Left completely continuous semi-algebras

by

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1. Introduction. A strict semi-algebra A in which for each element a the multiplication operator T_a: $x \to ax$ $(x \in A)$ is completely continuous is called a *left completely continuous* (lcc) *semi-algebra*. Thus, every strict locally compact semi-algebra is lcc. In § 2 we give an example of an lcc semi-algebra which is not locally compact. We also exhibit an example of an lcc semi-algebra A such that the Banach algebra $B = \mathrm{cl}(A-A)$ is not a cc algebra in the sense of Kaplansky [9]. More examples are discussed in § 9.

It turns out that much of the technique developed by Bonsall in [3] for locally compact semi-algebras can be carried over to lcc semi-algebras. In fact, if A is an lcc semi-algebra and a is an element of A whose spectral radius ν is different from zero, then the set $I = \{x \in A\colon ax = \nu x\}$ is a non-zero closed right ideal which is a locally compact semi-algebra. The importance of this observation will become clear when we come to discuss (in § 5) the existence of minimal closed right ideals and minimal idempotents in an lcc semi-algebra.

In § 3 we introduce the concepts of an sa-quasi-regular element and an sa-quasi-regular left (right) ideal which lead us to the definition of the radical in § 4. Our definition of semi-simplicity (i.e., radical $= (0)$) is different from that given by Bonsall in [3], but the two definitions are equivalent for strict locally compact semi-algebras. In § 6 the two definitions are compared and we discuss an example of an lcc semi-algebra which is equal to its radical but is semi-simple in the sense of [3]. In § 7 we study semi-simple lcc semi-algebras in general and in § 8 we restrict ourselves to those that are also commutative and prime.

There is one aspect of the theory of lcc semi-algebras that is not treated here; namely, the representation theory of lcc semi-algebras. Some results have been obtained in this direction, but much more spade work has to be done before one can see how far this aspect of the theory can be developed.

2. Notation and examples. Following [3] and [6] by a *semi-algebra* A we shall mean a non-empty subset A of a Banach algebra B

such that whenever a, b are in A then $ab \in A$ and $\alpha a+\beta b \in A$ for all real scalars $\alpha \geqslant 0$, $\beta \geqslant 0$. An element a of A is said to be *left completely continuous* if the mapping T_a: $x \to ax$ $(x \in A)$ is completely continuous in A, i.e., the sequence $\{T_a x_n\}$ has a convergent subsequence whenever $\{x_n\}$ is a bounded sequence in A. Similarly we define a right completely continuous element of A. If an element a of A is both left and right completely continuous, we shall say that a is *completely continuous*.

A semi-algebra A is said to be *strict* if $x \in A$ and $-x \in A$ imply that $x = 0$. A is called a *locally compact semi-algebra* if the set of x in A with $\|x\| \leqslant 1$ is a compact subset of B.

A semi-algebra A is called a *left completely continuous* semi-algebra if A is a non-zero, strict, closed semi-algebra in which every element is left completely continuous. Similarly we define a right completely continuous semi-algebra. If a semi-algebra A is both left and right completely continuous, we shall say that A is a completely continuous semi-algebra. It is clear that a commutative left completely continuous semi-algebra is completely continuous.

From now on we shall use the abbreviation lcc *semi-algebra* A for a left completely continuous semi-algebra. Throughout R will denote the real number field and R^+ the set of all non-negative real numbers.

We shall now give some examples of lcc semi-algebras. We borrow the notation and terminology from [3] and [6]. As our first example we shall show that if X is a partially ordered Banach space with a complete cone X^+ such that $X = \mathrm{cl}(X^+ - X^+)$, then every positive compact operator gives rise to an lcc semi-algebra. This will follow as a simple consequence of a result due to F. F. Bonsall (Theorem 1 in [5]). As his paper has not yet appeared in print, for the sake of completeness we state his result in Lemma A below and also include the proof as given in [5]. Let X be a Banach space and let $B(X)$ be the Banach algebra of continuous linear operators in X with the usual operator norm. Let t be an element of $B(X)$. The *centralizer* Y of t is the set of all elements of $B(X)$ which commute with t. It is clear that Y is a closed subalgebra of $B(X)$.

LEMMA A. *Let t be a compact linear operator in a Banach space X and let Y be the centralizer of t. Then the mapping $a \to ta$ $(a \in Y)$ is a compact linear operator in Y.*

Proof. Let X_1 denote the closed unit ball in X, and let $E = \overline{tX_1}$. Then E is a compact subset of X in the norm topology. Given $a \in Y$ with $\|a\| \leqslant 1$, we have

$$atX_1 = taX_1 \subset tX_1;$$

and therefore, by continuity, $aE \subset E$. Let $a_n \in Y$, $\|a_n\| \leqslant 1$, $(n = 1, 2, \ldots)$. Then, for each $x \in E$, the set $\{a_n x:\ n = 1, 2, \ldots\}$ is contained in the compact subset E of the Banach space X. Also

$$\|a_n x - a_n x'\| \leqslant \|x - x'\| \quad (x, x' \in E,\ n = 1, 2, \ldots)$$

which shows that the mappings $x \to a_n x$ $(x \in E,\ n = 1, 2, \ldots)$ form an equicontinuous sequence of mappings of the compact space E into the Banach space X. By Ascoli's theorem for Banach space valued functions, it follows that there exists a subsequence $\{a_{n_k}\}$ such that $\{a_{n_k} x\}$ converges uniformly for x in E. Consequently, $\{a_{n_k} tx\}$ converges uniformly for x in X_1, and so $\{a_{n_k} t\}$ converges with respect to the operator norm. Since $a_{n_k} t \in Y$ is closed, this shows that $\{a_{n_k} t\}$ converges in Y. Finally, $ta_n = a_n t$. Thus T_t: $a \to ta$ $(a \in Y)$ is compact in Y, and the proof is complete.

EXAMPLE I. Let T be a compact linear operator in a Banach space X. We shall denote by $P[T]$ the set of all operators in X of the form

$$\sum_{i=1}^{n} a_i T^i$$

with $a_i \geqslant 0$ $(i = 1, 2, \ldots, n)$. Let $A[T]$ be the closure of $P[T]$ with respect to the operator norm. $A[T]$ is the least closed semi-algebra containing T. From Lemma A it follows that, for each $a \in A[T]$, T_a: $x \to ax$ $(x \in A[T])$ is a completely continuous operator in $A[T]$. However, the semi-algebra $A[T]$ may fail to be strict. But if X is a partially ordered Banach space with a complete cone X^+ such that $X = \mathrm{cl}(X^+ - X^+)$ and T is a positive compact linear operator in X, i.e., $TX^+ \subset X^+$, then $A[T]$ is an lcc semi-algebra. In particular, if $X = C([0, 1])$, the Banach space of all continuous real-valued functions on the closed interval $[0, 1]$, and T is the positive compact operator defined by

$$(Tf)(s) = \int_0^s f(t)\,dt\,, \quad 0 \leqslant s \leqslant 1\,,$$

then $A[T]$ is an lcc semi-algebra which is not locally compact. In fact $A[T]$ is a prime radical lcc semi-algebra (cf. Theorem 11) and since every prime locally compact semi-algebra is semi-simple in the sense of [3] as well as in our sense, it follows that $A[T]$ is not locally compact.

EXAMPLE II. It is clear that the semi-algebra $A[T]$ generated by a compact linear operator T gives rise to a completely continuous (cc) algebra $B = \mathrm{cl}(A[T] - A[T])$ in the sense of Kaplansky [9]. As an example of a Banach algebra B which is not a cc algebra and which contains an lcc semi-algebra A such that $B = \mathrm{cl}(A - A)$, consider the set $\mathcal{C}$ of all non-negative increasing convex continuous real-valued functions on the closed interval $[0, 2]$. Let A be the restriction of the functions in $\mathcal{C}$ to the set $E = [0, 1] \cup \{2\}$. Then A is a prime commutative lcc semi-algebra with identity and $C_R(E) = \mathrm{cl}(A-A)$ (cf. Example on p. 68 in [3]). It is easy to see that $C_R(E)$ is not a cc algebra. (In fact, let $\{x_n\}$ be an infinite sequence in E and let $\{f_n\}$ be a sequence in $C_R(E)$ such that $f_n(x_n) = 1$ and $f_n(x_m) = 0$ for $m < n$. Then $\{f_n\}$ is a linearly independent set, and therefore $T_1\colon x \to x$ $(x \in C_R(E))$ is not cc.)

3. Quasi-regular elements in an lcc semi-algebra. Let a be an element in a semi-algebra A. We define the *left partial bound* $p_l(a)$ and the *left partial spectral radius* $\varrho_l(a)$ of a by

$$p_l(a) = \sup\{\|ax\|\colon x \in A \text{ and } \|x\| \leqslant 1\}$$

and

$$\varrho_l(a) = \lim_{n\to\infty} \big(p_l(a^n)\big)^{1/n}.$$

Similarly we define the right partial bound $p_r(a)$ and the right partial spectral radius $\varrho_r(a)$ of a. We shall denote by $\nu(a)$ the spectral radius of a, i.e., $\nu(a) = \lim\limits_{n\to\infty} \|a^n\|^{1/n}$.

