If $\operatorname{Int} D \cap P_1 = \emptyset$, we add to P_1 a 3-cell neighborhood of D to obtain a punctured 3-cell P_1' such that $F \subset \operatorname{Int} P_1' \cup \operatorname{Int} P_2$ and $\partial P_1' \cap \partial F'$ has fewer components than does $\partial P_1 \cap \partial F'$. The conclusion then follows by induction.

If $\operatorname{Int} D \subset \operatorname{Int} P_1$, we remove (as in lemma 3) a small product neighborhood of D from P_1 to obtain two disjoint punctured 3-cells P_{11} and P_{12}. We require that the neighborhood we remove is small enough that $F \subset \operatorname{Int} P_{11} \cup \operatorname{Int} P_{12} \cup \operatorname{Int} P_2$. If one of P_{1j} (say P_{12}) does not meet F' we let $P_1' = P_{11}$. If both meet F' we take an arc A in $\operatorname{Int} F'$ such that $\operatorname{Int} A \cap (P_{11} \cup P_{12}) = \emptyset$ and one end point of A is in ∂P_{11} and the other is in ∂P_{12}. In this case we let $P_1' = P_{11} \cup P_{12} \cup$ (small neighborhood of A). In either case the construction can be made so that P_1' is a punctured 3-cell such that $F \subset \operatorname{Int} P_1' \cup \operatorname{Int} P_2$ and $\partial P_1' \cap \partial F'$ has fewer components than does $\partial P_1 \cap \partial F'$. Again the conclusion follows by induction.

LEMMA 5. *Suppose that M is a closed 3-manifold and F is a (polyhedral, compact) contractible 3-manifold-with-boundary in M. If there exist punctured 3-cells P_1 and P_2 (polyhedral and in general position) in M such that $F \subset \operatorname{Int} P_1 \cup \operatorname{Int} P_2$ then F is a (combinatorial) 3-cell.*

Proof. We add a collar to F to obtain F' such that $F \subset \operatorname{Int} F'$ and $\overline{F' - F} = \partial F \times [0, 1]$. By lemma 4 we assume that $P_1 \cup P_2 \subset \operatorname{Int} F'$.

Then by lemma 3 (with $N = \operatorname{Int} F'$), $P_1 \cup P_2$ is a special punctured-cube-with-handles. Thus F can be piecewise-linearly embedded in S^3 and hence is a 3-cell.

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f-closure algebras

by

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1. Introduction. In the theory of closure algebras (cf. [2] and the references given there) several elementary algebraic problems can only be solved if appropriate completeness conditions hold for the algebras involved. The following are examples of this situation. A denotes a closure algebra, B a Boolean algebra.

1.1. Consider the set $\mathfrak{C}$ of all closure operators on B, ordered as usual. Whereas in topology the corresponding set of all topologies is a complete lattice, this will in general be no longer true for $\mathfrak{C}$. If B is complete, however, $\mathfrak{C}$ is a complete lattice.

1.2. Consider a Boolean epimorphism $A \to B$. In his paper [2], Sikorski solved the problem of defining a suitable closure operator on B as an analogue to the quotient topology. His construction makes use of a basis and of several assumptions about completeness properties of A.

1.3. The inverse problem of lifting a closure operator has also been investigated by Sikorski. Given a Boolean epimorphism $B \to A$, can a suitable closure operator be defined on B similar to the topology induced by a mapping on its domain? Sikorski (cf. [2], [3]) constructed a closure operator on B in such a way that its quotient operator on A coincides with the given one. Again he assumed the existence of a basis and σ-completeness.

1.4. If $A \to B$ or $B \to A$ are Boolean monomorphisms instead of epimorphisms, one is confronted with the problems of extension and of contraction of closure operators. Both problems have been investigated, though only incidentally, by McKinsey and Tarski (cf. [1]), using complete algebras.

1.5. Given a family A_i $(i \in I)$ of closure algebras, one might be interested in their product and in their coproduct. While the construction of the product is trivial, the coproduct will in general not exist. Let Q denote the Boolean coproduct of the Boolean algebras A_i (called "product" in [4]). A suitable closure operator can be defined on Q using the methods mentioned in 1.1 and 1.4, provided Q is complete.

In this paper we propose a generalization of the concept of closure algebra. Briefly, we shall define the closure of an element of a Boolean algebra not as another element, but as a filter, and we shall require the correspondence elements—filters to satisfy axioms formally identical to the axioms of Kuratowski. Making use of the fact that the lattice of filters of a Boolean algebra is complete we then show that problems 1.1-1.5 have natural solutions in terms of filter-closures. Moreover, in case the algebras involved satisfy certain completeness conditions, our constructions will be seen to coincide with those to be found in the literature.

Sections 2 and 3 of the present paper contain a brief review of those parts of the theory of filters on a Boolean algebra which will be needed in the sequel, followed by the definition and elementary properties of f-closure operators as well as some basic theorems. The second part of the paper (sections 4-6) contains the solutions of the problems mentioned above. All routine proofs will be omitted.

The author has presented the basic ideas of the present paper in a talk at the International Congress of Mathematicians at Moscow 1966.

2. Foundations of the theory of f-closure algebras.

2.1. Filters on Boolean algebras. Throughout this section, B, B^* will denote Boolean algebras, F, G, F^*, G^*, sometimes with subscripts, filters on B and B^*, respectively. $\mathcal{F}(B)$ will stand for the set of all filters on B.

We introduce an order relation on $\mathcal{F}(B)$ by defining

$$F \leqslant G \iff G \subseteq F .$$

As is well known, $\mathcal{F}(B)$ is a complete lattice with respect to this relation. If F_i $(i \in I)$ is a family of filters on B, its join and meet are given by

$$\bigvee F_i = \bigcap F_i ; \qquad \bigwedge F_i = \text{filter generated by } \bigcup F_i .$$

For $a \in B$, (a) will always denote the principal filter generated by a. The filter (0), consisting of all elements of B, will simply be written as 0.

