 4(ii), we show by induction on γ that $3 O_\gamma 2 \geqslant \bigcup_{\zeta<\gamma} (\zeta+1) = \gamma$. Another induction on γ, using 2(iii), 6, and 4(ii), shows that $2 O_\gamma 4 \geqslant \gamma$. The general result is then obtained by means of 4(i) and 6.

Theorem 12 does not hold in case $\alpha = 2$ and $\beta = 3$; e.g., $2 O_{\omega+1} 3 = \omega$.

THEOREM 13. *Assume that either* $\alpha \geqslant 3$ *and* $\beta \geqslant 1$, *or else* $\alpha, \beta \geqslant 2$ *and* $\gamma \neq \bigcup \gamma' + 1$. *Then* $\gamma > \gamma'$ *implies* $\alpha O_\gamma (\beta+1) > \alpha O_{\gamma'} (\beta+1)$.

Proof. If $\gamma' = 0$, the result follows from Theorem 8 and the traditional arithmetic; thus we may assume $\gamma' \geqslant 1$. Except when $\gamma' = \bigcup \gamma'$ it is sufficient, by Theorem 8, to assume $\gamma = \gamma'+1$.

In the case $\alpha \geqslant 3$ and $\beta \geqslant 1$ we have by 9 and 4(ii)

$$\alpha O_{\gamma'+1} (\beta+1) = (\alpha O_{\gamma'+1} \beta) O_{\gamma'} \alpha > (\alpha O_{\gamma'+1} \beta) O_{\gamma'} 2 . \tag{1}$$

But since $\gamma' > 0$, Definition 1 and Theorem 2 imply

$$(\alpha O_{\gamma'+1} \beta) O_{\gamma'} 2 = \bigcup_{\zeta<\gamma'} [(\alpha O_{\gamma'+1} \beta) O_\zeta (\alpha O_{\gamma'+1} \beta)] ,$$

from which, by means of 8, 6, 5(i), and 4(i), we obtain

(2) $$(\alpha O_{\gamma'+1}\beta) O_{\gamma'} 2 \geqslant \bigcup_{\zeta<\gamma'}[(\alpha O_{\gamma'}\beta) O_\zeta \alpha].$$

Now let $\eta \leqslant \beta$. Then, by 4(i) and 6,

$$(\alpha O_{\gamma'+1}\beta) O_\zeta \alpha \geqslant (\alpha O_{\gamma'+1}\eta) O_\zeta \alpha,$$

and hence

(3) $$\bigcup_{\zeta<\gamma'}[(\alpha O_{\gamma'}\beta) O_\zeta \alpha] = \bigcup_{\eta<\beta+1,\zeta<\gamma'}[(\alpha O_{\gamma'}\eta) O_\zeta \alpha].$$

The union on the right-hand side of (3) is just $\alpha O_{\gamma'}(\beta+1)$, so from (1), (2), and (3) we obtain

$$\alpha O_{\gamma'+1}(\beta+1) > \alpha O_{\gamma'}(\beta+1).$$

Next, assume $\gamma' \neq \bigcup \gamma'$ and $\alpha, \beta \geqslant 2$. Letting $\bigcup \gamma' = \zeta$, we get $\gamma' = \zeta+1$ and $\gamma = \zeta+2$. By 9, 10, and the monotony laws we readily obtain

$$\alpha O_{\zeta+2}(\beta+1) \geqslant (\alpha O_{\zeta+2}\beta) O_\zeta (\alpha O_{\zeta+2}\beta).$$

Now $\alpha O_{\zeta+2}\beta \geqslant \alpha O_{\zeta+1}\beta$ and $\alpha O_{\zeta+2}\beta > \alpha$ by 8 and 5(ii), respectively. Hence, by 6 and 4(ii),

$$\alpha O_{\zeta+2}(\beta+1) > (\alpha O_{\zeta+1}\beta) O_\zeta \alpha.$$

The right-hand side of this inequality is $\alpha O_{\zeta+1}(\beta+1)$; see Theorem 9.

Finally, consider the case $\gamma' = \bigcup \gamma'$ and $\alpha, \beta \geqslant 2$. Here, because $\gamma \neq \bigcup \gamma' + 1$, we cannot have $\gamma = \gamma'+1$ and we must assume $\gamma > \gamma'+1$. However, we can now replace γ' by $\gamma'+1$ in the argument for the case $\gamma' \neq \bigcup \gamma'$, obtaining $\alpha O_\gamma(\beta+1) > \alpha O_{\gamma'+1}(\beta+1)$, and then we need merely apply Theorem 8 to complete the proof.

The conclusion of Theorem 13 may fail in case $\alpha, \beta \geqslant 2$ and both $\gamma = \bigcup \gamma' + 1$, $\gamma' = \bigcup \gamma'$. In fact, it is easily seen that, e.g., $2 O_{\omega+1} 3 = 2 O_\omega 3 = \omega$.

By combining two or more monotony laws we obtain results of related character but more complicated structure, e.g.:

If $\alpha \geqslant 1$, $\alpha \geqslant \alpha'$, $\beta > \beta'$, *and* $\gamma \geqslant \gamma' \geqslant 1$, *then* $\alpha O_\gamma \beta > \alpha' O_{\gamma'} \beta'$.

The result just stated can be used, for instance, to simplify the proof of the following

THEOREM 14. *If* $\delta \neq 0$, *then the set* $\{\beta\colon \alpha O_\gamma \beta = \delta$ *for some* α *and* $\gamma\}$ *is finite.*

The proof of a much weaker result by which the set $\{\beta\colon \alpha+\beta = \delta$ *for some* $\alpha\}$ is finite can be found in the literature; see [12], page 277. Essentially the same argument can be applied to establish 14.

In opposition to Theorem 14, the set $\{\alpha\colon \alpha O_\gamma \beta = \delta$ *for some* β *and* $\gamma\}$, with a fixed δ, is in general infinite, and the same applies to the set

$\{\alpha\colon \alpha O_\gamma \beta = \delta$ *for some* $\beta\}$, with fixed γ and δ. It is known from the traditional arithmetic that, for any given $\delta > 0$ and for $\gamma = 0, 1, 2$, the set $\Gamma = \{\alpha\colon \alpha O_\gamma \beta = \delta$ *for some* $\beta\}$ is *closed* in the sense that $\bigcup \Delta \in \Gamma$ whenever Δ is a non-empty subset of Γ; for $\gamma = 1, 2$ see [4]. It turns out that this result extends to arbitrary γ; in a somewhat different form, the general result has been announced in [11].

Using the monotony laws we can simplify in various particular cases the recursive part of our definition of the operations O_γ. In other words, in addition to the formula 1(ii), we can establish several other *recursion formulas* of related but simpler structure and more restricted in scope. One such formula was given in Theorem 9. In the next theorem three other formulas of this kind will be established.

THEOREM 15. (i) *If* $\gamma \geqslant 1$, *then* $\alpha O_\gamma(\beta+1) = \bigcup_{\zeta<\gamma}[(\alpha O_\gamma \beta) O_\zeta \alpha]$.

(ii) *If* $\alpha, \beta \geqslant 2$, *then* $\alpha O_{\gamma+1}\beta = \bigcup_{\eta<\beta}[(\alpha O_{\gamma+1}\eta) O_\gamma \alpha]$.

(iii) *If* $\beta = \bigcup \beta$ *and either* $\beta \neq 0$ *or* $\gamma \neq 0$, *then* $\alpha O_\gamma \beta = \bigcup_{\eta<\beta}(\alpha O_\gamma \eta)$.

THEOREM 16. *If* $\gamma \geqslant 1$ *and either* $\alpha = \bigcup \alpha$ *or* $\beta = \bigcup \beta$, *then* $\alpha O_\gamma \beta = \bigcup (\alpha O_\gamma \beta)$, *i.e.*, $\alpha O_\gamma \beta$ *is a limit ordinal.*

Proof. In view of 2(i), we may assume $\alpha, \beta \neq 0$. If $\beta = \bigcup \beta$, the conclusion follows easily from 15(iii) and the strict monotony law 4(ii). If $\alpha = \bigcup \alpha \neq 0$ but $\beta \neq \bigcup \beta$, say $\beta = \eta+1$, then, by 15(i),

(1) $$\alpha O_\gamma \beta = \bigcup_{\zeta<\gamma}[(\alpha O_\gamma \eta) O_\zeta \alpha].$$

Now $\alpha = \bigcup \alpha \neq 0$, so by the preceding remarks concerning the case $\beta = \bigcup \beta$ (or by a property of addition in case $\zeta = 0$) we see that each term in the union on the right-hand side of (1) is a limit ordinal. Hence so is $\alpha O_\gamma \beta$.

Theorems 9 and 15 suggest various equivalent transformations of our basic definition 1. For instance, a commonly used definition of multiplication in terms of addition is:

(i) $\alpha \cdot 0 = 0$;

(ii) $\alpha \cdot (\beta+1) = \alpha \cdot \beta + \alpha$;

(iii) *if* $\beta = \bigcup \beta \neq 0$, *then* $\alpha \cdot \beta = \bigcup_{\eta<\beta}(\alpha \cdot \eta)$.

In an entirely analogous way exponentiation is defined in terms of multiplication. It is natural to attempt to apply the same schema in the general definition of O_γ; indeed, this was the present authors' original approach. Theorems 9 and 15 suggest (correctly) that the resulting sequence of operations will not differ essentially from our sequence $\langle O_\gamma \rangle_{\gamma \in \Omega}$. Specifically, the following theorem can be easily established:

THEOREM 17. *Consider the sequence of operations* $\langle O'_\gamma \rangle_{\gamma \in \Omega}$ *determined recursively by the following six conditions:*

(i) $\alpha O'_0 \beta = \alpha + \beta$,

(ii) $\alpha O'_\gamma 0 = 0 O'_\gamma \alpha = 0$ *for* $\gamma \geqslant 1$,

(iii) $\alpha O'_\gamma 1 = 1 O'_\gamma \alpha = \alpha$ *for* $\gamma \geqslant 1$,

(iv) $\alpha O'_{\gamma+1}(\beta+1) = (\alpha O'_{\gamma+1}\beta) O'_\gamma \alpha$ *for* $\alpha \geqslant 2,\ \beta \geqslant 1$,

(v) $\alpha O'_\gamma(\beta+1) = \bigcup_{\zeta<\gamma}[(\alpha O'_\gamma \beta) O'_\zeta \alpha]$ *for* $\beta \geqslant 1$ *and* $\gamma = \bigcup \gamma \neq 0$,

(vi) $\alpha O'_\gamma \beta = \bigcup_{\eta<\beta}(\alpha O'_\gamma \eta)$ *for* $\beta = \bigcup \beta \neq 0$ *and* $\gamma \geqslant 1$.

The sequence thus determined coincides with $\langle O_\gamma \rangle_{\gamma \in \Omega}$; *i.e., for all* $\alpha, \beta, \gamma \in \Omega$ *we have*

$$\alpha O_\gamma \beta = \alpha O'_\gamma \beta .$$

We see, therefore, that we could replace the original definition of O_γ by an equivalent one based upon Theorem 17. The new definition would be much more complicated, but would also be more closely related to the usual recursive definition of multiplication and exponentiation.

Notice that a recursion on β based upon the schema embodied in conditions (i)-(vi) of 17 begins with $\beta = 1$. Hence 17 (ii) plays an insignificant role in the whole development and can be modified almost at will. Essentially the same remarks apply to 17 (iii).

In general, the theory of the operations O_γ depends very little on the way in which the values of these operations have been fixed for the lowest values of the arguments.

We may mention here the possibility of taking for the initial term of the sequence of operations O'_γ, not addition, but the practically trivial successor operation O'_0: $\alpha O'_0 \beta$ does not depend on β and is simply the successor of α, in set-theoretical notation $\alpha \cup \{\alpha\}$. O'_1 will then be addition, and, in general, $O'_{1+\gamma}$ will coincide with the old O_γ. The definition of the sequence $\langle O'_\gamma \rangle_{\gamma \in \Omega}$ can be obtained from the one implicitly given in Theorem 17 in the following way: 17 (i) assumes the form $\alpha O'_0 \beta = \alpha \cup \{\alpha\}$, in 17 (ii) and 17 (iii) the formula $\gamma \geqslant 1$ is replaced by $\gamma \geqslant 2$, the phrases "$\alpha\ O'_\gamma 0 = 0\ O'_\gamma \alpha = \alpha$ *for* $\gamma = 1$" and "*when* $\gamma \geqslant 2$, *and for* $\alpha, \beta \geqslant 1$ *when* $\gamma = 1$", are added to 17 (ii) and 17 (iv), respectively, and in 17 (v) the subscript $\zeta < \gamma$ is replaced by $0 < \zeta < \gamma$. Using this definition of $\langle O'_\gamma \rangle_{\gamma \in \Omega}$ we can develop the whole ordinal arithmetic "from scratch", obtaining the theories of addition, multiplication, and exponentiation as particular cases. In the context of the present paper we see little to be gained by this procedure.