LEMMA 1. *Let A be a semi-algebra. Then $\varrho_l(a) = \varrho_r(a) = \nu(a)$ for every a in A.*

Proof. Since

$$\frac{\|a^{n+1}\|}{\|a\|} \leqslant p_l(a^n) \leqslant \|a^n\|$$

and since

$$\lim_{n\to\infty} \frac{\|a^{n+1}\|^{1/n}}{\|a\|^{1/n}} = \nu(a),$$

we have $\varrho_l(a) = \nu(a)$. Similarly we can show that $\varrho_r(a) = \nu(a)$.

DEFINITION. An element a in an lcc semi-algebra A is said to be *left* sa-*quasi-regular* if there exists an element b in A such that $a + ba = b$. The element b is called a *left* sa-*quasi-inverse* of a. Briefly we call a lsa-*quasi-regular* and b lsa-*quasi-inverse*.

Similarly we define an rsa-quasi-regular element and an rsa-quasi-inverse. If an element a in A is both lsa- and rsa-quasi-regular, we shall say that a is sa-quasi-regular. It is easy to check that if $a \in A$ has an lsa-quasi-inverse b and an rsa-quasi-inverse c, then $b = c$. It is also easy to see that if $\nu(a) < 1$ then a is sa-quasi-regular with the sa-quasi-inverse given by $\sum_{n=1}^{\infty} a^n$.

A semi-algebra J contained in a semi-algebra A is called a *left ideal* of A if $AJ \subset J$, a *right ideal* if $JA \subset J$, and a *two-sided-ideal* if it is both a left and a right ideal. An ideal J in an lcc semi-algebra A is called *sa-quasi-regular* if every element of J is sa-quasi-regular. If J is a left (right) ideal which is sa-quasi-regular, we shall say that J is an sa-quasi-regular left (right) ideal.

LEMMA 2. *Let A be an* lcc *semi-algebra. If J is a left (right) ideal of A in which every element is* lsa-(rsa-)*quasi-regular, then J is an* sa-quasi-*regular left (right) ideal.*

Proof. Let J be a left ideal in which every element is lsa-quasi-regular. We shall show that $\nu(a) = 0$ for every $a \in J$. In fact, suppose that $\nu(a) > 0$ for some $a \in J$. By Lemma 1, $\nu(a) = \varrho_l(a)$ and, since a is lcc, by Theorem 1 in [1] there exists a non-zero element u in A such that $au = \nu u$, where $\nu = \nu(a)$. Since αa is lsa-quasi-regular for all $\alpha \geqslant 0$, there exists b in A such that $\frac{1}{\nu}a + \frac{1}{\nu}ba = b$. We have

$$\frac{1}{\nu}au + \frac{1}{\nu}bau = bu, \quad \text{or} \quad u + bu = bu,$$

which gives $u = 0$. This is a contradiction. Hence $\nu(a) = 0$ and consequently a is sa-quasi-regular.

Let J be a right ideal in which every element is rsa-quasi-regular. As above we shall show that $\nu(a) = 0$ for every $a \in J$. Let $a \in J$ and suppose that $\nu(a) > 0$. For $\lambda > \nu = \nu(a)$, let

$$b_\lambda = \frac{1}{\lambda}a + \frac{1}{\lambda^2}a^2 + \ldots$$

Then

$$b_\lambda = \frac{1}{\lambda}a + \frac{1}{\lambda}b_\lambda a = \frac{1}{\lambda}a + \frac{1}{\lambda}ab_\lambda.$$

Since a is lcc, by (4.1) in [1], there exists a decreasing sequence $\{\lambda_n\}$ of real numbers λ_n converging to ν and such that $\|b_{\lambda_n}\| \to \infty$. Let $q_n = (\|b_{\lambda_n}\|)^{-1} b_{\lambda_n}$. By choosing a subsequence we may suppose that $aq_n \to c$. Since b_{λ_n} is the sa-quasi-inverse of a/λ_n and

$$(\|b_{\lambda_n}\|)^{-1}\frac{1}{\lambda_n}a \to 0,$$

we see that $q_n \to \frac{1}{\nu}c$; $c \neq 0$. We have

$$\frac{1}{\nu}c = \lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{1}{\lambda_n} q_n a = \frac{1}{\nu^2}ca, \quad \text{or} \quad \nu c = ca.$$

Since $\frac{1}{\nu}a$ is rsa-quasi-regular, there exists b in A such that $\frac{1}{\nu}a + \frac{1}{\nu}ab = b$. We have $\frac{1}{\nu}ca + \frac{1}{\nu}cab = cb$ or $c + cb = cb$, and so $c = 0$; a contradiction. Thus $\nu(a) = 0$ and therefore a is sa-quasi-regular.

COROLLARY. *Let J be an sa-quasi-regular left (right) ideal. Then for every $a \in J$, $\nu(a) = 0$ and the sa-quasi-inverse of a is given by $\sum_{n=1}^{\infty} a^n$.*

LEMMA 3. *Let I and J be sa-quasi-regular left (right) ideals of an* lcc *semi-algebra A. Then $I+J$ is an sa-quasi-regular left (right) ideal.*

Proof. Suppose that I and J are sa-quasi-regular left ideals. We argue as in [8], p. 302. Let $a \in I$ and $b \in J$, and let a_1 and b_1 be their sa-quasi-inverses, respectively. Since $b+a_1 b \in J$, $b+a_1 b$ is sa-quasi-regular. Let w be its sa-quasi-inverse. It is easy to check that a_1+w+wa_1 is an lsa-quasi-inverse of $a+b$ and therefore, by Lemma 2, it is the sa-quasi-inverse of $a+b$.

A similar proof holds for sa-quasi-regular right ideals I and J. In this case we may take $b+ba_1$.

COROLLARY. *If $\{I_a: a \in \Delta\}$ is a family of sa-quasi-regular left (right) ideals of A. Then the sum $\sum_{a\in\Delta} I_a$ is an sa-quasi-regular left (right) ideal.*

LEMMA 4. *Let a, x be elements of an* lcc *semi-algebra A, and let $\alpha \geqslant 0$. Then $a(\alpha+x)$ is* lsa-(rsa-) *quasi-regular if and only if $(\alpha+x)a$ is* lsa-(rsa-) *quasi-regular.*

Proof. It is easy to see that if y is an lsa-(rsa-) quasi-inverse of $a(\alpha+x)$, then $(\alpha+x)a+(\alpha+x)ya$ is an lsa-(rsa-) quasi-inverse of $(\alpha+x)a$. On the other hand, if z is an lsa-(rsa-) quasi-inverse of $(\alpha+x)a$, then it can be easily checked that $a(\alpha+x)+az(\alpha+x)$ is an lsa-(rsa-) quasi-inverse of $a(\alpha+x)$.

4. The radical of an lcc semi-algebra.

THEOREM 1. *Let A be an* lcc *semi-algebra and let $\mathfrak{R}$ be the set $\{a \in A$: $a(\alpha+x)$ is sa-quasi-regular for all $x \in A$ and all $\alpha \in R^+\}$. Then*

(i) *$\mathfrak{R} = \{a \in A$: $(\alpha+x)a$ is sa-quasi-regular for all $x \in A$ and all $\alpha \in R^+\}$.*

(ii) *$\mathfrak{R}$ is an sa-quasi-regular two-sided ideal and is equal to the sum of all sa-quasi-regular left (right) ideals of A.*

(iii) *$\mathfrak{R}$ is a topologically nil two-sided ideal and is equal to the sum of all topologically nil left (right) ideals of A.*

Proof. (i) follows from Lemma 4.

(ii). By the definition of the set $\mathfrak{R}$, every sa-quasi-regular right ideal belons to $\mathfrak{R}$. By (i), $\mathfrak{R}$ contains also every sa-quasi-regular left ideal of A. By Lemma 3 Corollary, $\mathfrak{R}$ contains the sum of all sa-quasi-regular left (right) ideals. But, if $a \in \mathfrak{R}$, then

$$I_a = \{a(\alpha+x): x \in A \text{ and } \alpha \geqslant 0\} \quad \text{and} \quad J_a = \{(\alpha+x)a: x \in A \text{ and } \alpha \geqslant 0\}$$

are sa-quasi-regular right and left ideals of A, respectively, and belong to $\mathfrak{R}$. Therefore $\mathfrak{R}$ is equal to the sum of all sa-quasi-regular left (right) ideals, and so $\mathfrak{R}$ is an sa-quasi-regular two-sided ideal.

(iii). By (ii) and Lemma 2 Corollary, $\mathfrak{R}$ is a topologically nil two-sided ideal. Since every topologically nil left (right) ideal of A is an sa-quasi-regular left (right) ideal and conversely, it follows that $\mathfrak{R}$ is the sum of all topologically nil left (right) ideals.

DEFINITION 2. The set $\mathfrak{R}$ defined in Theorem 1 is called the *radical of A*. If $A = \mathfrak{R}$, A is called a *radical* lcc *semi-algebra*, and if $\mathfrak{R} = (0)$, A is called a *semi-simple* lcc *semi-algebra*.