LEMMA 1. *Let F_i $(i \in I)$, G_k $(k \in K)$ be two families of filters. Then*

$$(\bigvee F_i) \vee (\bigwedge G_k) = \bigwedge (F_i \vee G_k) \qquad (i \in I,\ k \in K) .$$

Let $p\colon B \to B^*$ be a Boolean homomorphism. p induces mappings between $\mathcal{F}(B)$ and $\mathcal{F}(B^*)$, which will also be denoted by $p\colon \mathcal{F}(B) \to \mathcal{F}(B^*)$ and by $p^{-1}\colon \mathcal{F}(B^*) \to \mathcal{F}(B)$, viz.

$$F \in \mathcal{F}(B)\colon \quad pF = \text{filter generated by } \{pf \in B^*\colon f \in F\} ,$$
$$F^* \in \mathcal{F}(B^*); \quad p^{-1}F^* = \{f \in B\colon pf \in F^*\} .$$

We recall some of the elementary properties of these mappings in the following lemmas.

LEMMA 2. *For all $a \in B$, F, G, F_i $(i \in I)$ $\in \mathcal{F}(B)$, $F^*, G^* \in \mathcal{F}(B^*)$:*

$$(pa) = p(a) \quad (\textit{with the obvious use of the symbols } (\ )) ,$$
$$F \leqslant G \Rightarrow pF \leqslant pG; \qquad F^* \leqslant G^* \Rightarrow p^{-1}F^* \leqslant p^{-1}G^* ,$$
$$p(F \vee G) = pF \vee pG; \qquad p^{-1}(F^* \vee G^*) = p^{-1}F^* \vee p^{-1}G^* ,$$
$$p(\bigwedge F_i) = \bigwedge (pF_i) .$$

Let F_i $(i \in I)$ be a family of filters such that for any two subscripts $i, k \in I$ there is $m \in I$ with $F_i \wedge F_k \geqslant F_m$. Then F_i $(i \in I)$ will be called a *basic family* of filters.

LEMMA 3. *If F_i^* $(i \in I)$ is a basic family of filters belonging to $\mathcal{F}(B^*)$, then*

$$p^{-1}(\bigwedge F_i^*) = \bigwedge p^{-1}F_i^* .$$

LEMMA 4. *If $F \in \mathcal{F}(B)$, $F^* \in \mathcal{F}(B^*)$, then*

$$p^{-1}pF \leqslant F; \qquad F^* \leqslant pp^{-1}F^* .$$

Equality holds in the first relation, if $p\colon B \to B^*$ is a monomorphism, in the second one if it is an epimorphism.

2.2. f-closure operators. Let B be a Boolean algebra and C a mapping $C\colon B \to \mathcal{F}(B)$. C induces a mapping, again written as $C\colon \mathcal{F}(B) \to \mathcal{F}(B)$, defined by

$$F \in \mathcal{F}(B); \qquad CF = \bigwedge Cf \qquad (f \in F) .$$

C will be called a *f-closure operator on B*, and (B, C) a *f-closure algebra*, if C satisfies, for all $a, b \in B$, the following axioms:

Ax 1. $(a) \leqslant Ca$.

Ax 2. $C0 = 0$.

Ax 3. $C(a \vee b) = Ca \vee Cb$.

Ax 4. $C^2a = C(Ca) = Ca$.

If C satisfies only Ax 1-Ax 3, it will be called a *pre-f-closure operator*. Ax 3 implies

Ax 5. $a \leqslant b \Rightarrow Ca \leqslant Cb$.

Instead of Ax 1-Ax 4, one could take Ax 1, 2, 5 together with Ax 3 and Ax 4, where the equality sign has been changed to $\leqslant$, as an equivalent set of axioms. It is sometimes simpler to verify the latter set then the former.

The following lemma lists some easy consequences of the axioms and of the definition of $C\colon \mathcal{F}(B) \to \mathcal{F}(B)$:

LEMMA 5. *If C is a pre-f-closure operator, then for all $a \in B$, F, G, F_i $(i \in I) \in \mathcal{F}(B)$:*

$$C(a) = Ca ,$$
$$F \leqslant CF; \quad C(F \vee G) = CF \vee CG; \quad F \leqslant G \Rightarrow CF \leqslant CG; \quad C(F \wedge G) \leqslant CF \wedge CG.$$

If $a \to \bar{a}$ denotes an ordinary closure operator on B, define

$$Ca = (\bar{a});$$

obviously, C is a f-closure operator on B. Hence any ordinary closure algebra can be looked upon as a f-closure algebra. However, let F be a non-principal filter on B (provided there is one), and define C by

$$0 \neq a \in B; \quad Ca = (a) \vee F; \quad C0 = 0 .$$

C is a f-closure operator, but Ca will not in general be principal.

2.3. f-closure morphisms. In order to form a category, the objects of which are to be all f-closure algebras, we have to choose appropriate morphisms. Let (B, C), (B^*, C^*) be two f-closure algebras. A map $p\colon B \to B^*$ will be called a *f-closure morphism* if p is a Boolean homomorphism and if, for all $a \in B$,

$$C^* pa \leqslant pCa .$$

(Note that the letter p has different meanings on the two sides of this relation.)

If C and C^* are derived from ordinary closure operators, both expressed by bars, the relation reads $\overline{pa} \leqslant p\bar{a}$. Sikorski [2] calls a p with this property continuous; we prefer to use the term "closure morphism".

3. Some basic theorems. Throughout this section, B, B^* will denote Boolean algebras, $p\colon B \to B^*$ a Boolean homomorphism.

THEOREM 1. *Suppose that a f-closure operator C is given on B. Define the operator C^* on B^* by*

$$x \in B^*; \quad C^*x = pCp^{-1}(x) .$$

Then C^* is a f-closure operator on B^*.

