Covering three-manifolds with open cells

by

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1. Introduction and definitions. Let M be a piecewise-linear (combinatorial) n-manifold. If M is closed (i.e., compact and without boundary) and connected (as we always assume a manifold to be), then no more than $(n+1)$ piecewise-linear n-cells are required to cover M (see Zeeman [4], page 200). If two n-cells suffice, then M must be a topological n-sphere. We consider here the first instance in which there is a gap between these lower and upper bounds on the number of piecewise-linear n-cells needed to cover M: Which closed three-manifolds M can be obtained by gluing together three 3-cells?

It is not hard to see (lemma 1) that $\pi_1(M)$ must be a free group for such a 3-manifold M. This suggests the more precise result obtained here. That is, if we define a *punctured cube* to be a compact 3-manifold M such that M is embeddable in S^3 and such that ∂M is non-empty and consists entirely of 2-spheres, then we prove: A 3-manifold M (possibly non-orientable) can be covered by three open 3-cells if and only if M contains a finite disjoint collection of polyhedral 2-spheres $S_1, S_2, \ldots, S_h$ $(h \geqslant 0)$ such that if U_i is the interior of a thin regular neighborhood of S_i in M, then $M - \bigcup U_i$ is S^3 or a punctured cube. In particular M cannot contain a *fake* 3-*cell* (compact contractible 3-manifold which is not a 3-cell). Thus if the 3-dimensional Poincaré conjecture is false, then no counterexample can be covered by less than the maximum number (four) of open 3-cells. We note the contrast between this fact and the fact ([4], corollary 2) that any combinatorial homotopy 4-sphere can be covered by three open 4-cells. A 3-manifold M satisfying the conclusion of the result above is called a 3-*sphere-with-handles* (of genus $h \geqslant 0$). We note that a 2-sphere in a 3-manifold is always 2-sided so the regular neighborhood $\overline{U}_i$ of S_i (the ith handle) always has the form $S_i \times [0, 1]$.

If M is a punctured cube, the result M^* of adding a finite number

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of orientable handles of index one to ∂M (that is, one identifies pairs of disjoint polyhedral 2-cells in ∂M, in each case using an orientation-reversing homeomorphism) will be called a *special punctured-cube-with-handles* provided that M^* is embeddable in S^3. This simply means that the added handles do not join up different components of ∂M. We note that if M^* is a special punctured-cube-with-handles and M^{**} is obtained from M^* by identifying a disjoint pair of polyhedral 2-cells in the *same* component of ∂M^* by an orientation reversing homeomorphism, then M^{**} is also a special punctured-cube-with-handles.

By the *connected sum* $M_1 \# M_2$ of two disjoint closed piecewise-linear 3-manifolds M_1 and M_2, we shall mean the 3-manifold obtained by choosing a piecewise-linear closed 3-cell B_i $(i = 1, 2)$ in M_i and identifying $M_1 - \operatorname{Int} B_1$ with $M_2 - \operatorname{Int} B_2$ along ∂B_1 and ∂B_2 by a piecewise-linear homeomorphism h. If each M_i is oriented, h will always be chosen to be orientation-reversing, so that $M_1 \# M_2$ will be oriented. We note that if M is a 3-sphere-with-handles of genus $h > 0$, then we can write $M = M_1 \# \# M_2 \# \ldots \# M_h$ where either $M_i = S^2 \times S^1$ for $1 \leqslant i \leqslant h$ (in the case where M is orientable) or $M_i = S^2 \times S^1$ for $1 \leqslant i \leqslant h-1$ and M_h is a 3-sphere with one "non-orientable" handle.

By a *surface* we mean a closed 2-manifold. A polyhedral surface S in the interior of a 3-manifold M will be called *compressible* if there exists a polyhedral 2-cell D in M such that $D \cap S = \partial D$ and ∂D is not contractible in S. A non-compressible surface is *incompressible*. Finally, S^n denotes the n-sphere and Z the additive group of integers.