Another possible modification of the definition of the operations O'_γ should be considered at this point. We may replace parts (v) and (vi) of 17 by the following:

(v) $\alpha O'_{\gamma+1} \beta = \bigcup_{\eta<\lambda}(\alpha O'_{\gamma+1} \eta)$ *for* $\beta = \bigcup \beta \neq 0$,

(vi) $\alpha O'_\gamma \beta = \bigcup_{0<\zeta<\gamma}(\alpha O'_\zeta \beta)$ *for* $\gamma = \bigcup \gamma \neq 0$.

The relationship of the operations O'_γ thus defined to the original O_γ is simply expressed: for $\gamma < \omega$, O'_γ is the same as O_γ, and for $\gamma \geqslant \omega$, $O'_{\gamma+1}$ coincides with O_γ, but the operations O'_γ with $\gamma = \bigcup \gamma \neq 0$ are distinct from all the O_γ. The development of the theory of the new operations O'_γ presents some advantages and disadvantages as compared with that of the operations O_γ. Thus, the important monotony law 4 (ii) fails for the operations O_γ with $\gamma = \bigcup \gamma \neq 0$. In fact, we have, e.g., $\alpha O'_\omega \beta = \omega$ for any α, β such that $2 \leqslant \alpha, \beta < \omega$ and either $\alpha \neq 2$ or $\beta \neq 2$. As a consequence, $\alpha O'_\gamma \eta$ treated as a function of η (with α and γ fixed) is not normal in the sense of Definition 18 below, provided $\alpha > 1$ and $\gamma = \bigcup \gamma \neq 0$. On the other hand, $\alpha O'_\zeta \beta$ treated as a function in ζ is normal, assuming $\alpha \geqslant 3$ and $\beta = \beta' + 2$ for some $\beta' \geqslant 1$; this fact has some interesting consequences which will not be discussed here.

Theorems 4 (ii) and 15 (iii) show that each of the operations O_γ treated as a function of its second argument (with the first argument assumed to be fixed but different from 0) is a normal function in the following sense:

Definition 18. *A function* φ *on* Ω *to* Ω *is called* ***normal*** *if it satisfies the following two conditions*:

(i) φ *is strictly increasing, i.e., for any* $\eta, \eta' \in \Omega$, $\eta < \eta'$ *implies* $\varphi(\eta) < \varphi(\eta')$;

(ii) φ *is continuous, i.e., for any* $\eta \in \Omega$, $\eta = \bigcup \eta \neq 0$ *implies* $\varphi(\eta) = \bigcup_{\theta<\eta} \varphi(\theta)$.

Corollary 19. *For any given* $\alpha \geqslant 1$ *and* γ, *the function* $\varphi_{\alpha,\gamma}$ *determined by the formula* $\varphi_{\alpha,\gamma}(\eta) = \alpha O_\gamma \eta$ *for every* $\eta \in \Omega$ *is normal.*

Proof: by 4 (ii) and 15 (iii).

Several general laws concerning normal functions are commonly known and some of them can even be found in the literature; see in particular [2], pages 25 and 32 ff. For the convenience of the reader we shall state them explicitly in the next few theorems, and then apply them to the operations O_γ by means of Corollary 19.

Theorem 20. *Let* φ *be any normal function.*

(i) *If* $\Gamma \subset \Omega$ *is a nonempty set, then* $\varphi(\bigcup \Gamma) = \bigcup_{\eta \in \Gamma} \varphi(\eta)$.

(ii) $\varphi(\beta) \geqslant \beta$ *for every* β.

(iii) *If* $\delta \geqslant \varphi(0)$, *then there is exactly one* β *such that*

$$\varphi(\beta) \leqslant \delta < \varphi(\beta+1);$$

in fact, $\varphi(\beta)$ *is the largest ordinal* $\xi \leqslant \delta$ *and* $\varphi(\beta+1)$ *the least ordinal* $\xi > \delta$ *which belong to the range of* φ.

Proof. Let Γ be a nonempty subset of Ω, and let $\beta = \bigcup \Gamma$. If $\beta \in \Gamma$, then $\varphi(\beta) = \bigcup_{\eta \in \Gamma} \varphi(\eta)$ follows from 18 (i). If $\beta \notin \Gamma$, then $\beta = \bigcup \beta \neq 0$, and we have $\varphi(\beta) = \bigcup_{\eta<\beta}\varphi(\eta)$ by 18(ii). Now for each $\eta < \beta$ there exists an $\eta' \in \Gamma$ such that $\eta < \eta'$ and $\varphi(\eta) < \varphi(\eta')$. Hence,

$$\bigcup_{\eta<\beta}\varphi(\eta) \leqslant \bigcup_{\eta\in\Gamma}\varphi(\eta)\,,$$

i.e., $\varphi(\beta) \leqslant \bigcup_{\eta\in\Gamma}\varphi(\eta)$. On the other hand, $\beta \geqslant \eta$ for each $\eta \in \Gamma$, so that $\varphi(\beta) \geqslant \bigcup_{\eta\in\Gamma}\varphi(\eta)$. Thus, (i) is proved.

Our proof of (ii) is by induction on β. That $\varphi(0) \geqslant 0$ requires no proof; suppose $\beta > 0$, and $\varphi(\eta) \geqslant \eta$ for every $\eta < \beta$. Now, for any $\eta < \beta$, $\varphi(\eta+1) > \varphi(\eta) \geqslant \eta$, and hence

(1) $$\varphi(\eta+1) \geqslant \eta+1 \quad \text{for every } \eta < \beta\,.$$

It is well known that $\beta = \bigcup_{\eta<\beta}(\eta+1)$ for any ordinal β. Thus, by (i) and (1),

$$\varphi(\beta) = \bigcup_{\eta<\beta}\varphi(\eta+1) \geqslant \bigcup_{\eta<\beta}(\eta+1) = \beta\,.$$

Finally, we consider (iii). By (ii), $\delta < \delta+1 \leqslant \varphi(\delta+1)$, from which we conclude that the set $\{\eta\colon \delta < \varphi(\eta)\}$ is not empty. Let σ be the least element of this set. If we had $\sigma = \bigcup\sigma$, then (i) would imply

$$\delta < \varphi(\sigma) = \varphi(\bigcup\sigma) = \bigcup_{\eta<\sigma}\varphi(\eta)\,,$$

whence $\delta < \varphi(\eta)$ for some $\eta < \sigma$, thus contradicting the minimality of σ. Therefore, σ is not a limit number; $\sigma = \beta+1$ for some β, and, since $\beta < \sigma$, the definition of σ leads directly to

$$\varphi(\beta) \leqslant \delta < \varphi(\beta+1)\,.$$

Assume now that some other ordinal β' satisfies the same formula, i.e.,

$$\varphi(\beta') \leqslant \delta < \varphi(\beta'+1)\,.$$

We then have both $\varphi(\beta) < \varphi(\beta'+1)$ and $\varphi(\beta') < \varphi(\beta+1)$. It follows that $\beta < \beta'+1$ and $\beta' < \beta+1$, so that finally $\beta = \beta'$. $\varphi(\beta)$ is the largest ordinal $\xi \leqslant \delta$ in the range of φ, for if this were otherwise, say $\varphi(\beta) < \varphi(\eta) \leqslant \delta$, then φ would not be strictly increasing; i.e., we would have either $\eta < \beta$ and $\varphi(\beta) < \varphi(\eta)$, or $\beta+1 \leqslant \eta$ and $\varphi(\eta) < \varphi(\beta+1)$. Similarly, $\varphi(\beta+1)$ is the least ordinal $\xi > \delta$ in the range of φ.

From each of the three parts of Theorem 20 we can obtain as a particular case a corollary concerning the operations O_γ. The conclusion which can be derived this way from 20(ii) has already been stated as Corollary 7 and proved by a direct method. The corresponding conclusions from 20(i), (iii) will now be formulated explicitly:

Corollary 21. (i) *$\alpha O_\gamma(\bigcup\Gamma) = \bigcup_{\eta\in\Gamma}(\alpha O_\gamma\eta)$ for all α, γ and any nonempty set $\Gamma \subset \Omega$.*

(ii) *If $\alpha \geqslant 1$ and $\gamma \geqslant 1$, then for every δ there is exactly one β such that $\alpha O_\gamma\beta \leqslant \delta < \alpha O_\gamma(\beta+1)$.*

Proof: by 4(i), 19, and 20(iii).

Corollary 21(i) can be referred to as the *general continuity law.*

Notice that Corollary 21(ii) still holds for $\gamma = 0$, provided $\delta \geqslant \alpha$. Moreover, in this case the conclusion simplifies: it turns out that there is exactly one β such that $\alpha O_0\beta = \delta$, a well-known fact from the theory of addition.

Theorem 22. *Let $\Gamma \subseteq \Omega$. In order that there exist a normal function φ whose range is Γ it is necessary and sufficient that Γ satisfy the following two conditions:*

(i) *Γ is not a set (i.e., there is no $\xi \in \Omega$ for which $\Gamma \subseteq \xi$);*

(ii) *Γ is closed.*

Moreover, for every such class Γ there is just one normal function φ with range Γ.

Proof. The necessity of the two conditions is immediate from 20(ii) and 20(i). To establish sufficiency, we assume that Γ is a class satisfying (i) and (ii), and define a function φ on Ω by recursion:

$$\varphi(0) = \bigcap\Gamma\,,$$

$$\varphi(\beta+1) = \bigcap\{\xi\colon \xi\in\Gamma \text{ and } \xi > \varphi(\beta)\}\,,$$

$$\varphi(\beta) = \bigcup_{\eta<\beta}\varphi(\eta) \quad \text{in case} \quad \beta = \bigcup\beta \neq 0\,.$$

Now we prove simultaneously by induction on β:

(1) $$\text{if } \eta < \beta\,, \quad \text{then} \quad \varphi(\eta) < \varphi(\beta);$$

(2) $$\varphi(\beta) \in \Gamma\,.$$

In fact, if $\beta = 0$, then (1) is vacuously satisfied, while (2) follows immediately from the fact that Γ satisfies (i), and hence is not empty. Now assume that $\beta > 0$ and that (1) and (2) hold for every $\beta' < \beta$. In case $\beta = \beta'+1$ for some β', we have $\varphi(\beta') < \varphi(\beta)$ by the definition of φ, and $\varphi(\eta) \leqslant \varphi(\beta')$ for every $\eta < \beta$ by the inductive hypothesis. Hence, (1) holds for β, and (2) follows at once from the definition of φ. If, finally, $\beta = \bigcup\beta$, then for every $\eta < \beta$ there exists an η' such that $\eta < \eta' < \beta$, and by the inductive hypothesis we obtain at once

$$\varphi(\eta) < \varphi(\eta') \leqslant \varphi(\beta)\,.$$

Thus, (1) again holds for β, and (2) is an immediate consequence of the inductive hypothesis, the definition of φ, and the property (ii) of Γ.

That φ is a normal function follows from its definition and (1). To complete the proof we must show that Γ is included in the range of φ. Suppose that this is not the case; let δ be the least member of Γ not in the range of φ. By 20(iii), there exists exactly one β such that

$$\varphi(\beta) \leqslant \delta < \varphi(\beta+1)\,.$$

From (2) and our assumption concerning δ we obtain $\varphi(\beta) < \delta$. If there were any δ' such that $\varphi(\beta) < \delta' < \delta$ and $\delta' \in \Gamma$, then, by the minimality of δ, we could conclude that δ' is in the range of φ. But this is precluded by the definition of β and 20(iii). Hence, δ is the least ordinal of $\{\xi\colon\ \xi \in \Gamma$ and $\xi > \varphi(\beta)\}$. Then, by the definition of φ, $\delta = \varphi(\beta+1)$, which is a contradiction. Thus Γ is included in, and therefore equal to, the range of φ. The proof of 22 is completed by an easy induction, showing that, if φ' is any other normal function with range Γ, then $\varphi'(\beta) = \varphi(\beta)$ for every β.

It may be noticed that Theorem 22 continues to hold if we replace "*normal*" by "*strictly increasing*" and omit condition (ii).

If φ is the function correlated with the class Γ by Theorem 22, then for any ordinal η we refer to $\varphi(\eta)$ as the *η-th successive element of Γ* (in the natural order); we also say that the function φ *enumerates* the class Γ.