Proof. Using Lemmas 2, 4 and 5, C^* is easily seen to satisfy Ax 1, 2, 3 and 5. The proof of Ax 4 depends on the following relation:

$$F^* \in \mathcal{F}(B^*); \quad C^*F^* = pCp^{-1}F^*.$$

By definition, $C^*F^* = \bigwedge C^*y$ $(y \in F^*)$. Substituting for C^* on the right-hand side and using Lemma 2 leads to $C^*F^* = p\bigwedge Cp^{-1}(y)$ $(y \in F^*)$. But $Cp^{-1}(y) = \bigwedge Ca$ $(a \in p^{-1}(y))$, so that C^*F^* can be written as $C^*F^* = p\bigwedge Ca$ $(a \in p^{-1}(y),\ y \in F^*)$. On the other hand, it is easy to see that $\{a \in B\colon a \in p^{-1}(y),\ y \in F^*\} = p^{-1}F^*$. Hence $C^*F^* = p\bigwedge Ca$ $(a \in p^{-1}F^*) = pCp^{-1}F^*$, as required.

Ax 4 is now an easy consequence of Lemmas 2, 4 and 5:

$$C^*(C^*x) = pCp^{-1}pCp^{-1}(x) \leqslant pC^2p^{-1}(x) = C^*x .$$

THEOREM 2. *Suppose that a f-closure operator C^* is given on B^*. Define the operator C on B by*

$$a \in B; \quad Ca = p^{-1}C^*pa \vee (a) .$$

Then C is a pre-f-closure operator on B. If p is an epimorphism, C is a f-closure operator.

Proof. We prove only the second assertion. To this end, we first establish the relation

$$F \in \mathcal{F}(B); \quad CF = p^{-1}C^*pF \vee F ,$$

valid for any p. By definition, $CF = \bigwedge [p^{-1}C^*pf \vee (f)]$ $(f \in F)$. We assert:

$$\bigwedge [p^{-1}C^*pf \vee (f)] = \bigwedge [p^{-1}C^*pf \vee (g)] \quad (f, g \in F) .$$

Obviously, it suffices to show that the left-hand side is less than or equal to the right-hand side. For $f, g \in F$, write $h = f \wedge g$. Using Lemmas 2, 3 and 5, one obtains $p^{-1}C^*ph \vee (h) \leqslant p^{-1}C^*pf \vee (g)$, which proves our assertion.

Lemma 1 implies $\bigwedge [p^{-1}C^*pf \vee (g)] = [\bigwedge p^{-1}C^*pf] \vee [\bigwedge (g)]$ $(f, g \in F)$. If g runs through F, then $\bigwedge (g) = F$. The family of filters C^*pf $(f \in F)$ is a basic family and by Lemma 3, $\bigwedge p^{-1}C^*pf = p^{-1}\bigwedge C^*pf$. Moreover, $\bigwedge C^*pf = C^*pF$, so that finally

$$CF = \bigwedge [p^{-1}C^*pf \vee (g)] = p^{-1}C^*pF \vee F .$$

Ax 4 can now be seen to hold for C, if p is an epimorphism. It suffices to write $C^2a = C[p^{-1}C^*pa \vee (a)] = (p^{-1}C^*p)[p^{-1}C^*pa \vee (a)] \vee p^{-1}C^*pa \vee (a)$ $= p^{-1}C^*pp^{-1}C^*pa \vee p^{-1}C^*pa \vee (a) = p^{-1}C^*pa \vee (a) = Ca$, using Lemma 4.

In the sequel, we shall write $\mathfrak{P}$ for the set of all mappings $P\colon B \to \mathcal{F}(B)$ and $\mathfrak{C}$ for the set of all f-closure operators on B. We introduce the usual order relation on $\mathfrak{P}$ by

$$P_1 \leqslant P_2 \iff P_1 a \leqslant P_2 a \quad \text{for all } a \in B .$$

Given $P \in \mathfrak{P}$, let $\mathfrak{C}(P)$ denote the subset of $\mathfrak{C}$ consisting of all $M \in \mathfrak{C}$ with $P \leqslant M$.

THEOREM 3. *For any pre-f-closure operator E, the f-closure operator $E^* = \min \mathfrak{C}(E)$ exists,* (where the min has to be taken within $\mathfrak{C}$), *i.e. there is $E^* \in \mathfrak{C}$ such that* (1) *$E \leqslant E^*$,* (2) *if $C \in \mathfrak{C}$ and $E \leqslant C$, then $E^* \leqslant C$.*

Proof. Let $\varrho, \sigma, \ldots$ run through all ordinals up to a given, yet to be chosen ordinal α. We define recursively a α-sequence of operators $E_\varrho \in \mathfrak{P}$ as follows:

$$a \in B; \quad E_0 a = Ea; \quad E_{\varrho+1} a = E_\varrho^2 a;$$

if ϱ is a limit number,

$$E_\varrho a = \bigvee E_\lambda a \quad (0 \leqslant \lambda < \varrho) .$$

We claim that for all $\varrho, \sigma < \alpha$:

(1) Each E_ϱ is a pre-f-closure operator,

(2) $\varrho \leqslant \sigma \Rightarrow E_\varrho \leqslant E_\sigma$ (especially $E \leqslant E_\varrho$),

(3) $C \in \mathfrak{C}$, $E \leqslant C \Rightarrow E_\varrho \leqslant C$.

The proofs of (1)-(3) are straightforward applications of transfinite induction, making use of Lemma 5. As an example, we shall show (3). Let $a \in B$, $F \in \mathfrak{F}(B)$. Then for an ordinal ϱ, $E_\varrho \leqslant C$ implies $E_\varrho F \leqslant CF$ and $E_{\varrho+1}a = E_\varrho^2 a \leqslant E_\varrho Ca \leqslant C^2 a = Ca$, hence $E_{\varrho+1} \leqslant C$. One can now use transfinite induction, the case of ϱ, a limit number being trivial.