THEOREM 2. *Let A be a commutative* lcc *semi-algebra. Then the radical $\mathfrak{R}$ is a closed two-sided ideal of A.*

Proof. Let $a \in \mathrm{cl}(\mathfrak{R})$ and let $\{a_n\}$ be a sequence in $\mathfrak{R}$ such that $a = \lim_{n\to\infty} a_n$. Let b_n be the sa-quasi-inverse of a_n $(n = 1, 2, \ldots)$. Then $-b_n$ is the quasi-inverse of a_n in the Banach algebra B containing A and, since $b_n \in \mathfrak{R}$, $\nu(-b_n) = \nu(b_n) = 0$ $(n = 1, 2, \ldots)$. But $a_n a = a a_n$ $(n = 1, 2, \ldots)$. Therefore, by Theorem (1.4.23) in [11], a is quasi-regular. Similarly we can show that αa is quasi-regular for all $\alpha \geqslant 0$. Hence, by the argument in the first part of the proof of Lemma 2, we obtain $\nu(a) = 0$. Since A is commutative, $a(\alpha+x)$ is sa-quasi-regular for all $x \in A$, $\alpha \geqslant 0$, and so $a \in \mathfrak{R}$.

LEMMA 5. *Let A be a strict closed semi-algebra and let $\mathfrak{A} = A-A$. Let $S = \{x \in A: \|x\| \leqslant 1\}$ and $S^0 = \{\alpha x - \beta y: x, y \in S, \alpha, \beta \in R^+ \text{ and } \alpha+\beta = 1\}$. For each $x \in \mathfrak{A}$ let $\|x\|_A = \inf\{\lambda: \lambda > 0 \text{ and } x \in \lambda S^0\}$. Then*

(i) *$\|x\|_A$ is a norm on $\mathfrak{A}$ with $\|x\|_A = \|x\|$ for all $x \in A$, and $\|x\|_A \geqslant \|x\|$ for all $x \in \mathfrak{A}$.*

(ii) *$\mathfrak{A}$ is complete and A is a closed subset of $\mathfrak{A}$ with respect to $\|x\|_A$.*

(iii) *$\mathfrak{A}$ is a Banach algebra with respect to $\|x\|_A$.*

Proof. (i) and (ii) are proved in [4], pp. 64-65.

(iii). Let us first show that if z_1, z_2 are in S^0, then the product $z_1 z_2 \in S^0$. Let $z_i = \alpha_i x_i - \beta_i y_i$ with $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $\alpha_i+\beta_i = 1$, $x_i, y_i \in S$ $(i = 1, 2)$. Then

$$z_1 z_2 = \alpha_1\alpha_2 x_1 x_2 + \beta_1\beta_2 y_1 y_2 - \beta_1\alpha_2 y_1 x_2 - \alpha_1\beta_2 x_1 y_2$$

and

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \alpha_1\beta_2 + \alpha_2\beta_1 = 1 .$$

Let $\alpha_1\alpha_2+\beta_1\beta_2 = \beta$. If $\beta = 0$, then clearly $z_1 z_2 \in S^0$. If $\beta \neq 0$, we can write

$$z_1 z_2 = \beta\left(\frac{\alpha_1\alpha_2}{\beta} x_1 x_2 + \frac{\beta_1\beta_2}{\beta} y_1 y_2\right) - (1-\beta)\left(\frac{\alpha_1\beta_2}{1-\beta} x_1 y_2 + \frac{\alpha_2\beta_1}{1-\beta} y_1 x_2\right)$$

and it is easily verified that $z_1 z_2 \in S^0$. We have $\|z_1 z_2\|_A \leqslant 1$. Now let z_1, z_2 be any elements of $\mathfrak{A}$, and let λ, μ be positive real numbers such that

$\lambda^{-1}z_1 \in S^0$, $\mu^{-1}z_2 \in S^0$. Then, by what we have just proved and the fact that $\|\cdot\|_A$ is a norm in $\mathfrak{A}$, we have $\|z_1z_2\|_A \leqslant \lambda\mu$. It is now quite simple to conclude that $\|z_1z_2\|_A \leqslant \|z_1\|_A\|z_2\|_A$, so that $\mathfrak{A}$ is a Banach algebra.

DEFINITION 3. Let A be a semi-algebra. The norm $\|x\|_A$ defined in Lemma 5 is called the *cone norm* of the Banach algebra $\mathfrak{A}$.

THEOREM 3. *Let A be an* lcc *semi-algebra. Let $\mathfrak{R}$ be the radical of A and $\mathfrak{N}$ the radical of the Banach algebra $\mathfrak{A} = A - A$ with the cone norm $\|x\|_A$. Then $\mathfrak{R} = \mathfrak{N} \cap A$.*

Proof. Let $a \in \mathfrak{R}$, $x \in \mathfrak{A}$ and α a real number. We shall show that $a(\alpha + x)$ is a quasi-regular element of $\mathfrak{A}$. Let $x = x_1 - x_2$ with $x_1, x_2 \in A$, and write $\alpha = \alpha_1 - \alpha_2$ with $\alpha_1, \alpha_2 \in R^+$. Then

$$a(\alpha + x) = a(\alpha_1 + x_1) - a(\alpha_2 + x_2)\,.$$

Let $a_1 = a(\alpha_1 + x_1)$ and $a_2 = -a(\alpha_2 + x_2)$, and let $\nu_A(z)$ denote the spectral radius of z in $\mathfrak{A}$. Since $\|z\|_A = \|z\|$ for $z \in A$, we have $\nu_A(a_1) = \nu_A(a_2) = 0$. Therefore a_1 and a_2 are quasi-regular elements of $\mathfrak{A}$; in particular, a_1 is sa-quasi-regular. Let b_1 be the sa-quasi-inverse of a_1 and b_2 the quasi-inverse of a_2. Since $a(\alpha_2 + x_2) + a(\alpha_2 + x_2)b_1 \in \mathfrak{R}$, $\nu_A(a_2 + a_2b_1) = 0$. Thus $a_2 + a_2b_1$ is quasi-regular. Let w be the quasi-inverse of $a_2 + a_2b_1$. Using the fact that $a_1 + a_1b_1 = b_1$, it is easy to show that $-b_1 + w + b_1w$ is a right quasi-inverse of $a_1 + a_2 = a(\alpha + x)$. Similarly we can show that $-b_1 + w_1 + w_1b_1$ is a left quasi-inverse of $a(\alpha + x)$, where w_1 is the quasi-inverse of $a_2 + b_1a_2$. Thus $a(\alpha + x)$ is quasi-regular for all $x \in \mathfrak{A}$ and all real numbers α, and so $a \in \mathfrak{N}$. Hence $\mathfrak{R} \subset \mathfrak{N} \cap A$. On the other hand if $a \in \mathfrak{N} \cap A$, then $\nu\big(a(\alpha + x)\big) = \nu_A\big(a(\alpha + x)\big) = 0$ for all $x \in A$ and all $\alpha \geqslant 0$, and so $a \in \mathfrak{R}$. Thus $\mathfrak{R} = \mathfrak{N} \cap A$.

COROLLARY. *In the cone norm topology the radical of an* lcc *semi-algebra is a closed two-sided ideal.*

5. Existence of minimal idempotents in an lcc semi-algebra. An idempotent e in a semi-algebra A is said to be a *minimal idempotent* if eA is a minimal closed right ideal (cf. [3], p. 52).

THEOREM 4. *Let A be a non-radical* lcc *semi-algebra. Then A contains a minimal closed right ideal which is a locally compact semi-algebra.*

Proof. Since A is not equal to its radical, there exists an element a in A with $\nu(a) > 0$. By Theorem 1 in [1] and our Lemma 1, there exists a non-zero element u in A such that $au = \nu u$, where $\nu = \nu(a)$. Let $I = \{x \in A\colon ax = \nu x\}$. Then I is a non-zero closed right ideal and, since a is lcc, it is also a locally compact semi-algebra. By an argument analogous to that used in the proof of Lemma 1 in [3], we can show that A contains a minimal closed right ideal which belongs to I and therefore is locally compact.

COROLLARY 1. *Every minimal closed right ideal of A not contained in the radical is a locally compact semi-algebra.*

COROLLARY 2. *Every non-zero right ideal of A not belonging to the radical contains a minimal closed right ideal which is a locally compact semi-algebra.*

LEMMA 6. *Let E be a closed subset of an* lcc *semi-algebra A such that $\alpha E \subset E$ $(\alpha \geqslant 0)$ and such that $S(E) = \{x \in E\colon \|x\| \leqslant 1\}$ is a compact subset of the Banach algebra B containing A. Let a be an element of A such that $(a)_r \cap E = (0)$, where $(a)_r$ is the right annihilator of a in A. Then aE is a closed subset of A (and of B).*

Proof. The same as that of Lemma 2 in [3].