THEOREM 23. *Let φ be a normal function and let Γ be the class of all fixed points of φ, i.e., $\Gamma = \{\xi\colon\ \varphi(\xi) = \xi\}$. We then have:*

(i) *Γ satisfies the conditions* (i), (ii) *of Theorem* 22;

(ii) *there is just one normal function ψ whose range is Γ.*

Proof. Since (ii) is an immediate consequence of (i) and Theorem 22, we need only prove (i). Let ξ be an arbitrary ordinal; we seek a fixed point η of φ such that $\xi < \eta$. By 20(iii), there exists a β such that $\xi < \varphi(\beta)$. Let η be the least upper bound of the sequence $\beta, \varphi(\beta), \varphi\big(\varphi(\beta)\big), \ldots$; more precisely, let Δ be the least class containing β and such that $\varphi(\alpha) \in \Delta$ whenever $\alpha \in \Delta$, and then let $\eta = \bigcup \Delta$. (³) It is easy to show that $\varphi(\eta) = \eta$. Thus, Γ satisfies condition (i) of 22.

Now let Θ be any non-empty subset of Γ, and let $\eta = \bigcup \Theta$. By 20(i),

$$\varphi(\eta) = \textstyle\bigcup_{\xi\in\Theta}\varphi(\xi)\ .$$

Since each element of Θ is a fixed point of φ, we have

$$\varphi(\eta) = \textstyle\bigcup_{\xi\in\Theta}\xi = \bigcup \Theta = \eta\ ,$$

i.e., $\eta \in \Gamma$. Thus, Γ satisfies condition (ii) of 22 as well.

The function ψ correlated with a given function φ by Theorem 23 is sometimes called the *first derived function* of φ and denoted by φ'. For a discussion of the transfinite sequence of functions derived in this way from a normal function φ see [15].

COROLLARY 24. *For any $\alpha \geqslant 1$, β, and γ there is an η such that $\eta \geqslant \beta$ and $\alpha O_\gamma \eta = \eta$.*

In Section 3 we shall calculate, using the operations $O_{\gamma+1}$ and $O_{\gamma+2}$, the least ordinal η which is a solution of the equation $\alpha O_\gamma \eta = \eta$, i.e., the least fixed point of the function $\varphi_{\alpha,\gamma}$ defined by the formula $\varphi_{\alpha,\gamma}(\eta) = \alpha O_\gamma \eta$.

(³) The construction used here is the same as that used in [8] and [15].

We shall also give an explicit formula for the function that enumerates all fixed points of $\varphi_{\alpha,\gamma}$, i.e., for the first derived function $\varphi'_{\alpha,\gamma}$. Compare the remarks following Theorem 48.

Section 2. Identities

The discussion in this section leads to some of the most interesting, useful, and probably unexpected results of the extended ordinal arithmetic. We shall establish certain arithmetical equations involving the operations O_γ which prove to be satisfied by arbitrary ordinals provided that either the indices or some of the arguments of the operations involved are assumed to be limit ordinals. These facts will be stated in Theorems 27 and 32. The proof of these results is based upon several lemmas and theorems of lesser interest. We begin with a result closely related to Theorem 15(i); under more restrictive premises, it provides a formula that considerably generalizes the conclusion of the latter.

LEMMA 25. *If $\alpha \geqslant 2$, $\beta \geqslant 1$, and $\gamma = \bigcup \gamma \neq 0$, then*

$$\alpha O_\gamma(\beta+1) = \textstyle\bigcup_{\zeta<\gamma}[(\alpha O_\gamma \beta) O_\zeta \alpha']$$

for any ordinal α' such that $2 \leqslant \alpha' \leqslant \alpha O_\gamma \beta$.

Proof: by a straightforward application of the monotony laws, Corollary 10, and Theorem 15(i).

THEOREM 26. *If $\alpha \geqslant 3$, $\beta \geqslant 2$, and $\gamma = \bigcup \gamma \neq 0$, then*

$$(\alpha+1) O_\gamma \beta = \alpha O_\gamma \beta\ .$$

Proof. We first take up the case $\beta = 2$. By 15(i) and 2 we have

$$\alpha O_\gamma 2 = \textstyle\bigcup_{\zeta<\gamma}(\alpha O_\zeta \alpha)\ ,$$

while 25 gives

$$(\alpha+1) O_\gamma 2 = \textstyle\bigcup_{\zeta<\gamma}[(\alpha+1) O_\zeta \alpha]\ .$$

The equality of the right-hand sides in these two formulas follows from the inequalities $\alpha O_{\zeta+1} \alpha \geqslant \alpha O_{\zeta+1} 3 = (\alpha O_{\zeta+1} 2) O_\zeta \alpha$ and $\alpha O_{\zeta+1} 2 \geqslant \alpha+2 > \alpha+1$.

Having proved the theorem for $\beta = 2$, we proceed by induction on β without meeting any difficulty.

THEOREM 27. *If $\beta \geqslant 1$ and $\gamma = \bigcup \gamma \neq 0$, then*

(i) $\alpha O_\gamma(\beta+\beta') = (\alpha O_\gamma \beta) O_\gamma (1+\beta')$ *for* $\alpha \geqslant 2$,

whence

(i′) $\alpha O_\gamma(\beta+\beta') = (\alpha O_\gamma \beta) O_\gamma \beta'$ *for* $\alpha \geqslant 2$ *and* $\beta' \geqslant \omega$;

(ii) $(1+\alpha) O_{\gamma+1}(\beta+\beta') = [(1+\alpha) O_{\gamma+1} \beta] O_\gamma (1+\alpha\cdot\beta')$ *for* $\alpha \geqslant 1$,

whence

(ii′) $\alpha O_{\gamma+1}(\beta+\beta') = (\alpha O_{\gamma+1} \beta) O_\gamma (\alpha\cdot\beta')$ *for* $\alpha \geqslant \omega$ *and* $\beta' \geqslant 1$.

Proof. (i) is obtained by induction on β'. Using 17, (ii) can be derived from (i) by another induction on β'.

Note that conclusions (i′) and (ii′) of 27 also hold when $\gamma = 0$.

We proceed to establish results that are analogous to Theorems 26 and 27, but in which the operation O_γ is almost arbitrary while some of the arguments are supposed to be limit ordinals. The proof of these results is more difficult.

LEMMA 28. *For $\alpha \geqslant 2$ and any β we have*:

(i) $(\alpha+1)O_\gamma\beta \leqslant \alpha O_\gamma(1+2\cdot\beta)$;

(ii) *there exists $\varkappa < \omega$ such that* $(\alpha+1)O_\gamma\beta \leqslant \alpha O_\gamma(\beta+\varkappa)$.

Proof. For $\gamma = 0, 1$, (i) is a result from the traditional arithmetic of ordinals. For $\gamma \geqslant 2$ it is proved by induction on β. If $\beta = 0$, (i) follows from Corollary 2. Assume that (i) is true for every $\beta < \beta'$ and that $\beta' \geqslant 1$. If $\beta' = \bigcup \beta'$, (i) follows from the inductive hypothesis and Corollary 21. If $\beta' = \beta+1$ for some β, we distinguish two cases according to whether or not γ is a limit ordinal. First, suppose $\gamma = \gamma'+1$, $\gamma' \geqslant 1$. Then

$$(\alpha+1)O_\gamma(\beta+1) \leqslant [\alpha O_\gamma(1+2\cdot\beta)]O_{\gamma'}(\alpha+1)$$

by Theorem 9 and the inductive hypothesis. By Theorem 15(i),

$$(\alpha+1)O_\gamma(\beta+1) \leqslant \bigcup_{\zeta<\gamma'}\{[(\alpha O_\gamma(1+2\cdot\beta))O_{\gamma'}\alpha]O_\zeta[\alpha O_\gamma(1+2\cdot\beta)]\}\,.$$

Then, applying the inductive hypothesis, monotony laws, and 10, we have

$$(\alpha+1)O_\gamma(\beta+1) \leqslant \bigcup_{\zeta<\gamma'}[[\alpha O_\gamma(1+2\cdot\beta+1)]O_{\zeta+1}2]\,.$$

However, $\alpha \geqslant 2$, and hence

$$(\alpha+1)O_\gamma(\beta+1) \leqslant \bigcup_{\zeta<\gamma}[[\alpha O_\gamma(1+2\cdot\beta+1)]O_\zeta\alpha]\,.$$

The union on the right-hand side of the latter formula is just $\alpha O_\gamma(1+2\cdot(\beta+1))$. Now consider the case $\gamma = \bigcup\gamma \neq 0$. If $\beta' = 1$, (i) follows from 5(ii). If $\beta' \geqslant 2$, then, by Lemma 25,

$$(\alpha+1)O_\gamma(\beta+1) = \bigcup_{\zeta<\gamma}[[(\alpha+1)O_\gamma\beta]O_\zeta\alpha]\,;$$

an application of the inductive hypothesis and monotony laws yields (i).

(ii) follows immediately from (i), since, for any ordinal β, $1+2\cdot\beta$ is of the form $\beta+\varkappa$ for some $\varkappa<\omega$.

As an immediate consequence of Lemma 28 we obtain

THEOREM 29. *If $\alpha \geqslant 2$, $\beta = \bigcup\beta$, and $\gamma \geqslant 1$, then* $(\alpha+1)O_\gamma\beta = \alpha O_\gamma\beta$.

LEMMA 30. *Assume $\alpha \geqslant 2$, $\beta \geqslant 1$, and $\gamma \geqslant 2$. Then, for every β'*,

(i) $\alpha O_{2\cdot\gamma}(\beta+\beta') \leqslant (\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}(1+\beta')$,

(ii) $(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+\alpha\cdot\beta') \leqslant (1+\alpha)O_{2\cdot\gamma+1}(\beta+\beta')$.

Proof. (i) will be proved by induction on β', and then will be used to establish (ii), also by induction on β'.

For $\beta' = 0$ both (i) and (ii) are immediate consequences of Corollary 2. Assume that $\beta' > 0$, and that (i) with β' replaced by η holds for every $\eta < \beta'$. Now, by 4(i) and 6,

$$\alpha O_{2\cdot\gamma}(\beta+\beta') = \bigcup_{\eta<\beta+\beta',\zeta<2\cdot\gamma}[(\alpha O_{2\cdot\gamma}\eta)O_\zeta\alpha] = \bigcup_{\eta<\beta',\zeta<2\cdot\gamma}[[\alpha O_{2\cdot\gamma}(\beta+\eta)]O_\zeta\alpha]\,,$$

and hence

$$\alpha O_{2\cdot\gamma}(\beta+\beta') \leqslant \bigcup_{\eta<\beta',\zeta<2\cdot\gamma}[[(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}(1+\eta)]O_\zeta\alpha]$$

by the inductive hypothesis. Noting that $\alpha \leqslant \alpha O_{2\cdot\gamma}\beta$, we obtain in turn

$$\alpha O_{2\cdot\gamma}(\beta+\beta') \leqslant \bigcup_{\eta<\beta',\zeta<2\cdot\gamma}[[(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}(1+\eta)]O_\zeta(\alpha O_{2\cdot\gamma}\beta)]\,.$$

The right-hand side is just $(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}(1+\beta')$, independently of whether or not $\beta' = \bigcup\beta'$. Thus we have shown that (i) holds for every β'.

Now assume that $\beta' > 0$ and that (ii) with β' replaced by η holds for every $\eta < \beta'$. We distinguish two cases according to whether β' is a limit ordinal or not. If $\beta' = \bigcup\beta' \neq 0$, then (ii) is readily proved by an argument based on the continuity law, Corollary 21. Suppose $\beta' = \eta+1$; the inductive hypothesis states that (i), (ii) hold with β' replaced by η. Thus

$$\begin{aligned}(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+\alpha\cdot(\eta+1)) &= (\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+\alpha\cdot\eta+\alpha)\\ &\leqslant [(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+\alpha\cdot\eta)]O_{2\cdot\gamma}(1+\alpha)\\ &\leqslant [(1+\alpha)O_{2\cdot\gamma+1}(\beta+\eta)]O_{2\cdot\gamma}(1+\alpha)\,,\end{aligned}$$

and the last expression is equal to $(1+\alpha)O_{2\cdot\gamma+1}(\beta+\beta')$.

LEMMA 31. *If $\gamma \geqslant 2$, then for every $\alpha \geqslant 2$, $\beta \geqslant 1$, and β' we have*

(i) $(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}\beta' \leqslant \alpha O_{2\cdot\gamma}(\beta+3\cdot\beta')$,

(ii) $\alpha O_{2\cdot\gamma+1}(\beta+\beta') \leqslant (\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\beta')$.