We shall next prove that *to each $a \in B$ there corresponds at least one ordinal number $\tau < \alpha$ such that $E_\tau a = E_{\tau+1}a$.*

If this were not true, we would have, by (2), $E_\varrho a < E_{\varrho+1}a$, i.e. $E_{\varrho+1}a \subset E_\varrho a$, for all $0 \leqslant \varrho < \alpha$, the inclusion being proper. Hence there exists an element $b_\varrho \in E_\varrho a$, $b_\varrho \notin E_{\varrho+1}a$. Suppose that b_ϱ has been picked for each ϱ; the α-sequence $\{b_\varrho;\ 0 \leqslant \varrho < \alpha\}$ consists of pairwise different elements of B. Indeed, if $b_\varrho = b_\sigma$ for $\varrho < \sigma$, say, then by (2) $b_\varrho \in E_\sigma a \subseteq E_{\varrho+1}a$, contrary to our assumptions. Choosing the ordinal number α high enough, e.g. such that $\operatorname{card}\alpha > \operatorname{card}B$, one reaches a contradiction.

Let τ denote the least ordinal such that $E_\tau a = E_{\tau+1}a$. The foregoing proof shows that we must have $\operatorname{card}\tau \leqslant \operatorname{card}B$. Moreover, it is easy to see that $E_\tau a = E_\sigma a$ for all $\sigma \geqslant \tau$. Using these facts, one can choose an ordinal number β such that $E_\beta a = E_{\beta+1}a$ holds for all $a \in B$. Obviously, $E^* = E_\beta$ satisfies all the requirements of the theorem.

Given $p\colon B \to B^*$ as before, suppose that a f-closure operator C is defined on B. We shall call a f-closure operator C^* on B^* *coinduced* by p (given C) if it satisfies

(1) p is a f-closure morphism on (B, C) into (B^*, C^*),

(2) C^* is maximal with respect to this property in the lattice $\mathfrak{C}^*$ of f-closure operators on B^*.

THEOREM 4. *The f-closure operator C^*, defined for all $x \in B^*$ by $C^*x = pCp^{-1}(x)$, is coinduced by p (given C).*

Proof. By Theorem 1, C^* is a f-closure operator. For any $a \in B$, $C^*pa = pCp^{-1}(pa) = pCp^{-1}p(a) \leqslant pCa$, so that (1) holds. Suppose that $D^* \in \mathfrak{C}^*$ is such that $D^*pa \leqslant pCa$, for all $a \in B$. It follows from Lemma 2 that $D^*pF \leqslant pCF$, for any $F \in \mathfrak{F}(B)$. Let $x \in B^*$; by Lemma 4, $(x) \leqslant pp^{-1}(x)$, hence $D^*x \leqslant D^*pp^{-1}(x) \leqslant pCp^{-1}(x) = C^*x$ and (2) holds as well.

COROLLARY. *If p is a monomorphism, $C^*pa = pCa$, for all $a \in B$.* (Lemma 4.)

Given $p\colon B \to B^*$, suppose that a f-closure operator C^* is defined on B^*. We shall call a f-closure operator C on B *induced by* p (given C^*), if it satisfies

(1) p is a f-closure morphism on (B, C) into (B^*, C^*),

(2) C is minimal with respect to this property in the lattice $\mathfrak{C}$ of f-closure operators on B.

Consider the operator E, defined by $a \in B$, $Ea = p^{-1}C^*pa \vee (a)$. By Theorem 2, E is a pre-f-closure operator. By Theorem 3, there is a minimal f-closure operator C such that $E \leqslant C$. Again by Theorem 2, C equals E if p is an epimorphism.

THEOREM 5. *The operator C is induced by p (given C^*).*

Proof. Using Lemma 4, one infers for any $a \in B$ that $C^*pa \leqslant pp^{-1}C^*pa \leqslant pp^{-1}C^*pa \vee (pa) = pEa \leqslant pCa$, so that (1) holds. Let $D \in \mathfrak{C}$ be such that $C^*pa \leqslant pDa$. Then $Ea = p^{-1}C^*pa \vee (a) \leqslant p^{-1}pDa \vee (a) \leqslant Da \vee \vee (a) = Da$, hence $E \leqslant D$. By the minimality of C, this implies $C \leqslant D$, which proves (2).

COROLLARY. *If p is an epimorphism, $C^*pa = pCa$, for all $a \in B$.* (Lemma 4.)

As we mentioned in 2.2, any ordinary closure algebra can be turned into a f-closure algebra by writing $Ca = (\bar{a})$. In this case, all filters Ca are principal; if, conversely, all Ca are principal, i.e. if $\bar{a} = \min Ca$ exists for all $a \in B$, the given f-closure algebra can be obtained from an ordinary one in this way. We prove more generally:

THEOREM 6. *Let (B, C) be a f-closure algebra such that $a^* = \inf Ca$ exists for all $a \in B$. Then $a \to a^*$ is an ordinary closure operator on B.*

Proof. We have to verify Kuratowski's axioms

(1) $a \leqslant a^*$; (2) $0^* = 0$; (3) $(a \vee b)^* = a^* \vee b^*$; (4) $a^{**} = a^*$.

(1) and (2) are obvious. To prove (3), one uses Ax 3 for the operator C and the well-known fact: if a, b, c_i $(i \in I)$ are elements of a Boolean algebra, and if $c = \inf\{c_i;\ i \in I\}$ exists, then $a \leqslant b \vee c_i$ for all $i \in I$ implies $a \leqslant b \vee c$. The proof of (4) runs as follows: by (1) $a \leqslant a^*$, by construction, $(a^*) \leqslant Ca$. Applying C and using Lemma 5 yields $Ca \leqslant Ca^* \leqslant C^2 a = Ca$, hence $Ca^* = Ca$. Property (4) is an immediate consequence of this equation.