LEMMA 7. *Let A be an* lcc *semi-algebra and let M be a minimal closed right ideal in A. If M is a locally compact semi-algebra and $M^2 \neq (0)$, then M contains an idempotent e and $M = eA$.*

Proof. Since $M^2 \neq (0)$, there exists an element $a \in M$ with $aM \neq (0)$. Therefore the closed right ideal $(a)_r \cap M \neq M$. But M is a minimal closed right ideal. Terefore $(a)_r \cap M = (0)$. Since M is locally compact, by Lemma 6, aM is a closed right ideal; moreover $(0) \neq aM \subset M$. Therefore $aM = M$ and so there exists an element $e \in M$ such that $ae = a$. We may now follow the argument given in the proof of Theorem 1 in [3] to show that e is an idempotent. Clearly $M = eA$.

THEOREM 5. *Let A be an* lcc *semi-algebra and let $\mathfrak{R}$ be the radical of A. If I is a non-zero right ideal of A such that $\mathfrak{R} \cap I = (0)$, then I contains a minimal idempotent.*

Proof. By Theorem 4 Corollary 2, I contains a minimal closed right ideal M of A which is locally compact. Since $M \cap \mathfrak{R} \subset I \cap \mathfrak{R} = (0)$, $M^2 \neq (0)$ and so, by Lemma 7, there exists an idempotent e in I such that $M = eA$.

COROLLARY. *If A is a semi-simple* lcc *semi-algebra, then every non-zero right ideal of A contains a minimal idempotent.*

THEOREM 6. *Let e be a minimal idempotent in an* lcc *semi-algebra A. Then eAe a closed division semi-algebra; $eAe = R^+e$.*

Proof. Let $A_0 = eAe$. Then A_0 is a closed semi-algebra with unit element e (cf. proof of Theorem 2 in [3]). To show that A_0 is a division semi-algebra, let eae be a non-zero element of A_0 and let $(eae)_r$ be the right annihilator of eae in A. Then, since eA is a minimal closed right ideal of A and $e \notin (eae)_r$, we have

$$eA \cap (eae)_r = (0)\,.$$

Since eA is locally compact, Lemma 6 and the minimality property of eA together with the fact that $(0) \neq (eae)eA \subset eA$ imply that $(eae)eA = eA$.

Hence

$$eaeA_0 = A_0\,,$$

and therefore every non-zero element of A_0 has a right inverse. Using the usual group theoretic argument we can show that every non-zero element of A_0 has an inverse, and so A_0 is a division semi-algebra. From Theorem 3 in [3] it follows now that $eAe = R^+e$, and this completes the proof.

Later on (Theorem 13) we shall show that, if A is semi-simple, the converse of Theorem 6 also holds.

6. *w*-semi-simple lcc semi-algebras. In [3] a semi-algebra A was defined to be semi-simple if the zero ideal is the only closed two-sided ideal J with $J^2 = (0)$. An lcc semi-algebra with this property will be called *w-semi-simple.*

A semi-algebra A is said to be *prime* if $IJ \neq (0)$ whenever I and J are non-zero closed two-sided ideals of A.

THEOREM 7. *In a w-semi-simple* lcc *semi-algebra A the radical is a closed two-sided ideal.*

Proof. Let $\mathfrak{R}$ be the radical of A. The theorem is true for $\mathfrak{R} = (0)$ and $\mathfrak{R} = A$. So assume that $(0) \neq \mathfrak{R} \neq A$ and that $\mathfrak{R} \neq \mathrm{cl}(\mathfrak{R})$. Then, by Theorem 4 Corollary 2, $\mathrm{cl}(\mathfrak{R})$ contains a locally compact minimal closed right ideal M of A and, since A is w-semi-simple, Lemma 3 in [3] gives $M^2 \neq (0)$. Therefore, by Lemma 7, $\mathrm{cl}(\mathfrak{R})$ contains a minimal idempotent e and $M = eA$. Now, by Theorem 6, $e\mathfrak{R}e \subset R^+e$ and, since $e \notin \mathfrak{R}$ and $e\mathfrak{R}e \subset \mathfrak{R}$, we obtain $e\mathfrak{R}e = (0)$. Therefore $(e\mathfrak{R})^2 = (0)$ and so, by Lemma 3 in [3], $e\mathfrak{R} = (0)$. This shows that $e\,\mathrm{cl}(\mathfrak{R}) = (0)$ and in particular that $e^2 = 0$; a contradiction. Hence $\mathfrak{R}$ is closed.

THEOREM 8. *A w-semi-simple strict locally compact semi-algebra A is semi-simple.*

Proof. Let $\mathfrak{R}$ be the radical of A and suppose that $\mathfrak{R} \neq (0)$. By Theorem 7, $\mathfrak{R}$ is a closed two-sided ideal of A. Therefore, by Lemma 1 in [3], $\mathfrak{R}$ contains a minimal closed right ideal M of A and, by Lemma 3 in [3], $M^2 \neq (0)$. But, by Theorem 1 in [3] (or Lemma 7), this means that $\mathfrak{R}$ contains a minimal idempotent e, which is obviously impossible. Hence $\mathfrak{R} = (0)$.

THEOREM 9. *Let A be an* lcc *semi-algebra. If A is non-radical and w-semi-simple, then A contains a minimal idempotent.*

Proof. If A is different from its radical, Theorem 4 implies that A contains a locally compact minimal closed right ideal M. And if A is w-semi-simple, Lemma 3 in [3] gives $M^2 \neq (0)$. Therefore, by Lemma 7, A contains a minimal idempotent.

LEMMA 8. *Let A be a w-semi-simple* lcc *semi-algebra and let $\mathfrak{R}$ be its radical. Then for every minimal idempotent e in A we have $\mathfrak{R}e = e\mathfrak{R} = (0)$.*

Proof. If say $\mathfrak{R}e \neq (0)$, then, by Lemma 3 in [3], $(\mathfrak{R}e)^2 \neq (0)$. Therefore there exists $a \in \mathfrak{R}$ such that $eae \neq 0$. But, by Theorem 6, $eae = \lambda e$, $\lambda \in R^+$, so that $e \in \mathfrak{R}$; a contradiction. Hence $\mathfrak{R}e = (0)$. Similarly we can show that $e\mathfrak{R} = (0)$.

THEOREM 10. *Let A be a prime* lcc *semi-algebra. Then either A is equal to its radical $\mathfrak{R}$ or A is semi-simple.*

Proof. Since A is prime, A is w-semi-simple. If $A \neq \mathfrak{R}$, then, by Theorem 9, A contains a minimal idempotent e and, by Lemma 8, $\mathfrak{R}e = (0)$. Therefore $\mathfrak{R}(\overline{AeA}) = (0)$ and so, by the primeness of A, $\mathfrak{R} = (0)$. It follows now that if $\mathfrak{R} \neq (0)$, then $\mathfrak{R} = A$.

THEOREM 11. *Let X be the Banach space $C_R([0,1])$ of all continuous real-valued functions on the closed interval $[0,1]$, and let T be the compact linear operator on X defined by*

$$(Tf)(s) = \int_0^s f(t)\,dt\,, \qquad 0 \leqslant s \leqslant 1\,.$$

Then the closed semi-algebra $A[T]$ generated by T is a prime, radical lcc *semi-algebra.*

Proof. Let $E = [0,1]$. Since

$$(T^n f)(s) = \frac{1}{(n-1)!}\int_0^s (s-t)^{n-1} f(t)\,dt \qquad (s \in E,\ f \in X,\ n = 1, 2, \ldots)\,,$$

we obtain

$$(1)\qquad |(T^n f)(s)| \leqslant \frac{\|f\|}{n!}s^n \qquad (s \in E,\ f \in X,\ n = 1, 2, \ldots)\,,$$

which implies that $n!\|T^n\| \leqslant 1$. Now if e is the identity function, $e(t) = 1\,(t \in E)$, then

$$(2)\qquad (T^n e)(s) = \frac{s^n}{n!} \qquad (s \in E,\ n = 1, 2, \ldots)\,.$$

It follows that

$$(3)\qquad n!\|T^n\| = 1 \qquad (n = 1, 2, \ldots)\,.$$

Let l be the usual Banach space of all real sequences $\Phi = \{\zeta_n\}$ with $\|\Phi\| = \sum_{n=1}^{\infty}|\zeta_n| < \infty$, and let l^+ be the subset of l consisting of all sequences $\{\zeta_n\}$

with $\zeta_n \geq 0$ $(n = 1, 2, \ldots)$. We shall show that every element a in $A[T]$ is of the form

$$a = \sum_{n=1}^{\infty} a_n n! T^n \quad \text{with} \quad \{a_n\} \in l^+ .$$

In fact, for each $\Phi = \{a_n\} \in l^+$, equality (3) shows that $a_\Phi = \sum_{n=1}^{\infty} a_n n! T^n$ exists and

$$\|a_\Phi\| \leq \sum_{n=1}^{\infty} a_n n! \|T^n\| = \sum_{n=1}^{\infty} a_n ;$$

clearly $a_\Phi \in A[T]$. But equality (2) implies that $(a_\Phi e)(1) = \sum_{n=1}^{\infty} a_n$. Therefore