Proof. First we shall show that, for each γ, (ii) is actually a consequence of (i). This is done by induction on β'. Suppose that (i) holds for some $\gamma \geqslant 2$, $\alpha \geqslant 2$, and $\beta \geqslant 1$. If $\beta' = 0$, then (ii) follows immediately from Corollary 2. Assume that $\beta' > 0$ and that

$$(1)\qquad \alpha O_{2\cdot\gamma+1}(\beta+\eta) \leqslant (\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\eta)$$

for every $\eta < \beta'$. From (1) and the monotony laws we get

$$\begin{aligned}\alpha O_{2\cdot\gamma+1}(\beta+\beta') &= \bigcup_{\eta<\beta'}[[\alpha O_{2\cdot\gamma+1}(\beta+\eta)]O_{2\cdot\gamma}\alpha]\\ &\leqslant \bigcup_{\eta<\beta'}[[(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\eta)]O_{2\cdot\gamma}\alpha]\,.\end{aligned}$$

Now we apply (i) obtaining

$$(2)\qquad \begin{aligned}\alpha O_{2\cdot\gamma+1}(\beta+\beta') &\leqslant \bigcup_{\eta<\beta'}[(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\eta+3\cdot\alpha)]\\ &\leqslant \bigcup_{\eta<\beta'}[(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}[1+3\cdot\alpha\cdot(\eta+1)]]\,.\end{aligned}$$

The last union in (2) is just $(\alpha O_{2\cdot\gamma+1}\beta)O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\beta')$.

The proof of (i) is carried through by an induction of a more complicated nature: an induction on γ in which each of the cases $\gamma = 2$ and $\gamma \neq \bigcup\gamma \geqslant 2$ is handled by an "inner induction" on β'. We begin with the case $\gamma = 2$. If $\beta' = 0$, (i) becomes trivial. Assume that $\beta' > 0$ and take as the inductive hypothesis

(3) $\quad (\alpha O_4 \beta) O_4 \eta \leqslant \alpha O_4 (\beta + 3\cdot\eta) \quad$ for every $\alpha \geqslant 2$, $\beta \geqslant 1$, and $\eta < \beta'$.

Now, if $\beta' = \bigcup\beta' \neq 0$, we note that $3\cdot\beta' = \beta'$, and a simple argument using the continuity law 21 suffices to show that (3) still holds if η is replaced by β', i.e., to obtain (i) for $\gamma = 2$. If $\beta' \neq \bigcup\beta'$, say $\beta' = \eta+1$, then

$$\begin{aligned}(\alpha O_4\beta) O_4(\eta+1) &= [(\alpha O_4\beta)O_4\eta] O_3 (\alpha O_4 \beta)\\ &\leqslant [\alpha O_4(\beta+3\cdot\eta)] O_3(\alpha O_4\beta)\end{aligned}$$

by (3) and the monotony laws. Let $\xi = \alpha O_4(\beta+3\cdot\eta)$. Noting that $\alpha O_4 \beta \leqslant \xi$, we have by 3(iii)

$$(\alpha O_4\beta)O_4(\eta+1) \leqslant \xi O_3 \xi \leqslant \xi^{\xi^\xi} \leqslant (\xi O_3 2)^{(\xi O_3 2)} \leqslant (\xi O_3 2) O_3 2\,.$$

Since $\alpha \geqslant 2$, we then have

$$(\alpha O_4\beta)O_4(\eta+1) \leqslant (\xi O_3\alpha) O_3\alpha\,.$$

However,

$$(\xi O_3\alpha)O_3\alpha = \alpha O_4(\beta+3\cdot\eta+2) \leqslant \alpha O_4\big(\beta+3\cdot(\eta+1)\big)\,,$$

so that (i) again proves to hold for $\gamma = 2$.

Now assume that $\gamma = \zeta+1 \geqslant 3$ and that

(4) $\quad (\alpha O_{2\cdot\zeta}\beta) O_{2\cdot\zeta}\beta' \leqslant \alpha O_{2\cdot\zeta}(\beta+3\cdot\beta') \quad$ for every $\alpha \geqslant 2$, $\beta \geqslant 1$, and β'.

Since we have previously shown that (i) implies (ii), we immediately obtain from (4)

(5) $\quad \alpha O_{2\cdot\zeta+1}(\beta+\beta') \leqslant (\alpha O_{2\cdot\zeta+1}\beta) O_{2\cdot\zeta}(1+3\cdot\alpha\cdot\beta')$

for every $\alpha \geqslant 2$, $\beta \geqslant 1$, and β'.

We now proceed by induction on β'. Again, if $\beta' = 0$, (i) becomes trivial. Assume that $\beta' > 0$ and take as the inductive hypothesis

(6) $\quad (\alpha O_{2\cdot\gamma}\beta) O_{2\cdot\gamma}\eta \leqslant \alpha O_{2\cdot\gamma}(\beta+3\cdot\eta) \quad$ for every $\eta < \beta'$.

A continuity argument suffices to derive (i) in case $\beta' = \bigcup\beta'$. Otherwise let $\bigcup\beta' = \eta$, so that $\beta' = \eta+1$. Now we apply (6):

(7) $$\begin{aligned}(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}\beta' &= [(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}\eta] O_{2\cdot\zeta+1}(\alpha O_{2\cdot\gamma}\beta)\\ &\leqslant [\alpha O_{2\cdot\gamma}(\beta+3\cdot\eta)] O_{2\cdot\zeta+1}(\alpha O_{2\cdot\gamma}\beta)\,.\end{aligned}$$

Let $\xi = \alpha O_{2\cdot\gamma}(\beta+3\cdot\eta)$; we then get from (7) and the monotony laws

(8) $$(\alpha O_{2\cdot\gamma}\beta) O_{2\cdot\gamma}\beta' \leqslant \xi O_{2\cdot\zeta+1}\xi\,.$$

From (5), by setting $\alpha = \beta' = \xi$ and $\beta = 1$, we obtain

(9) $$\xi O_{2\cdot\zeta+1}(1+\xi) \leqslant \xi O_{2\cdot\zeta}(1+3\cdot\xi\cdot\xi)\,;$$

a fortiori, $\xi O_{2\cdot\zeta+1}\xi \leqslant \xi O_{2\cdot\zeta+1}(1+3\cdot\xi\cdot\xi)$, so that, by combining (8) and (9), we arrive at

(10) $$(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}\beta' \leqslant \xi O_{2\cdot\zeta}(1+3\cdot\xi\cdot\xi)\,.$$

Since $\xi \geqslant 2$, we have $3\cdot\xi\cdot\xi < (\xi O_2 2) O_2 2$. Noting that $\xi O_2 2 \geqslant 2$ and $2\cdot\zeta+1 > 2$, we apply the monotony law 8 and obtain

$$1+3\cdot\xi\cdot\xi \leqslant (\xi O_{2\cdot\zeta+1}2) O_{2\cdot\zeta+1} 2\,.$$

Then, by Corollary 10 and the monotony laws,

(11) $$\xi O_{2\cdot\zeta}(1+3\cdot\xi\cdot\xi) \leqslant [(\xi O_{2\cdot\zeta+1}2)O_{2\cdot\zeta+1}2] O_{2\cdot\zeta+1}2\,.$$

Noting that $\alpha \geqslant 2$, we use inequalities (10) and (11) to get

$$\begin{aligned}(\alpha O_{2\cdot\gamma}\beta)O_{2\cdot\gamma}\beta' &\leqslant [(\xi O_{2\cdot\zeta+1}\alpha)O_{2\cdot\zeta+1}\alpha] O_{2\cdot\zeta+1}\alpha\\ &\leqslant \alpha O_{2\cdot\gamma}(\beta+3\cdot\eta+3) = \alpha O_{2\cdot\gamma}[\beta+3\cdot(\eta+1)]\,.\end{aligned}$$

Thus (i) holds for $\beta' = \eta+1$. This completes the inner induction on β'; and hence (i) holds whenever $\gamma \neq \bigcup\gamma$.

In case $\gamma = \bigcup\gamma \neq 0$ the desired result follows at once from Theorem 27. This completes the proof of Lemma 31.

Under the assumption $\alpha \geqslant 3$, or $\alpha = 2$ and $\beta \geqslant 2$, the coefficient 3 in both parts of 31 may be replaced by 2. Notice that, for any ordinal δ, $3\cdot\delta = \delta+\varkappa$ for some finite $\varkappa$.

THEOREM 32. *Assume $\beta, \beta' \geqslant 1$ and $\gamma \neq 1$.*

(i) *If $\beta' = \bigcup\beta' \neq 0$ and $\alpha \geqslant 2$, then*

$$\alpha O_{2\cdot\gamma}(\beta+\beta') = (\alpha O_{2\cdot\gamma}\beta) O_{2\cdot\gamma}\beta'\,.$$

(ii) *If $\alpha = \bigcup\alpha$, or if $\alpha \geqslant 2$ and $\beta' = \bigcup\beta' \neq 0$, then*

$$\alpha O_{2\cdot\gamma+1}(\beta+\beta') = (\alpha O_{2\cdot\gamma+1}\beta) O_{2\cdot\gamma}(\alpha\cdot\beta')\,.$$

Proof. For $\gamma = 0$, (i) and (ii) are special cases of well-known laws of the traditional arithmetic. For $\gamma \geqslant 2$, (i) follows immediately from 30(i) and 31(i). If $\alpha = 0$, (ii) follows directly from 2(i). If either $\alpha = \bigcup\alpha \neq 0$, or else $\alpha \geqslant 2$ and $\beta' = \bigcup\beta' \neq 0$, we observe that $1+3\cdot\alpha\cdot\beta' = 1+\alpha\cdot\beta' = \alpha\cdot\beta'$ and obtain

(1) $$\alpha O_{2\cdot\gamma+1}(\beta+\beta') \leqslant (\alpha O_{2\cdot\gamma+1}\beta) O_{2\cdot\gamma}(\alpha\cdot\beta')\,,$$

(2) $$(\alpha O_{2\cdot\gamma+1}\beta) O_{2\cdot\gamma}(\alpha\cdot\beta') \leqslant (1+\alpha) O_{2\cdot\gamma+1}(\beta+\beta')$$

from 31 (ii) and 30 (ii), respectively. Next, we note that

(3) $$(1+\alpha)O_{2\cdot\gamma+1}(\beta+\beta') = \alpha O_{2\cdot\gamma+1}(\beta+\beta'),$$

for if $\alpha \geqslant \omega$, then $1+\alpha = \alpha$, while if $2 \leqslant \alpha < \omega$ and $\beta' = \bigcup \beta' \neq 0$, then (3) follows from 29. Taken together, (1), (2), and (3) establish (ii).

The restrictions of 32 (i), (ii) to limit ordinals are necessary. For instance, $2O_4(2+3) = 2^{2^{2059}}$ while $(2O_4 2)O_4 3 = 2^{2^{391}}$, so that 32 (i) fails for $\alpha = 2$, $\beta = 2$, $\beta' = 3$, and $\gamma = 2$.

Notice the connection between Theorems 27 and 32. In case γ is a limit number, we have, as is well known, $\gamma = 2\cdot\gamma$. Hence in this case the conclusion of 32 (i) goes into 27 (i′), and the conclusion of 32 (ii) into 27 (ii′); at the same time the hypotheses of 32 (i), (ii) can be considerably relaxed.

Setting $\beta = 1$ in 31 (ii), 27 (ii), and 32 (ii), we obtain

COROLLARY 33. *Assume $\alpha, \gamma \geqslant 2$. We then have:*

(i) $\alpha O_{2\cdot\gamma+1}\beta \leqslant \alpha O_{2\cdot\gamma}(1+3\cdot\alpha\cdot\beta)$;

(ii) *if* $\beta \neq 0$ *and* $\gamma = \bigcup \gamma \neq 0$, *then* $(1+\alpha)O_{\gamma+1}(1+\beta) = (1+\alpha)O_\gamma(1+\alpha\cdot\beta)$;

(iii) *if either* $\alpha = \bigcup \alpha$, *or else* $\alpha \geqslant 2$ *and* $\beta = \bigcup \beta \neq 0$, *or, finally,* $\alpha \geqslant \omega$, $\beta \neq 0$, *and* $\gamma = \bigcup \gamma \neq 0$, *then* $\alpha O_{2\cdot\gamma+1}(1+\beta) = \alpha O_{2\cdot\gamma}(\alpha\cdot\beta)$.

In connection with 33 (iii), recall that the formula $\gamma = \bigcup \gamma$ always implies $\gamma = 2\cdot\gamma$ (and conversely). Still simpler forms of 33 (ii), (iii) are obtained when $\beta \geqslant \omega$, since then $\beta = 1+\beta$.