4. The lattice of f-closure operators. As in the preceding section, $\mathfrak{C}$ will denote the ordered set of all f-closure operators on a given Boolean algebra B. The discrete operator D, defined by $a \in B$, $Da = (a)$ is the minimum, the indiscrete operator M, defined by $0 \neq a \in B$, $Ma = (1)$, $M0 = 0$, is the maximum of $\mathfrak{C}$.

THEOREM 7. *$\mathfrak{C}$ is a complete lattice.*

Proof. It suffices to show that $\sup \mathfrak{D}$ exists for any subset $\mathfrak{D}$ of $\mathfrak{C}$. Let $\mathfrak{D} = \{C_i;\ i \in I\}$ and define the operator $D \in \mathfrak{T}$ by

$$a \in B; \quad Da = \bigvee C_i a \ (i \in I).$$

It is a simple matter to check that D is a pre-f-closure operator. By Theorem 3, there is a f-closure operator D^*, minimal with respect to the inequality $D \leqslant D^*$. We assert $D^* = \sup \mathfrak{D}$.

Obviously, $C_i \leqslant D^*$ for all $i \in I$. If $C \in \mathfrak{C}$ is such that $C_i \leqslant C$ for all $i \in I$, then $\bigvee C_i a = Da \leqslant Ca$ and by the minimality of D^* we conclude $D^* \leqslant C$.

If $\mathfrak{D}$ consists of ordinary closure operators, D^* will in general still be a f-closure operator. Suppose, however, that $a^* = \inf D^* a$ exists for all $a \in B$. By Therem 6, $a \to a^*$ is an ordinary closure operator, which is evidently equal to $\sup \mathfrak{D}$. If this is true for any such $\mathfrak{D}$—as will happen, for instance, if B is complete—the set of all ordinary closure operators on B is itself a complete lattice.

5. Quotients and lifting of f-closure operators.

5.1. Quotients. Let $p\colon B \to B^*$ be a Boolean epimorphism, C a f-closure operator on B. The operator C^*, coinduced by p (given C) will be called the *quotient operator on* B^*.

In this section, we investigate quotients of ordinary closure operators, which are in general f-closure operators. For any $x \in B^*$, we shall write $[x] = \{a \in B\colon pa = x\}$. Obviously, $[x]$ is a residue class with respect to the ideal $\operatorname{Ker} p$ of B. Let $Ca = (\bar{a})$ $(a \in B)$.

PROPOSITION 1. *The filter C^*x, $x \in B^*$, is generated by the p-images of all elements $\bar{c}$, $c \in [x]$.*

Proof. By construction, $C^*x = pCp^{-1}(x) = p \bigwedge Ca$ $\quad (a \in p^{-1}(x))$. Choose $b \in [x]$; then for any $a \in p^{-1}(x)$, $p(a \wedge b) = pa \wedge pb = x$, while $Ca \geqslant C(a \wedge b)$. Hence $C^*x = p \bigwedge Cc$ $(c \in [x])$. Using Lemma 2 and $Cc = (\bar{c})$, one obtains $C^*x = \bigwedge (p\bar{c})$ $(c \in [x])$, as required.

Suppose that $x^* = \inf C^*x$ exists for all $x \in B^*$. By Theorem 6, $x \to x^*$ is an ordinary closure operator on B^*. Our arguments show that it is the quotient of the given closure operator on B, where quotient now refers to closure morphisms instead of f-closure morphisms.

There are several important cases for which $\inf C^*x$ can be shown to exist.

(1) $\operatorname{Ker} p$ *is a closure ideal*, i.e. $a \in \operatorname{Ker} p$ implies $\bar{a} \in \operatorname{Ker} p$. For $x \in B^*$, $a, b \in [x]$ implies $p\bar{a} = p\bar{b}$, as is easy to see by using the relation $\bar{a} - \bar{b} \leqslant \overline{a-b}$. By Proposition 1 therefore, C^*x is generated by a single element, i.e. is principal.

(2) $\operatorname{Ker} p$ *is a principal ideal, generated by an element* q, say. Then for any $x \in B^*$, the set $[x]$ possesses a minimum $m = m(x)$. In fact, choose $a \in [x]$ and write $m = a - q$. Then $m \in [x]$ and $m \leqslant b$ for any $b \in [x]$. Hence $m = \min [x]$. By Proposition 1, $C^*x = (p\bar{m})$.

(3) *B^* is complete.*

(4) The case treated by Sikorski (cf. [2], section 9) can be subsumed under the following proposition.

PROPOSITION 2. *If every family of closed elements of B possesses an infimum, and if p preserves such infima, i.e. if*

$$p \inf \{t_i\} = \inf \{pt_i\} \qquad (i \in I,\ \bar{t}_i = t_i \in B)\,,$$

then $x^* = \inf C^*x$ exists for all $x \in B^*$. In fact, $C^*x = (x^*)$.

Proof. For any $x \in B^*$, $x^0 = \inf\{\bar{a}\colon a \in [x]\}$ exists and $x^* = px^0 = \inf\{p\bar{a}\colon a \in [x]\}$. By Proposition 1, $x^* = \inf C^*x$.

As x^0 is a meet of closed elements, it is itself closed and $x \leqslant x^*$ implies $C^*x \leqslant C^*x^* = C^*px^0 \leqslant pCx^0 = p(x^0) = (px^0) = (x^*)$. Hence $C^*x = (x^*)$.

Sikorski considers a closure algebra A and an ideal J with the following properties: for some infinite cardinal m, A is m-complete, J is a m-ideal and A possesses a basis R (for the closed elements) with $\operatorname{card} R \leqslant m$. (Actually, Sikorski requires a basis for the open elements of A, a condition obviously equivalent to our assumptions). It is easily seen that Theorem 9.1 of [2] is a consequence of Proposition 2 and that the closure operator constructed by Sikorski is identical with the operator constructed above (cf. especially [2], 9.3).

5.2. Lifting. Let $p\colon B \to B^*$ be given as in 5.1, and let C^* be a f-closure operator on B^*. The operator C, induced by p (given C^*) will be called the *lifted operator on* B.