$$\|a_\Phi\| = \|\Phi\| = \sum_{n=1}^{\infty} a_n .$$

Let $u \in A[T]$. Then there exists a sequence $\{u_n\}$ in $P[T]$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$, and each u_n is of the form a_{Φ_n}, where $\Phi_n \in l^+$ and has only finitely many non-zero terms. Since $\|\Phi_n\| = \|u_n\|$, $\{\Phi_n\}$ is bounded and so, by the w^*-compactness of the closed unit ball in l, there is a w^*-cluster point Φ of $\{\Phi_n\}$; $\Phi \in l^+$. We shall now show that $u = a_\Phi$. For $f \in X$ and $0 \leq s < 1$, inequality (1) gives $n!(T^n f)(s) \to 0$. So for $f \in X$ and s in $0 \leq s < 1$ let $x_{f,s} = \{n!(T^n f)(s)\}$. Then the sequence $x_{f,s} \in c_0$ and $(u_n f)(s) = (a_{\Phi_n} f)(s) = \Phi_n(x_{f,s})$. Since Φ is a w^*-cluster point of $\{\Phi_n\}$, $\Phi(x_{f,s})$ is a cluster point of $\{\Phi_n(x_{f,s})\}$. But $\Phi_n(x_{f,s}) \to (uf)(s)$. Therefore $(uf)(s) = \Phi(x_{f,s}) = (a_\Phi f)(s)$ for all $0 \leq s < 1$ and $f \in X$ and so, by the continuity of the functions $(uf)(s)$ and $(a_\Phi f)(s)$ in E, we obtain $a_\Phi = u$. Thus every element a in $A[T]$ is of the form a_Φ with $\Phi \in l^+$. It follows that the mapping $\Phi \to a_\Phi$ is an isometric isomorphism of the semi-algebra l^+ onto the semi-algebra $A[T]$. (The multiplication operation in l^+ is the usual multiplication for series). This shows in partucilar that $A[T]$ is prime. Since T is a positive linear operator acting in a Banach lattice X, $A[T]$ is strict. Therefore $A[T]$ is a prime strict semi-algebra and, T being a compact operator, Lemma A implies that $A[T]$ is lcc. Since $\nu(T) = 0$, we have $\nu(a) = 0$ for every $a \in A[T]$ and so $A[T]$ is equal to its radical. This completes the proof of the theorem.

Remark. The proof of Theorem 11 given here was suggested to me by F. F. Bonsall. It is easy to see that the operator T above is a reducible positive operator (in the sense of [6]) acting in the Banach lattice $C_R([0, 1])$.

7. Semi-simple lcc semi-algebras.

THEOREM 12. *Let e be a minimal idempotent in a semi-simple* lcc *semi-algebra A. Then the least closed two-sided ideal I containing e is a minimal closed two-sided ideal.*

Proof. Let J be a closed two-sided ideal of A contained in I, and suppose that $J \neq I$. Then $e \notin J$ and so $eAJ \subset eA \cap J = (0)$. Thus $eA \subset J_l$, where J_l is the left annihilator of J. Since J_l is a closed two-sided ideal of A and $e \in J_l$, $I \subset J_l$. Therefore $J^2 \subset IJ \subset J_l J = (0)$, and so, since A is semi-simple, $J = (0)$. Thus I is a minimal closed two-sided ideal.

THEOREM 13. *Let e be an idempotent in a semi-simple* lcc *semi-algebra A. Then e is a minimal idempotent if and only if eAe is a division semi-algebra.*

Proof. If e is a minimal idempotent, then, by Theorem 6, eAe is a division semi-algebra; $eAe = R^+ e$. Suppose that eAe is a division semi-algebra, and let I be a non-zero right ideal contained in eA. Since $I^2 \neq (0)$, there exist elements ea and eb in I such that $eaeb \neq 0$. In particular $eae \neq 0$. Let c be an element of eAe such that $eaec = e$. Then

$$eA \supseteq I \supseteq eaA \supseteq eaecA = eA .$$

Thus $I = eA$ and so e is a minimal idempotent.

THEOREM 14. *Every non-zero left (right) ideal in a semi-simple* lcc *semi-algebra A contains a minimal idempotent.*

Proof. Since A is semi-simple, by Theorem 5 Corollary, every non-zero right ideal of A contains a minimal idempotent. Let J be a non-zero left ideal of A and let a be an element of J with $\nu(a) = 1$. Then $I = \{x \in A: ax = x\}$ is a non-zero closed right ideal (see proof of Theorem 4) and so contains a minimal idempotent, say f. Since $faf = f$, we have $(fa)^2 = fa$, so that fa is an idempotent, and $fa \in J$. The equality $(fa)A = fA$ shows that fa is a minimal idempotent.

LEMMA 9. *Let e be a minimal idempotent in a semi-simple* lcc *semi-algebra A. Then, for every $a \in A$, Aea (aeA) is either the (0) ideal or a minimal left (right) ideal.*

Proof. Suppose $Aea \neq (0)$. Since eAe is a division semi-algebra, by the argument of the proof of Lemma (2.1.8) in [11] it follows that Aea is a minimal left ideal. Similarly we can show that if $aeA \neq (0)$, then it is a minimal right ideal.

COROLLARY. *Let $\mathfrak{J}$ be the set of all minimal idempotents in A, and let $\mathfrak{J}A$ denote the set of all finite sums $e_1 a_1 + \ldots + e_n a_n$ with $e_i \in \mathfrak{J}$ and $a_i \in A$. Then $\mathfrak{J}A$ is a two-sided ideal of A.*

THEOREM 15. *Let A be a semi-simple* lcc *semi-algebra, and let e be an idempotent in A. Then eA is a minimal right ideal if and only if Ae is a minimal left ideal.*

Proof. If eA is a minimal right ideal, then, by Lemma 9, Ae is a minimal left ideal. Suppose that Ae is a minimal left ideal and let f be a minimal idempotent in Ae. Then $Ae = Af$ and so $e = af$ for some

$a \in A$. The equality $eA = afA$ and Lemma 9 show that e is a minimal idempotent.

LEMMA 10. *Let e, f be minimal idempotents in a semi-simple* lcc *semi-algebra A. Then $eAf \neq (0)$ if and only if e and f belong to the same minimal closed two-sided ideal.*

Proof. The same as that of Lemma 5 in [3].

LEMMA 11. *Let e, f be minimal idempotents in a semi-simple* lcc *semi-algebra A. Then there is an element u in eAf such that $eAf = R^+u$.*

Proof. We follow the proof of Lemma 6 in [3]. If e, f belong to different minimal closed two-sided ideals then, by Lemma 10, $eAf = (0)$ and we take $u = 0$. If e, f belong to the same minimal closed two-sided ideal then, by Lemma 10, $eAf \neq (0)$, and so there exists an element a of A with $eaf \neq 0$. Since $fAf = R^+f$ and $eA = eafA$, we have $eAf = eafAf = eaf(fAf) = eaf(R^+f) = R^+eaf = R^+u$, where $u = eaf$.

THEOREM 16. *Let $\{M_\alpha\colon \alpha \in \Delta\}$ be the family of all minimal closed two-sided ideals of a semi-simple* lcc *semi-algebra A. Let $\mathfrak{J}$ be the set of all minimal idempotents in A and let $\mathfrak{J}_\alpha = \mathfrak{J} \cap M_\alpha$ $(\alpha \in \Delta)$. Then the following statements are true:*

(i) *The sets $\mathfrak{J}_\alpha$ are mutually dispoint and non-empty, and their union is $\mathfrak{J}$.*

(ii) *For each pair of elements e, f in $\mathfrak{J}_\alpha$ there exists a non-zero element $w_{e,f}$ in eAf such that $eAf = R^+w_{e,f}$; moreover, for all elements e, f, g, h in $\mathfrak{J}_\alpha$, $w_{e,f}w_{g,h} = \lambda w_{e,h}$ for some $\lambda \geqslant 0$. Also $J_\alpha = \sum_{e,f \in \mathfrak{J}_\alpha} R^+w_{e,f}$ is a two-sided ideal of A contained in M_α, and $\mathrm{cl}(J_\alpha) = M_\alpha$.*

(iii) $\mathfrak{J}_\alpha A = A\mathfrak{J}_\alpha = \mathfrak{J}_\alpha A \mathfrak{J}_\alpha = J_\alpha$.

Proof. (i). That each $\mathfrak{J}_\alpha$ is not empty is clear from Theorem 5 Corollary. Also, since $M_\alpha \cap M_\beta = (0)$ for $\alpha \neq \beta$, $\mathfrak{J}_\alpha \cap \mathfrak{J}_\beta = \emptyset$, and, by Theorem 12, $\mathfrak{J} = \bigcup_{\alpha \in \Delta} \mathfrak{J}_\alpha$.