Corollary 33 (i) throws some interesting light on the relationship between the operations $O_{2\cdot\gamma}$ and $O_{2\cdot\gamma+1}$ for $\gamma \geqslant 2$. Roughly speaking, the two operations are equally powerful; neither increases essentially faster than the other. More precisely, although $O_{2\cdot\gamma+1}$ majorizes $O_{2\cdot\gamma}$ by Theorem 8, $O_{2\cdot\gamma+1}$ proves to be majorized by a simple composition of $O_{2\cdot\gamma}$, O_0, and O_1. (Notice that this last remark applies also to the case $\gamma = 1$; in fact, by 3 (iii), we have $\alpha O_3\beta \leqslant \alpha O_2(\alpha O_2\beta)$ for any α and β.) However, no analogous connections hold between O_0 and O_1, or between $O_{2\cdot\gamma+1}$ and $O_{2\cdot\gamma+2}$ for any value of γ; cf. the remarks following Theorem 52 in the next section.

The close relationship between operations $O_{2\cdot\gamma}$ and $O_{2\cdot\gamma+1}$ will be further emphasized by the discussion in Section 3; it will be seen, e.g. from Theorem 43 (iii), that in some constructions either of these operations can be replaced by the other.

The fact that the operations $O_{2\cdot\gamma}$ and $O_{2\cdot\gamma+1}$ have essentially the same power may seem to be a defect of our construction. It appears that each operation in our sequence beginning with O_2 is unnecessarily duplicated—an even operation $O_{2\cdot\gamma}$ by the corresponding odd operation $O_{2\cdot\gamma+1}$, and conversely. It might even seem at first sight that the defect could

easily be removed by a simple modification of our basic recursion schema, in fact, by changing Definition 1 in the following way:

(i) $\alpha O_0^*\beta = \alpha+\beta$;

(ii) $\alpha O_\gamma^*\beta = \bigcup_{\eta<\beta,\zeta<\gamma}[\alpha O_\zeta^*(\alpha O_\gamma^*\eta)]$.

(Another alternative would be to make corresponding changes in 17 (iv), (v).)

The matter, however, is not so simple. In the finite domain the operations O_γ^* with $\gamma < \omega$ indeed prove to be of interest and, since they avoid duplication, are probably more natural there than our original O_γ's. ([4]) In the transfinite domain, however, the new operations prove to be trivial. Consider, for instance, the case of $\gamma = 1$. As is easily seen, $\alpha O_1^*\beta = \alpha O_1\beta$ for every finite β and even for $\beta = \omega$ (α is arbitrary). However, it is also easily seen that $\alpha O_1^*\beta = \alpha O_1^*\omega$ for every $\beta > \omega$. The situation deteriorates even further when we consider higher operations O_γ^*.

In Theorem 32 we have established two *limit type identities*. These are arithmetical equations involving two or more operations O_γ, which are satisfied for all argument values (with the possible exception of 0 and 1), provided, however, that one of the arguments is a limit ordinal. In this connection we want to discuss here the problem of the existence—in the extended ordinal arithmetic—of *identities in the strict sense*, i.e., not subject to any restriction to limit ordinals.

Several such identities for the lowest operations are well known from the traditional arithmetic of ordinals. These are:

(I) $\alpha O_0(\beta O_0\delta) = (\alpha O_0\beta)O_0\delta$.

(II) $\alpha O_1(\beta O_0\delta) = (\alpha O_1\beta)O_0(\alpha O_1\delta)$.

(III) $\alpha O_1(\beta O_1\delta) = (\alpha O_1\beta)O_1\delta$.

(IV) $\alpha O_2(\beta O_0\delta) = (\alpha O_2\beta)O_1(\alpha O_2\delta)$.

(V) $\alpha O_2(\beta O_1\delta) = (\alpha O_2\beta)O_2\delta$.

In 3 (iii) an identity for O_3 was given in which a constant numeral 1 occurred in one argument place; we exclude such identities from the present discussion. However, with the help of 3 (iii) we can easily establish the following identity:

(VI) $\alpha O_3(\beta O_0\delta) = (\alpha O_3\beta)O_2(\alpha O_2\delta)$.

The equations (I)-(III) are satisfied by all values of the variables α, β, δ without exception. On the other hand, (IV)-(VI) may fail for the few smallest values of these variables, in fact, for 0 and 1. When referring here to some equations as identities (in the strict sense), we

([4]) In the finite domain, the functions $O_\varkappa^*$, $\varkappa < \omega$, are essentially the same as those used in Ackermann [1], in the construction of a computable function which is not primitive recursive.

only assume that they hold for all sufficiently large finite ordinals and for all infinite ordinals. It is instructive to examine similarities and differences between identities (I)-(VI) and the limit-type identities 32(i), (ii).

In addition to the identities listed above, there are some further, more special and less familiarly known, identities which also involve the operations O_0 and O_1. The commutative laws for O_0 and O_1 are known to fail in the ordinal arithmetic, but some particular cases and weak consequences of these laws turn out to hold, for instance,

$$(\alpha O_0 \beta) O_0 (\alpha O_0 \beta) = (\beta O_0 \alpha) O_0 (\alpha O_0 \beta)\,,$$
$$(\alpha O_0 \beta) O_0 (\alpha O_1 \beta) = (\beta O_0 \alpha) O_0 (\alpha O_1 \beta)\,.\ ^{(5)}$$

The situation changes radically when we turn to the next operation, O_4. We do not know a single identity involving exclusively operations $O_0, \ldots, O_4$, which is not a purely logical consequence of those identities that involve exclusively operations $O_0, \ldots, O_3$ (and which therefore remains valid if O_4 is replaced by any other binary operation on and to ordinals). The problem whether such identities exist is open. This problem can be formulated more sharply if we are interested in all identities in the finite domain, i.e., equations which are identically satisfied by arbitrary (sufficiently large) finite ordinals. We can then supplement the list of identities (I)-(VI) by three new formulas:

(VII) $\alpha O_0 \beta = \beta O_0 \alpha$,

(VIII) $\alpha O_1 \beta = \beta O_1 \alpha$,

(IX) $(\alpha O_1 \alpha') O_2 \beta = (\alpha O_2 \beta) O_1 (\alpha' O_2 \beta)$.

The problem now assumes the form: are there any identities in the finite domain involving the operations $O_0, \ldots, O_4$ which are not purely logical consequences of (I)-(IX)?

Our problem, in both infinite and finite domains, remains open if, instead of $O_0, \ldots, O_4$, we consider operations $O_0, \ldots, O_\gamma$ for any finite $\gamma \geqslant 4$. However, the answer to the problem proves to be affirmative if we turn to operations with infinite indices. Here essentially new identities do appear. We give some instances of such identities in Theorem 35; they are simply particular instances of the limit-type identities from Theorem 32, derived with the help of the following

LEMMA 34. *If $\gamma \geqslant \omega$, $\alpha \geqslant 2$, $\beta \geqslant 2$, and $\alpha = \beta = 2$ does not hold, then $\alpha O_\gamma \beta$ is a limit ordinal.*

Proof. The proof is carried through by induction on γ with an "inner induction" on β. We begin with $\gamma = \omega$. If $\alpha \geqslant 3$ and $\beta = 2$, then $\alpha O_\omega \beta = \bigcup_{\zeta<\omega} (\alpha O_\zeta \alpha)$; that this union is a limit ordinal follows from 13. If $\alpha = 2$ and $\beta = 3$, we have

$$2 O_\omega 3 = \bigcup_{\zeta<\omega} [(2 O_\omega 2) O_\zeta 2] = \bigcup_{\zeta<\omega} (4 O_\zeta 2)\,,$$

and again we refer to 13. Next, assume $\beta = \eta + 1$ and $\alpha O_\omega \eta$ is a limit ordinal. Then $\alpha O_\omega \beta = \bigcup_{\zeta<\omega} [(\alpha O_\omega \eta) O_\zeta \alpha]$, and, since $(\alpha O_\omega \eta) O_0 \alpha \leqslant (\alpha O_\omega \eta) O_1 \alpha$, we also have

$$\alpha O_\omega \beta = \bigcup_{0<\zeta<\omega} [(\alpha O_\omega \eta) O_\zeta \alpha];$$

thus $\alpha O_\omega \beta$ is a union of limit ordinals by 16. The case $\beta = \bigcup \beta \neq 0$ presents no difficulty. Finally, assume that $\gamma > \omega$ and that the theorem holds for each ζ such that $\omega \leqslant \zeta < \gamma$. Again we proceed by induction on β. In the basic cases, $\beta = 2$ and $\beta = 3$, we apply the inductive hypothesis together with Definition 1(ii), and in the inductive step we use the inductive hypothesis in combination with Corollary 16.

THEOREM 35. *If $\gamma \geqslant \omega$, $\beta \geqslant 1$, $\alpha, \alpha', \beta', \beta'' \geqslant 2$, and neither $\alpha = \alpha' = 2$ nor $\beta' = \beta'' = 2$ holds, then*

$$\alpha O_{2\cdot\gamma} [\beta O_0 (\beta' O_{2\cdot\gamma} \beta'')] = (\alpha O_{2\cdot\gamma} \beta) O_{2\cdot\gamma} (\beta' O_{2\cdot\gamma} \beta'')\,,$$
$$(\alpha O_{2\cdot\gamma} \alpha') O_{2\cdot\gamma+1} (\beta O_0 \beta') = [(\alpha O_{2\cdot\gamma} \alpha') O_{2\cdot\gamma+1} \beta] O_{2\cdot\gamma} [(\alpha O_{2\cdot\gamma} \alpha') O_1 \beta']\,.$$

Proof: by 34 and 32.

It may be pointed out that none of the operations O_γ is commutative, i.e., for no value of γ is the equation

$$\alpha O_\gamma \beta = \beta O_\gamma \alpha$$

identically satisfied by all ordinals. For instance, it can easily be shown that

$$(\omega+1) O_\gamma \omega \neq \omega O_\gamma (\omega+1)$$

for every γ (cf. 29 and 4(ii)).

So far we have been concerned with the existence of identities in which no constant symbols denoting particular ordinals appear. In our earlier discussion, however, we have come across several identities involving constant symbols; cf., e.g., 9 and 26. We shall conclude this section by establishing in Theorem 37 another simple, but not trivial, example of such identities. To this end, we begin with a technical lemma.

LEMMA 36. *If $\gamma = \bigcup \gamma \neq 0$, then for every $\gamma' < \gamma$ and any $\varkappa, \lambda < \omega$ there is a $\gamma'' < \gamma$ such that $\varkappa O_{\gamma'} \lambda \leqslant 3 O_{\gamma''} 2$.*

Proof. Let $\varkappa'$ be the largest of the ordinals 2, $\varkappa$, and λ, i.e., $\varkappa' = \bigcup \{2, \varkappa, \lambda\} = 2 \cup \varkappa \cup \lambda$. We have $3 O_\gamma 2 \geqslant \omega$ by Theorem 12, and $3 O_\gamma 2 = \bigcup_{\zeta<\gamma} (3 O_\zeta 2)$. Thus, $3 O_\zeta 2 \geqslant \varkappa'$ for some $\zeta < \gamma$. Now we obtain

$$\varkappa O_{\gamma'} \lambda \leqslant \varkappa' O_{\gamma'} \varkappa' = \varkappa' O_{\gamma'+1} 2 \leqslant (3 O_\zeta 2) O_{\gamma'+1} 2 \leqslant (3 O_{\zeta'} 3) O_{\zeta'} 3\,,$$

where $\zeta' = (\gamma'+1) \cup \zeta$. However, $(3 O_{\zeta'} 3) O_{\zeta'} 3$ is simply $3 O_{\zeta'+1} 3 = 3 O_{\zeta'+2} 2$.

(5) Such "unorthodox" identities for addition and multiplication were noticed by Tarski several years ago.

THEOREM 37. *For every γ such that $\gamma = \bigcup \gamma \neq 0$ and every β we have*

$$2\,O_\gamma(3+\beta) = 3\,O_\gamma(2+\beta)$$

and, more generally,

$$2\,O_\gamma(3+\beta) = \varkappa\,O_\gamma(2+\beta)\,,$$

where $\varkappa$ is any finite ordinal $\geqslant 3$.