We investigate lifting of ordinary closure operators. Let $C^*x = (\bar{x})$, for all $x \in B^*$. Recall that $[x]$ is the set of all elements of B mapped onto x.

PROPOSITION 3. *For $a \in B$, Ca is generated by all $b \in B$ with $b \in [\overline{pa}]$, $b \geqslant a$.*

Proof. By construction,

$$Ca = p^{-1}C^*pa \vee (a) = p^{-1}(\overline{pa}) \vee (a) = (a) \vee \bigwedge (b) \qquad (pb = \overline{pa})\,.$$

If $a^* = \inf Ca$ exists for all $a \in B$, then $a \to a^*$ is an ordinary closure operator on B, as shown by Theorem 6, but it will only be the lifted operator if $a^* = \min Ca$, in consequence of the Corollary to Theorem 5.

If $\operatorname{Ker} p$ is a principal ideal, every one of its residue classes possesses a minimum, as shown in 5.1. Therefore $m(a) = \min\{b \in B\colon b \in [\overline{pa}]\}$ exists and by Proposition 3, $\min Ca = m(a) \vee a$.

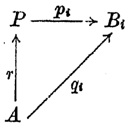
There do not seem to be other simple conditions which guarantee the existence of $\min Ca$. Even completeness of B does not necessarily imply $\inf Ca = \min Ca$. As is easily seen, this will be the case if and only if $p \inf Ca = \inf pCa$, for all $a \in B$. Requiring the homomorphism p to be complete is of no help, as with B and p complete the ideal $\mathrm{Ker}\,p$ is again principal.

Sikorski, in his papers [2] and [3], uses the following conditions: there exists an enumerable basis R on B^*, B is a σ-algebra and $\mathrm{Ker}\,p$ a σ-ideal. Again it does not follow that $\min Ca$ exists. As a result, the closure operator constructed by Sikorski on the basis of his assumptions does not coincide with the operator $a \to a^*$ obtained by assuming $a^* = \min Ca$. (For $a \in \mathrm{Ker}\,p$, $a^* = a$, which does not hold for Sikorski's operator.) Therefore, the closure operator constructed in [2] does not, in general, satisfy the conditions of lifting as defined in the present paper. (It does, however, satisfy a reciprocity condition: its quotient operator on B^* is equal to the one given there.)

5.3. Products. We use the results of section 5.2 to construct the product of an arbitrary family of f-closure algebras $\{(B_i, C_i);\ i \in I\}$. Let $P = \prod B_i$ denote the product of the Boolean algebras B_i, with the canonical projections $p_i\colon P \to B_i$. (Note that products are called "disjoint unions" in [4]). As each p_i is a Boolean epimorphism, the corresponding f-closure operator C_i can be lifted to P; call the lifted operator C_i^0. Let $C = \sup C_i^0$ $(i \in I)$ be the f-closure operator determined as in Theorem 7.

THEOREM 8. *(P, C) is the product of the algebras (B_i, C_i) in the category of f-closure algebras and f-closure morphisms.*

Proof. Obviously, the mappings $p_i\colon (P, C) \to (B_i, C_i)$ are f-closure morphisms. Let (A, D) be any f-closure algebra and let $q_i\colon (A, D) \to (B_i, C_i)$ $(i \in I)$ be f-closure morphisms. Because P is a product in the category of Boolean algebras and Boolean homomorphisms, there is a unique Boolean homomorphism $r\colon A \to P$ such that the diagrams



commute for all $i \in I$. We have to show that r is a f-closure morphism.

For $a \in A$, consider $C_i^0 ra$. Using the commutativity of the diagrams, we can write

$$C_i^0 ra = p_i^{-1} C_i p_i ra \vee (ra) = p_i^{-1} C_i q_i a \vee (ra) \leqslant p_i^{-1} q_i Da \vee (ra)\,.$$

For any filter $F \in \mathcal{F}(A)$, $p_i^{-1} q_i F = p_i^{-1} p_i rF \leqslant rF$, according to Lemma 4, so that

$$C_i^0 ra \leqslant rDa\,.$$

Let D^* be the operator coinduced by r (given D). As D^* satisfies $D^* ra \leqslant rDa$ for all $a \in A$ and as it is maximal with respect to this relation, we have $C_i^0 \leqslant D^*$ for all $i \in I$, hence $C \leqslant D^*$. Choose $a \in A$; then $Cra \leqslant D^* ra = rDr^{-1}(ra) = rDr^{-1}r(a) \leqslant rDa$. This shows r to be a f-closure morphism.

Suppose the given (B_i, C_i) are ordinary closure algebras; then (P, C) turns out to be the usual product obtained by defining, for any $b \in P$, the closure Cb as the result of applying the operators C_i to the respective components of b.

For any $a \in B_i$, $p_i^{-1}(a)$ is a principal filter on P, generated by an element a^* with $p_i a^* = a$, $p_k a^* = 0$ $(k \neq i)$. From this follows easily, for any $b \in P$, that $C_i^0 b$ is a principal filter on P, generated by an element $b_i \in P$ with $p_k b_i = p_k b$ $(k \neq i)$, while $p_i b_i$ is equal to the closure of $p_i b$ in B_i. By the definition of C, this shows Cb to be principal, generated by an element $\bar{b}$ with $p_i \bar{b}$ equal to the closure of $p_i b$ in B_i, for all $i \in I$.

6. Extension and contraction of f-closure operators.

6.1. Extensions. Let $p\colon B \to B^*$ denote a Boolean monomorphism, and let C be a f-closure operator on B. The operator C^*, coinduced by p (given C) will be called the *extended operator on* B^*.

To study the extension of ordinary closure operators, we shall assume p to be an embedding, i.e. $p\colon B \subseteq B^*$. All filters, infima etc. will refer to B^*.

Let C be derived from an ordinary closure operator on B, denoted by a bar. By the Corollary to Theorem 4, $C^* a = (\bar{a})$, for any $a \in B$, so that C^* can be said to coincide with C on B. For $x \in B^*$, $x \notin B$, $C^* x$ will in general not be principal.