(ii). The first assertion follows from Lemma 11 and the second is verified in the proof of Theorem 7 in [3]. It is clear that J_α is a semi-algebra contained in M_α. To show that it is an ideal of A, let $a \in A$ and $e, f \in \mathfrak{J}_\alpha$. If $fa = 0$ and $ae = 0$, then clearly $aw_{e,f}$ and $w_{e,f}a \in J_\alpha$. Suppose that $fa \neq 0$. Then, since A is semi-simple, $Afa \neq (0)$ and so, by Theorem 14 and Lemma 9, $Afa = Ap$ for some minimal idempotent p. Since $Afa \subset M_\alpha$, $p \in \mathfrak{J}_\alpha$. We have $eAfa = eAp = R^+w_{e,p} \subset J_\alpha$. Similarly, if $ae \neq 0$, then $aeAf = qAf = R^+w_{q,f} \in J_\alpha$, where q is a minimal idempotent in $\mathfrak{J}_\alpha$. Thus $aw_{e,f}$ and $w_{e,f}a$ belong to J_α, and so J_α is a two-sided ideal of A. Since $\mathrm{cl}(J_\alpha)$ is a closed two-sided ideal of A contained in M_α, the minimality property of M_α gives $\mathrm{cl}(J_\alpha) = M_\alpha$.

(iii). We show that $\mathfrak{J}_\alpha A$ is a two-sided ideal of A. Since each element of $\mathfrak{J}_\alpha A$ is of the form $e_1a_1 + \ldots + e_na_n$ with $e_i \in \mathfrak{J}_\alpha$ and $a_i \in A$ $(i = 1, 2, \ldots, n)$, it clearly suffices to prove the following: given $e \in \mathfrak{J}_\alpha$ and $a \in A$, there exist $f \in \mathfrak{J}_\alpha$ and $b \in A$ such that $ae = fb$. In fact, this is immediate if $ae = 0$, for then we can take $f = e$ and $b = 0$. If $ae \neq 0$, then, by Theorem 14 and Lemma 9, $aeA = fA$ for some minimal idempotent f and $f \in \mathfrak{J}_\alpha$ because $aeA \subset M_\alpha$; whence the assertion. Thus $\mathfrak{J}_\alpha A$ is a two-sided ideal of A, and similarly we can show that $A\mathfrak{J}_\alpha$ and $\mathfrak{J}_\alpha A \mathfrak{J}_\alpha$ are two-sided ideals of A. Also, each of these ideals, as well as J_α, is the smallest ideal containing $\mathfrak{J}_\alpha$.

LEMMA 12. *Let A be a semi-simple commutative* lcc *semi-algebra. Then every minimal closed two-sided ideal I of A contains a single minimal idempotent e and $I = eA = eAe = R^+e$.*

Proof. Let e, f be minimal idempotents in I. Since A is commutative, $I = eA = eAe = R^+e = fA$. Thus $f = \lambda e$ and, since $f^2 = f$, $\lambda = 1$.

THEOREM 17. *Let A be a semi-simple commutative* lcc *semi-algebra, and let $\mathfrak{J} = \{e_\alpha\colon \alpha \in \Delta\}$ be the set of all minimal idempotents in A. If a is a non-zero element of A, then $\nu(a) > 0$ and for each e_α in $\mathfrak{J}$ there exists a non-negative real number λ_α such that $ae_\alpha = \lambda_\alpha e_\alpha$. Moreover, $\sup\{\lambda_\alpha\colon \alpha \in \Delta\} = \nu(a)$ and there exists an e_{α_0} in $\mathfrak{J}$ such that $ae_{\alpha_0} = \nu(a)e_{\alpha_0}$.*

Proof. Since A is commutative and semi-simple, $\nu(a) > 0$ for every non-zero $a \in A$ and, by Theorem 6, $ae_\alpha = e_\alpha a e_\alpha = \lambda_\alpha e_\alpha$ with $\lambda_\alpha \geqslant 0$ $(e_\alpha \in \mathfrak{J})$ Since $\|(ae_\alpha)^n\|^{1/n} \leqslant \|a^n\|^{1/n}\|e_\alpha\|^{1/n}$ and $\|e_\alpha\| \geqslant 1$, we obtain $\lambda_\alpha \leqslant \nu(a)$, which implies that $\sup\{\lambda_\alpha\colon \alpha \in \Delta\} \leqslant \nu(a)$. To show that we actually have $\sup\{\lambda_\alpha\colon \alpha \in \Delta\} = \nu(a)$, we shall prove that there exists an $e_{\alpha_0} \in \mathfrak{J}$ such that $ae_{\alpha_0} = \nu(a)e_{\alpha_0}$. Now, by Theorem 1 in [1], for every non-zero $a \in A$ there exists a non-zero $u \in A$, such that $au = \nu(a)u$; moreover, the commutative and semi-simplicity properties of A imply that $(\mathfrak{J}A)u \neq (0)$. Therefore there exists e_{α_0} in $\mathfrak{J}$ and μ in R^+, $\mu \neq 0$, such that $e_{\alpha_0}u = \mu e_{\alpha_0}$. Thus

$$\mu a e_{\alpha_0} = \nu(a) u e_{\alpha_0} = \nu(a)\mu e_{\alpha_0}, \quad \text{and so} \quad ae_{\alpha_0} = \nu(a)e_{\alpha_0}.$$

COROLLARY 1. *Let A be a semi-simple commutative* lcc *semi-algebra. Then, for every non-zero element a in A, the set of minimal idempotents e in A for which $ae = \nu(a)e$ is non-empty and finite.*

Proof. For every $a \neq 0$, by Theorem 17, $\nu(a) > 0$ and the set of minimal idempotents e with $ae = \nu(a)e$ is not empty. Moreover, since the ideal $\{x \in A\colon ax = \nu(a)x\}$ is locally compact, by Theorem 6 in [3] and Lemma 12, this set is finite.

COROLLARY 2. *If A is a semi-simple commutative prime* lcc *semi-algebra, then A contains a single minimal idempotent p and $ap = \nu(a)p$ for every a in A.*

Proof. Suppose that e, f are distinct minimal idempotents in A. Then, by Theorem 12 and Lemma 12, A contains two distinct minimal closed ideals $I = eA$ and $J = fA$, and $IJ = (0)$, contradicting the primeness of A. Hence there exists only one minimal idempotent p in A, and, by Theorem 17, $ap = \nu(a)p$ for all $a \in A$.

8. Semi-simple commutative prime lcc semi-algebras.

LEMMA 13. *Let A be a semi-simple commutative prime* lcc *semi-algebra. Then there exists a positive real number M such that, for every $a \in A$,*

$$\|a\| \leqslant M\nu(a) .$$

Proof. By Theorem 17 Corollary 2, A contains a single minimal idempotent p and $ap = \nu(a)p$ for every a in A. Since p is lcc and $p^2 = p$, the set $W = \{pa:\ a \in A \text{ and } \|a\| = 1\}$ is a closed compact subset of A (and hence of the Banach algebra B containing A), and $0 \notin W$. Let

$$\varkappa = \inf\{\|x\|:\ x \in W\} .$$

Then

$$\varkappa = \inf\{\|px\|:\ x \in W\} .$$

By the compactness of W, $\varkappa$ is attained and, by Theorem 17 Corollary 2, $px \neq 0$ for every x in W, and so $\varkappa > 0$. We have

$$\varkappa\|a\| \leqslant \|pa\| = \nu(a)\|p\| ,$$

and the proof is complete.

THEOREM 18. *Let A be a semi-simple commutative prime* lcc *semi-algebra, and let $S = \{a \in A:\ \nu(a) = 1\}$. Then*

(i) *$S = \{a \in A:\ ap = p\}$, where p is the unique minimal idempotent in A;*

(ii) *S is a bounded closed convex set;*

(iii) *S is a semi-group with respect to the multiplication given in A;*

(iv) *S is a base for A in the sense that every non-zero element a of A has a unique representation in the form $a = \lambda s$ with $\lambda > 0$ and $s \in S$.*

Proof. (i). By Theorem 17 Corollary 2, $ap = \nu(a)p$ for all a in A, and so $ap = p$ if and only if $\nu(a) = 1$.

(ii). By (i), S is a closed convex subset of A and hence of B. By Lemma 13, S is bounded.

(iii). That S is a semi-group is immediate from (i).

(iv). Let a be a non-zero element of A. Then $\nu(a) \neq 0$, and so $a = \lambda s$ with $\lambda = \nu(a)$ and $s = \lambda^{-1}a \in S$. The uniqueness of the representation is easily verified.

THEOREM 19. *Let A and S be as in Theorem 18. Let $a \in S$ and let $G(a)$ be the set of all limit points of the sequence $\{a^n\}$. Then $G(a)$ is a non-empty compact abelian group contained in the semi-group S. If u is the identity element of $G(a)$, then $G(a)$ is the closure of the set $\{a^k u:\ k = 1, 2, \ldots\}$.*

Proof. We recall that an element b in A is a limit point of the sequence $\{a^n\}$ if for every $\varepsilon > 0$ there exists a strictly increasing sequence of integers, $n_1, n_2, \ldots$ such that $\|a^{n_k} - b\| < \varepsilon$ $(k = 1, 2, \ldots)$. (In [6], such a point b is called a *cluster point*.) Since S is a semi-group and $a \in S$, $a^n \in S$ $(n = 1, 2, \ldots)$ and the sequence $\{a^n\}$ is bounded. Now either (1) $\{a^n\}$ has only a finite number of distinct terms (this happens when $a^m = a^n$ for some positive integers m, n), or (2) $\{a^n\}$ has all its terms different. In case (1), $G(a)$ is clearly a non-empty closed set. In case (2), the lcc property of the element a and the boundedness of the sequence $\{a^n\}$ show that $G(a)$ is not empty. Since we are working in a metric topology, $G(a)$ is the derived set of $\{a^n\}$ and therefore is closed. By the argument given in the proof of Theorem 4 in [6], $G(a)$ is a group.