Proof. Only the first equality need be proved, since the second follows from it by Theorem 26 and an easy induction. We proceed by induction on β. If $\beta = 0$, then, on the one hand, we have

$$(1) \qquad 3\,O_\gamma 2 = \bigcup_{\zeta<\gamma}(3\,O_\zeta 2)$$

by Lemma 25. On the other hand, 25 also yields

$$2\,O_\gamma 3 = \bigcup_{\zeta<\gamma}[(2\,O_\gamma 2)\,O_\zeta 2] = \bigcup_{\zeta<\gamma}(4\,O_\zeta 2)\,.$$

Hence, from 36 and the monotony laws we obtain

$$(2) \qquad 2\,O_\gamma 3 = \bigcup_{\zeta<\gamma}(3\,O_\zeta 2)\,.$$

(1) and (2) imply $2\,O_\gamma 3 = 3\,O_\gamma 2$. Now suppose $\beta = \eta+1$ for some η, and $2\,O_\gamma(3+\eta) = 3\,O_\gamma(2+\eta)$. Then, by 25,

$$\begin{aligned} 3\,O_\gamma(2+\eta+1) &= \bigcup_{\zeta<\gamma}\big[\big(3\,O_\gamma(2+\eta)\big)\,O_\zeta 2\big] \\ &= \bigcup_{\zeta<\gamma}\big[\big(2\,O_\gamma(3+\eta)\big)\,O_\zeta 2\big] \\ &= 2\,O_\gamma(3+\eta+1)\,. \end{aligned}$$

The case $\beta = \bigcup \beta \neq 0$ is handled by a simple continuity argument, thereby completing the induction.

Section 3. Main numbers

The notion of a main number was first introduced by Hessenberg in [7], page 578, for the operation of addition and then extended by Jacobsthal in [8], page 153, to an extensive class of binary operations O on and to ordinals which, in particular, comprehends all the operations O_γ. The definition which we formulate below refers to all operations O on $\Omega\times\Omega$ to Ω. In the general case it is not equivalent to the definition in [8]. However, the two definitions prove to be equivalent when applied to the operations O_γ.

DEFINITION 38. *Let O be an arbitrary binary operation on $\Omega\times\Omega$ to Ω. A* **main number** *of O is any ordinal $\delta \geqslant \omega$ such that $\alpha, \beta < \delta$ always implies $\alpha O \beta < \delta$. $M(O)$ denotes the class of all main numbers of O.*

The restriction of main numbers to transfinite ordinals is more a matter of convenience than necessity. Removing it would sometimes

result in the inclusion of finite ordinals as main numbers; in fact 0 would be (trivially) a main number of every operation O on $\Omega\times\Omega$ to Ω, 1 would be a main number of every operation O_γ, and 2 of every operation O_γ with $\gamma \geqslant 1$. No operation O_γ would have, however, finite main numbers $\geqslant 3$.

THEOREM 39. *Let O be any operation on $\Omega\times\Omega$ to Ω.*

(i) *For every ordinal α there exists an ordinal $\delta \geqslant \alpha$ such that $\delta \in M(O)$; in other words, $M(O)$ is not a set.*

(ii) *If Δ is a non-empty subset of $M(O)$, then $\bigcup \Delta \in M(O)$; in other words, $M(O)$ is closed.*

Proof. (i) Given any ordinal α, we define a sequence of ordinals $\langle\sigma_\varkappa\rangle_{\varkappa<\omega}$ by recursion:

$$\sigma_0 = \alpha\,,$$

$$\sigma_{\varkappa+1} = (\sigma_\varkappa+1) \cup \bigcup_{\xi,\eta<\sigma_\varkappa}(\xi\,O\,\eta) \qquad \text{for each } \varkappa<\omega\,.$$

Clearly, $\sigma_\varkappa < \sigma_{\varkappa+1}$ for $\varkappa = 0, 1, \ldots$ Now let $\delta = \bigcup_{\varkappa<\omega}\sigma_\varkappa$. Then $\delta \geqslant \sigma_\varkappa$ for every $\varkappa<\omega$ and, in particular, $\delta \geqslant \alpha = \sigma_0$. Moreover, if $\xi, \eta < \delta$, there are $\varkappa, \lambda < \omega$ such that $\xi < \sigma_\varkappa$ and $\eta < \sigma_\lambda$. Setting $\mu = \varkappa \cup \lambda$, we obtain sucessively

$$\xi, \eta < \sigma_\mu\,,$$

$$\xi\,O\,\eta \leqslant \sigma_{\mu+1} < \sigma_{\mu+2}$$

and, finally,

$$\xi\,O\,\eta < \delta\,.$$

Thus $\delta \in M(O)$.

(ii) Let Δ be a non-empty subset of $M(O)$. We have $\bigcup \Delta \geqslant \varrho \geqslant \omega$ for every $\varrho \in \Delta$, whence $\bigcup \Delta \geqslant \omega$. If $\xi, \eta < \bigcup \Delta$, then there is a $\varrho \in \Delta$ such that $\xi, \eta < \varrho$; since $\varrho \in M(O)$, we conclude that $\xi\,O\,\eta < \varrho$ and hence $\xi\,O\,\eta < \bigcup \Delta$. Thus $\bigcup \Delta \in M(O)$, and the proof is complete.

According to 39 the class $M(O)$ satisfies the premises of Theorem 22, and hence there is exactly one normal function with range $M(O)$. We now define:

DEFINITION 40. *For any operation O on $\Omega\times\Omega$ to Ω and any ordinal η, the value at η of the unique normal function with range $M(O)$ is denoted by $\mu(\eta, O)$. The ordinal $\mu(\eta, O)$ is referred to as the η-th* **successive main number of O.**

COROLLARY 41. *For any operation O on $\Omega\times\Omega$ to Ω we have:*

(i) *$\mu(\bigcup\Gamma, O) = \bigcup_{\eta\in\Gamma}\mu(\eta, O)$ for every non-empty set $\Gamma \subseteq \Omega$;*

(ii) *$\mu(\eta, O) \geqslant \eta$ for every ordinal η;*

(iii) *if $\delta \geqslant \mu(0, O)$, then there is exactly one η such that $\mu(\eta, O) \leqslant \delta < \mu(\eta+1, O)$; in fact, $\mu(\eta, O)$ is the largest main number of O which is $\leqslant \delta$, and $\mu(\eta+1, O)$ is the least main number of O which is $> \delta$;*

(iv) *for every δ there exists $\eta > \delta$ such that $\mu(\eta, O) = \eta$.*

Proof: by 20, 23, and 40.

We now turn to the discussion of properties specific to the main numbers of the operations O_γ.

THEOREM 42. (i) *All main numbers of O_γ are limit numbers.*

(ii) *If $\gamma < \omega$, then the least main number of O_γ is ω; in other words $\mu(0, O_\gamma) = \omega$.*

Proof. These are easy consequences of the monotony laws and 2(iv)

THEOREM 43. (i) *If $\gamma \leqslant \gamma'$, then $M(O_\gamma) \supseteq M(O_{\gamma'})$.*

(ii) $M(O_0) \supset M(O_1) \supset M(O_2)$.

(iii) *If $\gamma \geqslant 1$, then $M(O_{2\cdot\gamma}) = M(O_{2\cdot\gamma+1})$.*

Proof. (i) is immediate from the monotony law 8, and (ii) is a result from the traditional arithmetic. $M(O_{2\cdot\gamma}) = M(O_{2\cdot\gamma+1})$ for $\gamma \geqslant 1$ follows from (i), 3(iii) (in case $\gamma = 1$), and Lemma 31 (in case $\gamma \geqslant 2$).

Some important supplements to this theorem will be provided later in 52 and 57.

In the next three theorems we give new characterizations of the classes $M(O_\gamma)$, different from the one that was used in our general definition of main numbers.

THEOREM 44. *For any ξ and γ the following conditions are equivalent:*

(i) $\xi \in M(O_\gamma)$;

(ii) *$\xi \geqslant 2$ and $\xi \neq \alpha O_\zeta \beta$ for all $\alpha, \beta < \xi$ and $\zeta \leqslant \gamma$.*

Proof. If (i) holds, then $\xi \geqslant \omega$ by 38 and, a fortiori, $\xi \geqslant 2$; moreover, for any $\alpha, \beta < \xi$ and $\zeta \leqslant \gamma$ we have $\xi \in M(O_\zeta)$ by 43(i), whence $\alpha O_\zeta \beta < \xi$ and $\xi \neq \alpha O_\zeta \beta$. Thus (i) implies (ii).

To establish the implication in the opposite direction, suppose to the contrary that, for a given γ, (ii) holds while (i) fails; we may assume that γ is the least ordinal such that

$$\xi \notin M(O_\gamma)\,. \tag{1}$$

By (ii), $\xi \geqslant 2$, and $\xi \neq \alpha + \beta$ for any $\alpha, \beta < \xi$. Hence, clearly, $\xi \geqslant \omega$. Together with (1) this implies the existence of two ordinals α and β such that

$$\alpha, \beta < \xi \tag{2}$$

and

$$\xi \leqslant \alpha O_\gamma \beta\,. \tag{3}$$

By fixing α we can assume that β is the least ordinal satisfying (3). By (2), there is an ordinal β' such that $\xi = \alpha + \beta'$. If γ were equal to 0, (3) would give $\alpha + \beta' \leqslant \alpha + \beta$ and hence $\beta' \leqslant \beta$; thus, in view of (2), we would have $\xi = \alpha + \beta'$ with $\alpha, \beta' < \xi$, which contradicts (ii). Therefore $\gamma \neq 0$. Hence, by 1(ii),

$$\alpha O_\gamma \beta = \bigcup\nolimits_{\eta<\beta, \zeta<\gamma} [(\alpha O_\gamma \eta) O_\zeta \alpha]\,. \tag{4}$$

(ii) and (3) imply

$$\xi < \alpha O_\gamma \beta\,. \tag{5}$$

From (4) and (5) we conclude that

$$\xi = (\alpha O_\gamma \eta) O_\zeta \alpha \tag{6}$$

for some $\eta < \beta$ and $\zeta < \gamma$. We have, however, $\xi \in M(O_\zeta)$ by (1) and the minimality of γ; also, $\alpha O_\gamma \eta < \xi$ by (3) and the minimality of β, and $\alpha < \xi$ by (2). Consequently, in view of 38,

$$(\alpha O_\gamma \eta) O_\zeta \alpha < \xi\,. \tag{7}$$

By (6) and (7) we have again arrived at a contradiction. Thus we must assume that (ii) always implies (i), and the proof has been completed.

COROLLARY 45. (i) *$\xi \in M(O_\gamma)$ if and only if $\xi \neq \alpha O_\gamma \beta$ for all $\alpha, \beta < \xi$, and $\xi \in M(O_\zeta)$ for every $\zeta < \gamma$.*

(ii) *$\xi \in M(O_{\gamma+1})$ if and only if $\xi \in M(O_\gamma)$ and $\xi \neq \alpha O_{\gamma+1} \beta$ for all $\alpha, \beta < \xi$.*

Let us say that an ordinal ξ is *O_γ-indecomposable* if $\xi \neq \alpha O_\gamma \beta$ for all $\alpha, \beta < \xi$. Theorem 44 can then be expressed in this way: the main numbers of O_γ are just those ordinals which are O_ζ-indecomposable for every $\zeta \leqslant \gamma$. From Corollary 45(i) we obtain at once the following result of the traditional arithmetic (cf. [12], p. 279): the main numbers of addition are just those ordinals which are additively indecomposable. It is known that this result cannot be extended to O_1, i.e., multiplication; for example, $\omega+1$ is multiplicatively indecomposable, but clearly $\omega+1 \notin M(O_1)$. A generalization of this counter-example shows that the result in question does not extend to any operation O_γ, $\gamma \geqslant 1$.

THEOREM 46. *For any ξ and γ the following conditions are equivalent:*

(i) $\xi \in M(O_\gamma)$;

(ii) *$\xi \geqslant 3$ and $\alpha O_\gamma \xi = \xi$ for every α such that $2 \leqslant \alpha < \xi$.*

Proof. In case $\xi \in M(O_\gamma)$ and $2 \leqslant \alpha < \xi$, we have by 42(i)

$$\alpha O_\gamma \xi = \bigcup\nolimits_{\eta<\xi} (\alpha O_\gamma \eta)\,.$$

Now $\alpha O_\gamma \eta < \xi$ for each $\eta < \xi$, so $\alpha O_\gamma \xi \leqslant \xi$; the equality then follows from 7. Of course, $\xi \in M(O_\gamma)$ implies $\xi \geqslant 3$.

Next assume that (ii) holds. That (ii) implies (i) in case $\gamma = 0$ is known from the traditional arithmetic; accordingly, we may assume $\gamma \geqslant 1$. Let $\alpha, \beta < \xi$; if $\alpha < 2$, then $\alpha O_\gamma \beta < \xi$ follows from Corollary 2. If $\alpha \geqslant 2$, then, by the strict monotony law 4(ii) and our assumption,

$$\alpha O_\gamma \beta < \alpha O_\gamma \xi = \xi\,.$$

To establish $\xi \in M(O_\gamma)$, it remains to prove that $\xi \geqslant \omega$; in fact, we shall show that $\xi = \bigcup \xi$. For, if $\xi = \eta+1$ for some $\eta < \xi$, then, by the monotony laws,

$$2\,O_\gamma \xi \geqslant 2\,O_1(\eta+1) = 2\cdot\eta+2 > \xi ,$$

which contradicts our assumption.