PROPOSITION 4. *For $x \in B^*$, $C^* x$ is generated by the set $\{\bar{a} \in B^*\colon x \leqslant a,\ a \in B\}$.*

Proof. An obvious consequence of the construction of C^*.

Suppose that $x^* = \inf C^* x$ exists for all $x \in B^*$. By Proposition 4, $x^* = \inf\{\bar{a} \in B^*\colon x \leqslant a,\ a \in B\}$; for $a \in B$, $a^* = \bar{a}$. Hence the operator $x \to x^*$ coincides with the closure operator constructed in [1], Lemma 2.3 (cf. also [5], III, 4.1).

$\inf C^* x$ will exist in case B^* is complete (as assumed in [1]). One might also require B^* to be m-complete for some infinite cardinal m and B to possess a basis R for the closed elements with $\mathrm{card}\,R \leqslant m$. Every $\bar{a}$ $(x \leqslant a,\ a \in B)$ will then be a meet of elements of R, so that $\inf C^* x$ will exist by virtue of the m-completeness of B^*.

6.2. Contractions. As in 6.1, assume $p\colon B \to B^*$ to be a Boolean monomorphism. Suppose that a f-closure operator C^* is given on B^*. The operator C induced by p (given C^*) will be called the *contracted operator on* B.

Assume p to be an embedding and C^* to be derived from an ordinary closure operator, $C^*x = (\bar{x})$, $x \in B^*$. For the remainder of this section, filters, infima etc. refer to B.

PROPOSITION 5. *Let $\bar{B}$ denote the set of elements $a \in B$ such that $\bar{a} \in B$. Then for all $a \in \bar{B}$, $Ca = (\bar{a})$.*

Proof. The operator E, defined by $a \in B$, $Ea = p^{-1}C^*pa \vee (a)$ is easily seen to satisfy

$$Ea = \{b \in B\colon\ b \geqslant \bar{a}\}\ .$$

Hence if $a \in \bar{B}$, then $Ea = (\bar{a})$ and $E^2a = \bigwedge Eb$ $(b \geqslant \bar{a}, b \in B)$ turns out to be equal to $E\bar{a} = (\bar{\bar{a}}) = Ea$. Therefore $Ea = Ca = (\bar{a})$ for all $a \in \bar{B}$.

In their paper [1], Lemma 4.14, McKinsey and Tarski used a different operator. When constructed in terms of f-closure, it also satisfies Proposition 5 and is, moreover, maximal with respect to this property. Define

$$a \in B; \quad Ka = \bigwedge(\bar{b}) \quad (a \leqslant b,\ b \in \bar{B})\ .$$

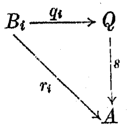
K is a f-closure operator on B. Ax 1-3 and 5 are readily verified, using Lemma 1. To prove Ax 4, note that $K^2a = \bigwedge(\bar{c})$, where c runs through the set of all elements of $\bar{B}$ dominating elements of Ka. The elements $\bar{b}$, which generate Ka belong to this set, so that $K^2a \leqslant Ka$.

Obviously, for any $a \in \bar{B}$, $Ka = (\bar{a})$. If L is any f-closure operator on B satisfying the same relation, then for $a \leqslant b$, $b \in \bar{B}$ we have $La \leqslant Lb = (\bar{b})$, hence $L \leqslant K$. Proposition 5 implies, especially, that $C \leqslant K$.

6.3. Coproducts. By means of the results of section 6.1 we are going to construct the coproduct of an arbitrary family $\{(B_i, C_i);\ i \in I\}$ of f-closure algebras. Let $Q = \coprod B_i$ denote the Boolean coproduct of the Boolean algebras B_i, while $q_i\colon B_i \to Q$ stands for the canonical injections. Given C_i on B_i, the extended operator with respect to the monomorphism q_i will be called C_i^0. Let $C = \inf C_i^0$ $(i \in I)$.

THEOREM 9. *(Q, C) is the coproduct of the (B_i, C_i) in the category of f-closure algebras and f-closure morphisms.*

Proof. Obviously, $q_i\colon (B_i, C_i) \to (Q, C)$ is a f-closure morphism for every $i \in I$. Let (A, D) be any f-closure algebra and let $r_i\colon B_i \to A$ $(i \in I)$ be f-closure morphisms. There is a unique Boolean homomorphism s such that the diagrams



commute for all $i \in I$. It must be shown that s is a f-closure morphism.

For $x \in Q$, $sC_i^0x = sq_iC_iq_i^{-1}(x) = r_iC_iq_i^{-1}(x) \geqslant Dr_iq_i^{-1}(x) \geqslant Dsx$, using commutativity of the diagrams and Lemma 4. Hence

$$x \in Q;\ Dsx \leqslant sC_i^0x\ .$$

Let D^* be the operator induced by s (given D). As D^* satisfies $Dsx \leqslant sD^*x$ for any $x \in Q$, and as it is minimal with respect to this inequality, we have $D^* \leqslant C_i^0$ for all $i \in I$ and $D^* \leqslant C$. This implies $Dsx \leqslant sD^*x \leqslant sCx$, as required.

Contrary to the product, if all the operators C_i are ordinary closure operators, the coproduct will in general not be an ordinary closure algebra. We shall describe the filters Cx for this case.

Assume all q_i to be embeddings, i.e. $q_i\colon B_i \subseteq Q$. For the sake of brevity, we shall call the B_i the *constituents of* Q. As is well known, $\{B_i;\ i \in I\}$ is an independent family of subalgebras of Q, i.e. if $b_1, b_2, \ldots, b_n$ are elements of Q belonging to constituents with pairwise different subscripts, and if $b_1 \wedge \ldots \wedge b_n = 0$, then at least one of the b_i is equal to 0. Two constituents with different subscripts have only the elements 0 and 1 in common (cf. [4], § 13).