Let u be the identity of $G(a)$. Since u is lcc and $ux = x$ $(x \in G(a))$ and since $G(a)$ is a closed subset of A (and hence of the Banach algebra B containing A), it follows that $G(a)$ is compact. That $G(a)$ is the closure of the set $\{a^k u:\ k = 1, 2, \ldots\}$ follows from the proof of Theorem 8(v) in [6]. (In case (1), this is immediate.)

COROLLARY. *Every element t of $G(a)$ generates a prime strict locally compact semi-algebra $A[t]$. Moreover, if p is the unique minimal idempotent in $A[t]$, then*

$$p = \lim_{n\to\infty} \frac{1}{n}(t + t^2 + \ldots + t^n) .$$

Proof. It is clear that $A[t]$ is strict and prime. Also, since $tu = t$, $A[t]$ is locally compact. Let

$$t_n = \frac{1}{n}(t + t^2 + \ldots + t^n)$$

and let $S[t]$ be the set of all elements s of A such that $sp = p$. Then, by Theorem 18 and the fact that A is locally compact, $S[t]$ is a convex compact semi-group. Since $t \in S[t]$, $t^n \in S[t]$ $(n = 1, 2, \ldots)$ and, since $S[t]$ is compact, the sequence $\{t_n\}$ has at least one limit point. Let q be a limit point of $\{t_n\}$. We have

$$tt_n - t_n = n^{-1}(t^{n+1} - t) ,$$

so that $\lim(tt_n - t_n) = 0$. Thus $tq = q$. Moreover, the sequence $\{\|t^n\|\}$ is bounded. Therefore, by Lemma 1 in [6], $pq = \alpha q$, $\alpha > 0$. Since p is idempotent, $\alpha = 1$, and so $pq = q$. But $pq = p$, since $q \in S[t]$. Therefore $p = q$.

It follows now that the sequence $\{t_n\}$ has a unique limit point p, and so $\lim_{n\to\infty} t_n = p$.

Theorem 20. *Let A be a semi-simple commutative prime* lcc *semi-algebra and let t be an element of A with spectral radius $\nu(t) = 1$. Let $A[t]$ be the semi-algebra generated by t. Then $A[t]$ is a semi-simple commutative prime* lcc *semi-algebra and, if p is its unique minimal idempotent, then*

$$p = \lim_{n\to\infty} \frac{1}{n}(t^2 + t^3 + \ldots + t^{n+1}).$$

Proof. That $A[t]$ has all the properties claimed in the theorem above is clear. Let $t_n = n^{-1}(t + t^2 + \ldots + t^n)$ and $\tilde{t_n} = n^{-1}(t^2 + \ldots + t^{n+1})$ $(n = 1, 2, \ldots)$. Then, by Lemma 13, the sequence $\{t_n\}$ is bounded and therefore, by the lcc property of t, the sequence $\{\tilde{t_n}\}$ has at least one limit point. Let q be a limit point of $\{\tilde{t_n}\}$. Since $\nu(t) = 1$, $p\tilde{t_n} = p$ $(n = 1, 2, \ldots)$. Therefore $pq = p$. As in the proof of the corollary above we can show that $p = q$. Thus the sequence $\{\tilde{t_n}\}$ has a unique limit point p, and so

$$\lim_{n\to\infty} \tilde{t_n} = \lim_{n\to\infty} n^{-1}(t^2 + \ldots + t^{n+1}) = p,$$

which completes the proof of the theorem.

See [6] for applications of the group $G(a)$ to the study of spectral properties of compact linear operators acting in a Banach space for which $A[T]$ is a prime strict locally compact semi-algebra.

9. More examples. The following examples were suggested by F. F. Bonsall.

Example III. Let G be a compact group and let B be a (complex) Banach algebra. Let μ be a positive Haar measure on G such that $\mu(G) = 1$. Let $\mathfrak{B}$ be the vector space of all continuous mappings of G into B. Then $\mathfrak{B}$ is a Banach space under the sup norm

$$\|f\| = \sup\{\|f(x)\|:\ x \in G\}.$$

Let B' be the conjugate space of B and let $C(G)$ be the space of all continuous complex-valued functions defined on G. If $\varphi \in B'$ and $f \in \mathfrak{B}$, the mapping $\varphi\circ f$, given by $(\varphi\circ f)(x) = \varphi(f(x))$ $(x \in G)$, belongs to $C(G)$. Given $f \in \mathfrak{B}$, we define the linear functional L_f on B' by

$$L_f(\varphi) = \int_G (\varphi\circ f)\,d\mu \qquad (\varphi \in B').$$

By Prop. 2, chap. III, § 4, No. 1 in [7], there exists an element a_f belonging to the closed convex hull of $f(G)$ in B such that $L_f(\varphi) = \varphi(a_f)$ $(\varphi \in B')$.

As usual, we denote a_f by $\int_G f\,d\mu$. We have

$$\Big\|\int_G f\,d\mu\Big\| \leqslant \int_G \|f(x)\|\,d\mu(x).$$

Now if $f, g \in \mathfrak{B}$, then $f(y)g(y^{-1}x)$ is a continuous mapping of G into B for every fixed $x \in G$ and so the integral

$$(f*g)(x) = \int_G f(y)g(y^{-1}x)\,d\mu(y)$$

exists and belongs to B for each $x \in G$. Using the right uniform continuity of g, it is easy to show that $f*g$ is a continuous mapping of G into B. Thus $f*g \in \mathfrak{B}$ and it is clear that $\|f*g\| \leqslant \|f\|\,\|g\|$. Consequently $\mathfrak{B}$ is a Banach algebra under the sup norm and multiplication given by the convolution.

Let A be a strict locally compact semi-algebra in B, and let

$$\mathcal{A} = \{f \in \mathfrak{B}:\ f(x) \in A \text{ for every } x \in G\}.$$

It is clear that $\mathcal{A}$ is a closed subset of $\mathfrak{B}$ and that if f, g are in $\mathcal{A}$ and α, β are non-negative real numbers, then $\alpha f + \beta g \in \mathcal{A}$. That $\mathcal{A}$ is closed under convolution is also clear. In fact, let $f, g \in \mathcal{A}$ and let A_1 be the closed unit ball of A. We may assume without any loss of generality that $f, g \in \mathcal{A}_1$, the closed unit ball of $\mathcal{A}$. Then $f(y)g(y^{-1}x) \in A_1$ for all $x, y \in G$ and, since A_1 is a closed convex compact set in B, it follows that $(f*g)(x) \in A_1$ for every $x \in G$. Therefore $f*g \in \mathcal{A}$ and so $\mathcal{A}$ is a semi-algebra.

Next we show that $\mathcal{A}$ is an lcc semi-algebra. Since μ is left invariant

$$(f*g)(x) = \int_G f(xy)g(y^{-1})\,d\mu(y).$$

Let $f \in \mathcal{A}$. Then

$$f*\mathcal{A}_1 = \{f*g:\ g \in \mathcal{A}_1\}$$

is an equicontinuous family of mappings on G into B. In fact, we have

$$\|(f*g)(x) - (f*g)(x')\| \leqslant \int_G \|f(xy) - f(x'y)\|\,\|g(y^{-1})\|\,d\mu(y)$$
$$\leqslant \int_G \|f(xy) - f(x'y)\|\,d\mu(y).$$

Let $\varepsilon > 0$ be given. Then, by the left uniform continuity of f, there exists a neighborhood $N(x)$ of x such that

$$\|(f*g)(x) - (f*g)(x')\| \leqslant \varepsilon$$

for all $x' \in N(x)$ and all $g \in \mathcal{A}_1$. Since for each $x \in G$ the set

$$\{(f*g)(x):\ g \in \mathcal{A}_1\}$$

lies in the compact set $\|f\|A_1$, by Ascoli's theorem for vector-valued functions, the closure of the set $f*\mathcal{A}_1$ is compact. Thus the operator T_f: $g\to f*g$ $(g\in\mathcal{A})$ is completely continuous in $\mathcal{A}$. Moreover, the strictness of A implies that of $\mathcal{A}$, so that $\mathcal{A}$ is an lcc semi-algebra.