This theorem shows that our general definition of main numbers, when applied to the operations O_γ, is essentially equivalent to the definition in [8], p. 153. For $\gamma = 0, 1, 2$, Theorem 46 is well known from the literature.

In application to the operations O_γ with $\gamma \geqslant 2$ Theorem 46 can be improved as follows:

THEOREM 47. *Let $\gamma \geqslant 2$. If α is any ordinal such that $2 \leqslant \alpha < \xi$, then the conditions*

(i) $\xi \in M(O_\gamma)$,

(ii) $\alpha\,O_\gamma \xi = \xi$

are equivalent. In particular, for $\gamma \geqslant 2$ condition (i) *is equivalent to*

(ii′) $2\,O_\gamma \xi = \xi \neq 0$.

Proof. Assuming $\gamma \geqslant 2$, we shall first show that (ii′) implies (i). For $\gamma = 2$ this is a familiar result of the traditional arithmetic. Let $\gamma = 2\cdot\zeta$ for some $\zeta \geqslant 2$, and assume (ii′); we wish to prove that ξ and γ satisfy condition 46 (ii). We have $\xi \geqslant 3$ by 2 (ii), (iii). Consider any ordinal α such that $2 \leqslant \alpha < \xi$. Since, by 7 and 8,

$$\xi = 2\,O_{2\cdot\zeta}\xi \geqslant 2\,O_2 \xi \geqslant \xi ,$$

we conclude that $\xi \in M(O_2)$ and therefore $\xi \in M(O_0)$. With the help of 32 and 42 (i) we get

$$\alpha\,O_{2\cdot\zeta}\xi \leqslant (2\,O_{2\cdot\zeta}\alpha)\,O_{2\cdot\zeta}\xi = 2\,O_{2\cdot\zeta}(\alpha+\xi);$$

since $\alpha < \xi \in M(+)$, we have $\alpha+\xi = \xi$ (by 46 with $\gamma = 0$), and we obtain

$$\alpha\,O_{2\cdot\zeta}\xi \leqslant 2\,O_{2\cdot\zeta}\xi = \xi \leqslant \alpha\,O_{2\cdot\zeta}\xi ,$$

so that $\alpha O_{2\cdot\zeta}\xi = \xi$. Thus, for $\gamma = 2\cdot\zeta$ and $\zeta \geqslant 2$, (ii′) indeed implies 46 (ii), and hence, by 46, it implies condition (i) of our theorem. Finally, let $\gamma = 2\cdot\zeta+1$ for some $\zeta \geqslant 1$. We notice that

$$2\,O_{2\cdot\zeta+1}\xi \geqslant 2\,O_{2\cdot\zeta}\xi \geqslant \xi .$$

Therefore, assuming $2\,O_{2\cdot\zeta+1}\xi = \xi$, we obtain $2\,O_{2\cdot\zeta}\xi = \xi$, hence $\xi \in M(O_{2\cdot\zeta})$ by what was proved above, and $\xi \in M(O_{2\cdot\zeta+1})$ by 43 (iii).

We see that (ii′) always implies (i) for $\gamma \geqslant 2$. The implication in the opposite direction follows immediately from 46, so that (i) and (ii′) turn out to be equivalent.

We can now easily establish the equivalence of (i) and (ii) for any given α such that $2 \leqslant \alpha < \xi$. On the one hand, (i) implies (ii) by 46. On the other hand, (ii) yields

$$\xi \leqslant 2\,O_\gamma \xi \leqslant \alpha\,O_\gamma \xi = \xi ,$$

whence

$$2\,O_\gamma \xi = \xi ,$$

and we get (i) by the first part of our proof.

The ordinals $\xi > \omega$ which satisfy condition 47 (ii′) for $\gamma = 2$ are referred to in the literature as epsilon numbers. Thus, in case $\gamma = 2$, Theorem 47 expresses the well-known result that the main numbers of exponentiation are just ω and the epsilon numbers.

It is a familiar fact of the traditional arithmetic that an enumeration of the main numbers of addition and multiplication can be obtained with the aid of exponentiation; specifically, $\mu(\eta, O_0) = \omega^{1+\eta}$ and $\mu(\eta, O_1) = \omega^{\omega^\eta}$. The following Theorem 48 and its corollary show that the ability of higher operations to express the main numbers of lower ones is preserved throughout the entire transfinite hierarchy of operations O_γ.

To understand properly the meaning of our next theorem, recall that for any given ordinals α, γ there is a uniquely determined ordinal ν such that $\mu(\nu, O_\gamma)$ is the smallest main number of O_γ which exceeds α; cf. Corollary 41. Hence the function $\psi_{\alpha,\gamma}$ such that $\psi_{\alpha,\gamma}(\eta) = \mu(\nu+\eta, O_\gamma)$ for every η enumerates all main numbers of O_γ that exceed α.

THEOREM 48. *Given $\alpha \geqslant 2$ and γ, let $\mu(\nu, O_\gamma)$ be the least main number of O_γ exceeding α. For every ordinal η we then have*

(i) $\mu(\nu+\eta, O_\gamma) = \alpha\,O_{\gamma+1}\,\omega^{1+\eta}$ *for* $\gamma = 0, 1$;

(ii) $\mu(\nu+\eta, O_\gamma) = \mu(\nu+\eta, O_{\gamma+1}) = \alpha\,O_{\gamma+2}[\omega\cdot(1+\eta)]$ *in case* $\gamma = 2\cdot\zeta$ *for some* $\zeta \geqslant 1$.

Proof. We first assume $\eta = 0$. For (i) we appeal to the traditional arithmetic; we recall that $\alpha\,O_1\,\omega$, resp. $\alpha\,O_2\,\omega$, is the least main number of O_0, resp. O_1, which exceeds α. Now suppose that $\zeta \geqslant 1$ and $\gamma = 2\cdot\zeta$. Using 3 (iii) and 31 (ii), it is easy to show by induction that, for any $\varkappa < \omega$,

$$\alpha\,O_{\gamma+2}\varkappa < \mu(\nu, O_\gamma) ,$$

whence

$$\alpha\,O_{\gamma+2}\,\omega \leqslant \mu(\nu, O_\gamma) .$$

Of course, $\alpha O_{\gamma+2}\omega > \alpha$, so we need only show that $\alpha O_{\gamma+2}\omega \in M(O_\gamma)$. If $\alpha', \beta < \alpha O_{\gamma+2}\omega$, then there is a $\varkappa < \omega$ such that both $\alpha' < \alpha O_{\gamma+2}\varkappa$ and $\beta < \alpha O_{\gamma+2}\varkappa$. Hence

$$\begin{aligned}\alpha'\,O_\gamma \beta &\leqslant (\alpha O_{\gamma+2}\varkappa)\,O_\gamma(\alpha O_{\gamma+2}\varkappa) = (\alpha O_{\gamma+2}\varkappa)\,O_{\gamma+1}2\\ &\leqslant (\alpha O_{\gamma+2}\varkappa)\,O_{\gamma+1}\alpha = \alpha\,O_{\gamma+2}(\varkappa+1)\\ &< \alpha\,O_{\gamma+2}\,\omega ,\end{aligned}$$

and it follows that $\alpha O_{\gamma+2}\omega \in M(O_\gamma)$.

We proceed by induction on η. Suppose that $\eta = \eta'+1$ for some η', and that

$$\mu(\nu+\eta', O_\gamma) = \alpha\,O_{\gamma+2}[\omega\cdot(1+\eta)] .$$

Now $\mu(\nu+\eta'+1, O_\gamma)$ is the least main number of O_γ which exceeds $\mu(\nu+\eta', O_\gamma)$; hence, by the inductive hypothesis and the argument above for the case $\eta = 0$;

$$\mu(\nu+\eta'+1, O_\gamma) = [\alpha O_{\gamma+2}[\omega\cdot(1+\eta')]]O_{\gamma+2}\omega\,.$$

Since $\gamma+2 = 2\cdot(\zeta+1)$, the limit type identity 32(i) now yields:

$$\mu(\nu+\eta, O_\gamma) = \alpha O_{\gamma+2}[\omega\cdot(1+\eta')+\omega] = \alpha O_{\gamma+2}[\omega\cdot(1+\eta)]\,.$$

A simple continuity argument suffices for the case $\eta = \bigcup\eta \neq 0$. To complete the proof we notice that, in case $\gamma = 2\cdot\zeta$ and $\zeta \geqslant 1$, $\mu(\nu+\eta, O_\gamma) = \mu(\nu+\eta, O_{\gamma+1})$ by 40 and 43(iii).

Theorems 46 and 47 have revealed the close connections between main number of the operations O_γ and fixed points of the functions $\varphi_{\alpha,\gamma}$ defined by $\varphi_{\alpha,\gamma}(\eta) = \alpha O_\gamma\eta$. In fact, Theorem 46 shows that an ordinal ξ belongs to $M(O_\gamma)$ if and only if it is a common fixed point of all functions $\varphi_{\alpha,\gamma}$ with $2 \leqslant \alpha < \xi$. In case $\gamma \geqslant 2$, Theorem 47 gives a stronger result: $\xi \in M(O_\gamma)$ if and only if ξ is a fixed point of any one function $\varphi_{\alpha,\gamma}$ with $2 \leqslant \alpha$; in particular, for $\alpha = 2$ (or for any finite $\alpha \geqslant 2$), $M(O_\gamma)$ coincides with the class of all fixed points of $\varphi_{\alpha,\gamma}$. In view of this, Theorem 48 provides an enumeration of the fixed points of any function $\varphi_{\alpha,\gamma}$ with $\alpha, \gamma \geqslant 2$. In fact, the ηth consecutive fixed point of $\varphi_{\alpha,\gamma}$ is $\alpha O_{\gamma+1}[\omega\cdot(1+\eta)]$ in case γ is of the form $2\cdot\zeta+1$, and is $\alpha O_{\gamma+2}[\omega\cdot(1+\eta)]$ otherwise; and, in particular, the smallest fixed point of $\varphi_{\alpha,\gamma}$ is $\alpha O_{\gamma+1}\omega$ or $\alpha O_{\gamma+2}\omega$. In case $\gamma = 0$ or $\gamma = 1$, the enumerating formulas are different: it can easily be shown that the ηth fixed points of $\varphi_{\alpha,0}$ and $\varphi_{\alpha,1}$ with $\alpha \geqslant 2$ are, respectively, $\alpha\cdot\omega+\eta$ and $\alpha^\omega\cdot(1+\eta)$.

Corollary 49. (i) *If* $\gamma = 0, 1$, *then* $\mu(\eta, O_\gamma) = 2O_{\gamma+2}[\omega\cdot(1+\eta)] = \omega O_{\gamma+2}(1+\eta)$;

(ii) *If* $\gamma = 2\cdot\zeta$ *for some* $\zeta \geqslant 1$, *then* $\mu(\eta, O_\gamma) = \mu(\eta, O_{\gamma+1}) = 2O_{\gamma+2}[\omega\cdot\cdot(1+\eta)]$. *If, in addition,* $\gamma < \omega$, *then* $\mu(\eta, O_\gamma) = \mu(\eta, O_{\gamma+1}) = \omega O_{\gamma+3}(1+\eta)$, *while if* $\gamma \geqslant \omega$, *then* $\mu(\eta, O_\gamma) = \mu(\eta, O_{\gamma+1}) = \omega O_{\gamma+3}(2+\eta)$.

Part (i) of 49 is well known from the traditional arithmetic of ordinals; recall that $\omega O_3(1+\eta) = \omega^{\omega^\eta}$. From part (ii) with $\zeta = 1$ it follows that the ηth epsilon number is simply $\omega O_5(2+\eta)$.

Corollary 50. *Assume that* $\beta = \bigcup\beta$ *and that* ϱ *is the largest main number of* O_γ *such that* $\varrho \leqslant \alpha$. *We then have:*

(i) $\alpha O_{\gamma+1}\beta = \varrho O_{\gamma+1}\beta$ *if* $\gamma = 0, 1$;

(ii) $\alpha O_{\gamma+2}\beta = \varrho O_{\gamma+2}\beta$ *if* $\gamma = 2\cdot\zeta$ *for some* $\zeta \geqslant 1$.

Proof: by 48, using the elementary fact that $\beta = \bigcup\beta$ iff β is of the form $\omega\cdot\eta$.

From 49 and 47 we readily obtain a new characterization of the main numbers of $O_{\gamma+2}$:

Theorem 51. *If* $\xi > \mu(0, O_\gamma)$, *then* $\xi \in M(O_{\gamma+2})$ *if and only if* $\mu(\xi, O_\gamma) = \xi$.

Proof: by 47, 49, and 43.