Consider the set $B \subseteq Q$, consisting of all finite joins of elements chosen from among the constituents. Each $a \in B$, $a \neq 0, 1$ has a unique standard representation $a = b_1 \vee \ldots \vee b_n$, where the elements b belong to constituents with pairwise different subscripts and neither of them is 0 or 1. Moreover, if 1 is a join of elements belonging to constituents with pairwise different subscripts, then at least one of these elements is itself equal to 1.

Let a be any element of B, $a = b_1 \vee \ldots \vee b_n$ a representation of a by elements b of the constituents. We may assume that none of the b is equal to 1 and that not all b are equal to 0. Drop all b equal to 0; collect all b belonging to the same consituents and substitute their joins. Let $a = c_1 \vee \ldots \vee c_r$ be the result of this process. We may assume that no 0 and no 1 occurs among the elements c; in this way we have reached a standard representation.

Suppose that $a = d_1 \vee \ldots \vee d_s$ is another standard representation. Then $c_1 \leqslant d_1 \vee \ldots \vee d_s$, which is equivalent to $c_1 \wedge d_1' \wedge \ldots \wedge d_s' = 0$ (the prime denoting complement in Q). Because none of the elements in this equation is 0 or 1, there must be one d, say d_1, which belongs to the same constituent as c_1. Therefore $c_1 \wedge d_1' = 0$, i.e. $c_1 \leqslant d_1$. Pairing off in this way the c and the d and starting all over with the d yields the desired result. A similar argument leads to a proof of the second assertion.

On B we introduce an operator, denoted by a bar, as follows: let $a \in B$, $a \neq 0, 1$, and let $a = b_1 \vee \ldots \vee b_n$ be the standard representation of a. $b_1^*, \ldots, b_n^*$ will stand for the closures of the b in their respective constituents. Define

$$\bar{a} = b_1^* \vee \ldots \vee b_n^*; \quad \bar{0} = 0; \quad \bar{1} = 1\ .$$

It is easy to verify the following properties:

$$b \in B \Rightarrow \bar{b} \in B; \quad b \leqslant \bar{b}; \quad \overline{b \vee c} = \bar{b} \vee \bar{c}; \quad \bar{\bar{b}} = \bar{b} \quad (b, c \in B).$$

We consider again the operators C_i, C_i^0, C defined in this section. Recall that each C_i is assumed to be an ordinary closure operator on B_i; by the preceding definition, C_i can now be expressed by a bar.

Lemma 6. *For* $a \in B$, $C\bar{a} = (\bar{a})$.

Proof. First, let a be an element of a constituent B_i. Because $C \leqslant C_i^0$, the equality $CC_i^0 a = C_i^0 a$ holds. The definition of C_i^0 implies $C_i^0 a = (\bar{a})$. (Use the Corollary to Theorem 4.) Hence $C\bar{a} = (\bar{a})$ is true in this special case. Now suppose that a is any element of B, $a \neq 0, 1$, and $a = b_1 \vee \ldots \vee b_n$ the standard representation of a. Then $\bar{a} = \bar{b}_1 \vee \ldots \vee \bar{b}_n$ and $C\bar{a} = C\bar{b}_1 \vee \ldots \vee C\bar{b}_n = (\bar{b}_1) \vee \ldots \vee (\bar{b}_n) = (\bar{a})$, as required.

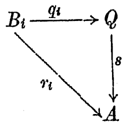
We are now able to prove

Proposition 6. *For* $x \in Q$, *the filter* Cx *is generated by all elements* $\bar{a} \in B$ *with* $x \leqslant \bar{a}$.

Proof. Let $Dx = \bigwedge (\bar{a})$ $(x \leqslant \bar{a}, \bar{a} \in B)$. We shall first show that D is a f-closure operator on Q. Ax 1-3 and 5 follow easily from the properties of the operator $a \to \bar{a}$. To prove Ax 4, consider $D^2 x = \bigwedge Dy$ $(y \in Dx)$. Obviously, $D^2 x$ is generated by all $\bar{b} \in B$ dominating elements of Dx. As each generator $\bar{a}$ of Dx has this property too, $D^2 x \leqslant Dx$ for all $x \in Q$.

The definition of C_i^0 yields $C_i^0 x = \bigwedge (\bar{b})$ $(x \leqslant b, b \in B_i)$. Hence $D \leqslant C_i^0$ for all $i \in I$, and $D \leqslant C$. On the other hand, by Lemma 6, $x \leqslant \bar{a}$, $\bar{a} \in B$ implies $Cx \leqslant C\bar{a} = (\bar{a})$, so that $C \leqslant D$. This proves our assertion.

We consider the case of a complete Q. In this case, $x^* = \inf Cx = \inf\{\bar{a} \in B\colon x \leqslant \bar{a}\}$ exists for all $x \in Q$ and $x \to x^*$ is an ordinary closure operator on Q (Theorem 6). This operator satisfies the following conditions: (1) with respect to it, all the injections $q_i\colon B_i \to Q = B_i \subseteq Q$ are closure morphisms, (2) it is maximal with respect to this property. Obviously, these conditions are similar to the usual non-categorical requirements for a coproduct in topology. To investigate the categorical situation, let A be any closure algebra with a closure operator $a \to \hat{a}$, and let $r_i\colon B_i \to A$ be closure morphisms. Finally, let



be the corresponding commutative diagram of Boolean homomorphisms.

As shown in the proof of Theorem 9, the homomorphism s satisfies

$$x \in Q\colon (\widehat{sx}) \leqslant sCq.$$

If A is assumed to be complete, this relation implies $\widehat{sx} \leqslant \inf sCx$, but does not allow to infer $\widehat{sx} \leqslant sx^* = s \inf Cx$. If, however, all the Boolean algebras involved belong to the category of complete algebras and complete homomorphisms, s will be a closure morphism and $(Q, {}^*)$ the categorical coproduct of the B_i.

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