Let a be a non-zero element of A and consider the mapping τ of $C(G)$ into $\mathcal{B}$ defined by

$$\tau\colon\ \beta\to\beta(\ )a\qquad(\beta\in C(G))$$

It is easy to see that τ is a homeomorphism of $C(G)$ into $\mathcal{B}$ which maps $C^+_R(G)$ into $\mathcal{A}$, where $C^+_R(G)$ is the set of all non-negative real-valued functions in $C(G)$. Thus if G is not a finite group, $C^+_R(G)$ is not a locally compact semi-algebra and therefore $\mathcal{A}$ is not a locally compact semi-algebra. For each $a\in B$ let $\hat{a}$ be the mapping given by $\hat{a}(x)=a$ for all $x\in G$. It is obvious that $\hat{a}\in\mathcal{B}$. If $\hat{a}$ and $\hat{b}$ are two such constant functions, then $\hat{a}*\hat{b}\in\mathcal{B}$ and $(\hat{a}*\hat{b})(x)=ab$ for all $x\in G$ because $\hat{a}(y)\hat{b}(y^{-1}x)=ab$ for all $x,y\in G$ and the set $\{ab\}$ is compact and convex. Thus $a\to\hat{a}$ is an isometric isomorphism of B into $\mathcal{B}$, and so if B is not a cc algebra then neither is $\mathcal{B}$. We observe that if B contains an identity element, then $\mathcal{B}$ is a cc algebra if and only if B is finite dimensional. In particular, if we take $A=R^+$ and $B=C$ the algebra of complex numbers, then $\mathrm{cl}(\mathcal{A}-\mathcal{A})$ is a cc algebra.

We should observe that if G is a commutative compact group and A is a commutative strict locally compact semi-algebra then $\mathcal{A}$ is a commutative lcc semi-algebra. If, moreover, A is prime, then $\mathcal{A}$ is a prime commutative lcc semi-algebra. In fact, let us show that if $f,g\in\mathcal{A}$ and $f\neq 0$, $g\neq 0$, then $f*g\neq 0$. Since A is commutative and prime, there exist points $x,y\in G$ such that $f(y)g(y^{-1}x)\neq 0$. To simplify notation, let $h(y)=f(y)g(y^{-1}x)$; we keep x fixed. Since $h\neq 0$, there exists a point $a_0\in h(G)\subset A$ such that $a_0\neq 0$, and, since A is strict, $a_0\notin -A$, where $-A=\{-a\colon a\in A\}$. But $-A$ is a closed convex subset of B. Hence there exists $\varphi\in B'$ and a constant γ such that

$$\varphi(b)\leqslant\gamma<\varphi(a_0)\qquad(b\in -A)\,.$$

Since $\varphi(0)=0$, $\gamma\geqslant 0$ and so $\varphi(a_0)\geqslant 0$. Moreover, it is easily seen that $\varphi(b)\leqslant 0$ for all $b\in -A$. Thus $\varphi(a)\geqslant 0$ for all $a\in A$, and consequently $(\varphi\circ h)(x)\geqslant 0$ for all $x\in G$. Since $\varphi\circ h\in C^+_R(G)$ and $(\varphi\circ h)(a_0)\neq 0$,

$$\int_G(\varphi\circ h)\,d\mu=L_h(\varphi)\neq 0\,.$$

Thus $\int_G h\,d\mu\neq 0$, $f*g\neq 0$, and so $\mathcal{A}$ is prime. It is interesting to note that $\mathcal{A}$ is also semi-simple. In fact, since A is semi-simple, A contains a minimal idempotent e. Therefore $\mathcal{A}$ contains a non-zero idempotent; namely $\hat{e}$, and so, by Theorem 10, $\mathcal{A}$ is semi-simple.

In general, if A is not commutative and prime, we do not know whether semi-simplicity of A implies semi-simplicity of $\mathcal{A}$. However, it can be shown that whenever $\mathcal{A}$ is w-semi-simple, then semi-simplicity of A implies that of $\mathcal{A}$. In fact, let $a\in A$ and $f\in\mathcal{B}$. Then, using the decomposition of the identity for compact Hausdorff spaces (see § 15, 2, *II* in [10] p. 220) and the fact that if σ is a μ-integrable complex-valued function on G and r is any element of B, then

$$\int_G\sigma(y)\hat{r}(y)\,d\mu(y)=\Big\{\int_G\sigma(y)\,d\mu(y)\Big\}r,$$

we can easily show that

$$(f*\hat{a})(x)=\Big\{\int_G f(y)\,d\mu(y)\Big\}a\,,$$

$$(\hat{a}*f)(x)=a\int_G f(y)\,d\mu(y)\,.$$

It follows now that if e is a minimal idempotent in A and $f\in\mathcal{A}$, then

$$(\hat{e}*f*\hat{e})(x)=e\Big\{\int_G f(y)\,d\mu(y)\Big\}e=\lambda e$$

with $\lambda\in R^+$ (because $\int_G f(y)\,d\mu(y)\in A$). Therefore $\hat{e}*\mathcal{A}*\hat{e}=R^+e$.

Let $\mathcal{R}$ be the radical of $\mathcal{A}$. Then $\hat{e}*\mathcal{R}*\hat{e}\subset R^+e$. So if for some $f\in\mathcal{R}$, $\hat{e}*f*\hat{e}\neq 0$, then $e\in\mathcal{R}$. This is impossible since $\mathcal{R}$ does not contain any idempotents and $\hat{e}$ is clearly an idempotent in $\mathcal{A}$. Therefore $\hat{e}*\mathcal{R}*\hat{e}=(0)$ and so $(\mathcal{R}*\hat{e})^2=(0)$. Now if we assume that $\mathcal{A}$ is w-semi-simple, Lemma 3 in [3] gives $\mathcal{R}*\hat{e}=(0)$. But

$$(f*\hat{e})(x)=\Big\{\int_G f(y)\,d\mu(y)\Big\}e\quad\text{ and }\quad\int_G f(y)\,d\mu(y)\in A$$

for every $f\in\mathcal{A}$. Hence if $\mathcal{R}\neq(0)$, then

$$J=\{a\in A\colon\ ae=0\text{ for every minimal idempotent }e\text{ in }A\}$$

is a non-zero left ideal of A. Since A is semi-simple, Theorem 14 implies that J contains a minimal idempotent q. But by the definition of J, $q^2=0$; a contradiction. Therefore $\mathcal{R}=(0)$ and so $\mathcal{A}$ is semi-simple.

Example IV. We shall now exhibit a method of constructing an lcc semi-algebra out of a given family of locally compact semi-algebras.

Let $\{B_\alpha\colon \alpha \in \Delta\}$ be a family of Banach algebras and suppose, that for each $\alpha \in \Delta$, A_α is a strict locally compact semi-algebra contained in B_α. Let B be the $B(\infty)$-sum of B_α, i.e., B is the set of all functions on Δ such that $f(\alpha) \in B_\alpha$ for each $\alpha \in \Delta$, and such that for every $\varepsilon > 0$ the set $\{\alpha \in \Delta\colon \|f(\alpha)\| \geqslant \varepsilon\}$ is finite. B is a Banach algebra under the norm $\|f\| = \sup\{\|f(\alpha)\|\colon \alpha \in \Delta\}$ and with multiplication given by $(fg)(\alpha) = f(\alpha)g(\alpha)$. In the notation of [11], $B = (\sum B_\alpha)_0$.

Let A be the subset of B consisting of those f in B for which $f(\alpha) \in A_\alpha$ for each $\alpha \in \Delta$. Then A is a closed semi-algebra in B and, the strictness of each A_α implies that of A. We shall now show that A is an lcc semi-algebra. Let t be an element of A such that $t(\alpha) = 0$ except on a finite subset Γ of Δ, i.e., $t(\alpha) = 0$ for $\alpha \in \Delta/\Gamma$. Let $\{a_n\}$ be a sequence in A, $\|a_n\| \leqslant 1$ $(n = 1, 2, \ldots)$. Since Γ is finite and each A_α locally compact, there exists a subsequence $\{a_{n_k}\}$ such that $ta_{n_k}(\alpha)$ converges to an element $b_\alpha \in A_\alpha$ for each $\alpha \notin \Gamma$. Let b be the element of B such that $b(\alpha) = b_\alpha$ for $\alpha \in \Gamma$ and $b(\alpha) = 0$ for $\alpha \notin \Gamma$. Then $b \in A$, ta_{n_k} converges to b, and so $T_t\colon x \to tx$ $(x \in A)$ is completely continuous in A. Since the set of all $t \in A$ which vanish off finite subsets of Δ is dense in A, it follows that for each $a \in A$, the operator $T_a\colon x \to ax$ $(x \in A)$ is completely continuous in A. Therefore A is an lcc semi-algebra. We write $A = (\sum A_\alpha)_0$.

It is easy to see that if $\{A_\alpha\colon \alpha \in \Delta\}$ is a family of lcc semi-algebras, then $(\sum A_\alpha)_0$ is also an lcc semi-algebra. Let X be a Banach space with a complete cone X^+ such that $X = \mathrm{cl}(X^+ - X^+)$. Let $\{T_\alpha\colon \alpha \in \Delta\}$ be a family of positive compact operators in X, and let A_α be the lcc semi-algebra generated by T_α for each $\alpha \in \Delta$. Then $(\sum A_\alpha)_0$ is an lcc semi-algebra which contains all the operators T_α.

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