Theorem 52. *For every* $\gamma \geqslant 1$ *we have:*

(i) $M(O_{2\cdot\gamma}) = M(O_{2\cdot\gamma+1}) \supset M(O_{2\cdot\gamma+2})$;

(ii) $\xi \in M(O_{2\cdot\gamma})$ *if and only if* $\xi = \alpha O_{2\cdot\gamma+2}\beta$ *for some* $\alpha \geqslant 2$ *and some* β *such that* $\beta = \bigcup\beta \neq 0$.

Proof: by 43(iii), 51, and 49.

From Theorem 52 it is easily seen that no operation $O_{2\cdot\gamma+2}$ can be majorized by a composition of operations O_ζ with lower indices. This is in striking opposition to the property of the operations $O_{2\cdot\gamma+1}$ expressed in Corollary 33(i) and discussed in the remarks following that Corollary.

In the last portion of this section we deal operations O_γ in which the index γ is a limit ordinal. We shall be interested in main numbers common to all operations O_ζ, $\zeta < \gamma$, and we shall exhibit a function enumerating all such main numbers. As a consequence, it will turn out that, contrary to what one could except, the class $\bigcap_{\zeta<\gamma}M(O_\zeta)$ does not coincide with the class $M(O_\gamma)$, but includes the latter as a proper part. In fact, we shall see that practically all ordinals in the range of O_γ belong to $\bigcap_{\zeta<\gamma}M(O_\zeta)$.

Lemma 53. *If* $\gamma = \bigcup\gamma \neq 0$ *and* $\alpha \geqslant 3$, *then* $\alpha O_\gamma 2$ *is the least element of* $\bigcap_{\zeta<\gamma}M(O_\zeta)$ *which exceeds* α.

Proof. Let $\xi = \bigcup_{\zeta<\gamma}(\alpha O_\zeta\omega)$. It is easy to show that, if $\xi' \in \bigcap_{\zeta<\gamma}M(O_\zeta$ and $\xi' > \alpha$, then $\xi' \geqslant \xi$. Using the monotony laws, 47, 43(i), and 39(ii), we can also prove without difficulty that $\xi \in \bigcap_{\zeta<\gamma}M(O_\zeta)$. Now, in case $\alpha \geqslant \omega$, we get by Lemma 25 and the monotony laws

$$\alpha O_\gamma 2 = \bigcup_{\zeta<\gamma}(\alpha O_\zeta\omega) = \xi\,.$$

If, on the other hand, $\alpha < \omega$, then, by Theorem 37, $\alpha O_\gamma 2 = 3O_\gamma 2$, while by 29 and an easy induction (and because $\omega \in M(O_0)$ in case $\zeta = 0$), we obtain

$$\alpha O_\zeta\omega = 3O_\zeta\omega \quad \text{for any } \zeta\,.$$

Thus, to establish $\alpha O_\gamma 2 = \xi$ in case $\alpha < \omega$, it suffices to show that

$$(1) \qquad 3O_\gamma 2 = \bigcup_{\zeta<\omega}(3O_\zeta\omega)\,.$$

By the continuity law 21(i) we have

$$\bigcup_{\zeta<\omega}(3O_\zeta\omega) = \bigcup_{\zeta<\omega,\varkappa<\omega}(3O_\zeta\varkappa)\,.$$

Lemma 36 shows that for all $\zeta, \varkappa < \omega$ there exists $\zeta' < \omega$ such that $3O_\zeta\varkappa \leqslant 3O_{\zeta'}2$; hence,

$$\bigcup_{\zeta<\omega}(3O_\zeta\omega) = \bigcup_{\zeta<\omega}(3O_\zeta 2)\,.$$

The right-hand side of this equation is equal to $3O_\gamma 2$ by Lemma 25; this establishes (1) and completes the proof.

THEOREM 54. *For every $\alpha \geqslant 3$ and every γ such that $\gamma = \bigcup \gamma \neq 0$ the following conditions are equivalent:*

(i) $\xi \in \bigcap_{\zeta<\gamma} M(O_\zeta)$ *and* $\xi > \alpha$;

(ii) $\xi = \alpha O_\gamma(2+\eta)$ *for some* η.

Proof. First, assume that ξ satisfies (i). The function φ defined by

$$\varphi(\eta) = \alpha O_\gamma(2+\eta)$$

is a normal function, so by 20(iii) there exists exactly one η_0 such that

$$\alpha O_\gamma(2+\eta_0) \leqslant \xi < \alpha O_\gamma(2+\eta_0+1)\,.$$

According to Lemma 53, the least element of $\bigcap_{\zeta<\gamma} M(O_\zeta)$ which exceeds $\alpha O_\gamma(2+\eta_0)$ is $[\alpha O_\gamma(2+\eta_0)]O_\gamma 2$, and by Theorem 27(i) we have

$$[\alpha O_\gamma(2+\eta_0)]O_\gamma 2 = \alpha O_\gamma(2+\eta_0+1)\,.$$

Thus, $\xi > \alpha O_\gamma(2+\eta_0)$ cannot hold, and we conclude that $\xi = \alpha O_\gamma(2+\eta_0)$.

To show that (ii) implies (i), we must prove that $\alpha O_\gamma(2+\eta) \in \bigcap_{\zeta<\gamma} M(O_\zeta)$ for every η. This is easily done by induction on η, using 53 and 27.

From Theorem 54 we see that, under the hypothesis of this theorem, $\alpha O_\gamma(2+\eta)$ is the ηth successive element of $\bigcap_{\zeta<\gamma} M(O_\zeta)$ which exceeds α.

COROLLARY 55. *For every γ such that $\gamma = \bigcup\gamma \neq 0$ the following conditions are equivalent:*

(i) $\xi \in \bigcap_{\zeta<\gamma} M(O_\zeta)$;

(ii) $\xi = 3O_\gamma(2+\eta)$ *for some* η.

In connection with Corollary 55 compare 37.

COROLLARY 56. *If $\gamma = \bigcup\gamma \neq 0$, and either $\alpha \geqslant 3$ and $\beta \geqslant 2$, or $\alpha \geqslant 2$ and $\beta \geqslant 3$, then $\alpha O_\gamma \beta \in \bigcap_{\zeta<\gamma} M(O_\zeta)$.*

Proof: by 54 and 37.

From this corollary it follows immediately that, under its hypothesis, the formulas $\alpha', \beta' < \alpha O_\gamma \beta$ and $\gamma' < \gamma$ always imply $\alpha' O_{\gamma'} \beta' < \alpha O_\gamma \beta$. This consequence is clearly related in its character to the monotony laws of Section 1 and can be compared with Theorem 9 and Corollary 2(iv).

COROLLARY 57. *Whenever $\gamma = \bigcup\gamma \neq 0$, we have $\bigcap_{\zeta<\gamma} M(O_\zeta) \supset M(O_\gamma)$.*

Theorems 43, 52, and Corollary 57 exhaustively describe the inclusion relations between the classes $M(O_\gamma)$.

Appendix

The results of this paper form a base for a study of a number of metamathematical problems concerning the extended arithmetic of ordinals. To formulate these problems precisely, we need some additional notations. For more information concerning the notions used in this appendix the reader may consult [13].

With any given ordinal γ we correlate a definite algebraic structure $\mathfrak{O}_\gamma$ formed by the class Ω of ordinals and the sequence of operations $O_0, \ldots, O_\gamma$; symbolically,

$$\mathfrak{O}_\gamma = \langle \Omega, O_0, \ldots, O_\gamma \rangle = \langle \Omega, O_\xi \rangle_{\xi \leqslant \gamma}\,.$$

By T_γ we denote the elementary (first order) theory of $\mathfrak{O}_\gamma$. In addition, let $\mathfrak{O}$ denote the relational structure formed by Ω and the ordinary $\leqslant$ relation between ordinals, $\mathfrak{O} = \langle \Omega, \leqslant \rangle$, and let T be the elementary theory of $\mathfrak{O}$.

Mostowski and Tarski have jointly obtained a number of results concerning the theory T, some of which are stated in their abstract [10]. In particular they have established the following facts:

(I) Every ordinal $\alpha < \omega^\omega$ is definable in T. In other words, for every $\alpha < \omega^\omega$ there exists in the language of T a formula φ with one free variable such that α is the only ordinal that satisfies φ. Moreover, α is *intrinsically definable* in T in the following sense: the formula φ can be constructed in such a way that the ranges of all bound variables are restricted to ordinals smaller than α. For instance, if $\alpha = 2$, we can take for φ the formula:

$$\exists_{\xi,\eta}[\xi < \alpha \wedge \eta < \alpha \wedge \xi \neq \eta \wedge \forall_\zeta(\zeta < \alpha \rightarrow \zeta = \xi \vee \zeta = \eta)]\,.$$

(II) No ordinal $\alpha \geqslant \omega^\omega$ is definable in T.

(III) There are proper substructures of $\mathfrak{O}$ which are models of T, and the smallest of them is $\langle \omega^\omega, \leqslant \rangle$. Thus, $\mathfrak{O}$ and $\langle \omega^\omega, \leqslant \rangle$ are *elementarily equivalent*; actually $\mathfrak{O}$ is an *elementary extension* of $\langle \omega^\omega, \leqslant \rangle$, in the sense of [14].

(IV) For any two ordinals α and β, the structures $\langle \alpha, \leqslant \rangle$ and $\langle \beta, \leqslant \rangle$ are elementarily equivalent if and only if α and β are congruent modulo ω^ω, i.e., there are ordinals ξ, η, δ such that $\delta < \omega^\omega$, $\alpha = \omega^\omega \cdot \xi + \delta$, $\beta = \omega^\omega \cdot \eta + \delta$, and either $\xi = \eta = 0$ or both $\xi \neq 0$, $\eta \neq 0$.

The main question in which we are interested here concerns whether and in what form the results (I)-(IV) can be extended to the theories T_γ. Tarski has shown that (I) holds for every theory T_γ if the ordinal ω^ω is replaced in it by $\mu(\omega^\omega, O_\gamma)$ everywhere, and, of course, T is replaced by T_γ. He also has conjectured that the same applies, with appropriate changes, to the results (II)-(IV). The changes are as follows: in (II) they are the same as in (I); in (III) $\mathfrak{O}$, T, and $\langle \omega^\omega, \leqslant \rangle$ are respectively replaced by $\mathfrak{O}_\gamma$, T_γ, and $\langle \mu(\omega^\omega, O_\gamma), O_\zeta \rangle_{\zeta \leqslant \gamma}$; in (IV) $\langle \alpha, \leqslant \rangle$ and $\langle \beta, \leqslant \rangle$ are respectively replaced by $\langle \mu(\alpha, O_\gamma), O_\zeta \rangle_{\zeta \leqslant \gamma}$ and $\langle \mu(\beta, O_\gamma), O_\zeta \rangle_{\zeta \leqslant \gamma}$. Everything else in (II)-(IV) remains unchanged.

In [5] Ehrenfeucht confirmed Tarski's conjectures for $\gamma = 0$ and $\gamma = 1$. Actually, he has obtained stronger results by showing that (II)-(IV) apply not only to elementary theories T, T_0, and T_1, but also to the corresponding weak second-order theories (with two kinds of variables, those of the first kind ranging over ordinals, and those of the second kind over finite sets of ordinals).

Doner has extended all the results of Ehrenfeucht to theories T_γ with arbitrary indices and to the corresponding weak second-order theories. A paper with a detailed presentation of Doner's results is being prepared for publication.

In our final remarks we restrict ourselves to structures $\mathfrak{O}_\gamma$ and theories T_γ with finite indices γ. From the results of Ehrenfeucht and Doner (and some other results well known from the metamathematical literature) it easily follows that with every sentence σ in the language of T_γ we can recursively correlate a sentence σ^* with the following properties: σ^* is formulated in the language of T_1, the ranges of all variables are restricted to ordinals $< \omega$, i.e., natural numbers, and the equivalence $\sigma \leftrightarrow \sigma^*$ is valid in T_γ. (Roughly speaking, σ^* is a sentence in the language of elementary number theory which is equivalent to σ.) It seems natural in this context to consider a subtheory T'_γ of T_γ whose construction is entirely analogous to that of Peano's arithmetic, the familiar axiomatic subtheory of elementary number theory. Just as Peano's arithmetic, T'_γ is based upon a recursive axiom system whose main components are recursive definitions of the operations $O_0, \ldots, O_\gamma$ and the schema of induction (in this case, of transfinite induction). Tarski has conjectured that the equivalence $\sigma \leftrightarrow \sigma^*$ is not only valid in T_γ, but also provable in T'_γ. This conjecture has also been confirmed by Doner.

To conclude, we may call to the reader's attention that the problems on the existence of identities discussed in Section 2 have also a "metamathematical flavor" and their solution may require metamathematical methods.

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