2. The union of two punctured cubes contains no fake cube.

THEOREM. *Suppose that M is a closed piecewise-linear 3-manifold which can be covered by three open 3-cells. Then M is a 3-sphere-with-handles. In particular if M is simply-connected, then $M = S^3$.*

Proof. We note that the converse is also true; for if M is a 3-sphere-with-handles, then M has a spine which is the union of two collapsible complexes; thus M can be covered by three open 3-cells.

The proof is assembled from the subsequent lemmas as follows. We can shrink the open cells covering M to obtain polyhedral, closed 3-cells B_1, B_2, B_3 whose interiors cover M. By lemma 1, $\pi_1(M)$ is a free group; hence by the corollary to lemma 2 we can write M as a connected sum, $M = M' \# \Sigma$, where M' is a 3-sphere-with handles and Σ is a homotopy 3-sphere. To complete the proof we must show that $\Sigma = S^3$. This is equivalent to showing that M contains no fake 3-cell. If there were a polyhedral fake 3-cell $F \subset M$, we could assume, by performing an isotopy on M if necessary, that $F \cap B_3 = \emptyset$; hence $F \subset \operatorname{Int} B_1 \cup \operatorname{Int} B_2$. It then follows from lemma 5 with $B_i = P_i$ $(i = 1, 2)$ that F is indeed a real 3-cell.

We note that the generality in stating lemma 5 (also lemmas 3 and 4) for punctured 3-cells rather than merely for 3-cells facilitated their proof, but is not needed for the proof of this theorem.

LEMMA 1. *If M is a closed, combinatorial n-manifold, $n \geqslant 3$, which can be covered by the interiors of three closed polyhedral combinatorial n-cells, then $\pi_1(M)$ is a free group (possibly trivial).*

Proof. Let B_1, B_2, B_3 be closed, polyhedral combinatorial n-cells in general position, and whose interiors cover M. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(M - \operatorname{Int} B_3) & \xrightarrow{\;i_*\;} & \pi_1(M) \\ & {\scriptstyle j_*}\searrow \quad \nearrow{\scriptstyle k_*} & \\ & \pi_1(B_1 \cup B_2) & \end{array}$$

where i, j, and k are inclusions. Since $n \geqslant 3$, i_* is an isomorphism; thus j_* is a monomorphism. Since B_1 and B_2 are simply connected, it follows from van Kampen's theorem that $\pi_1(B_1 \cup B_2)$ is a free group of rank one less than the number of components of $B_1 \cap B_2$. Thus $\pi_1(M)$ is isomorphic to a subgroup of a free group and, hence, is a free group.

The following result has been proven by Papakyriakopoulos (see Theorem 32.1 of [2]) in the case where M is orientable. We could give a proof similar to his in the non-orientable case, using the theory of ends, but we prefer the more elementary proof below.

LEMMA 2. *Let M be a closed piecewise-linear 3-manifold whose fundamental group is a free group on h free generators. Then M contains a disjoint collection of polyhedral 2-spheres $S_1, S_2, \ldots, S_h$ such that $M - \bigcup S_i$ is connected and simply-connected.*

Proof. We proceed by induction on h, noting that the case $h = 0$ is trivial. Assume $h > 0$. In this case, we claim that M contains a 2-sided incompressible surface S which fails to separate M. To construct such a surface, we use the techniques of [3]. A sketch of the proof of the existence of S follows in the next paragraph.

Since $H_1(M; Z)$ is infinite, there is a piecewise-linear mapping $f\colon M \to S^1$ such that the induced mapping f_* is a homomorphism of $H_1(M; Z)$ *onto* $Z = H_1(S^1; Z)$. For appropriate choice of $p \in S^1$, $f^{-1}(p)$ will be a collection of 2-sided surfaces in M, and some component S_0 of $f^{-1}(p)$ will fail to separate M. If S_0 is compressible, we apply the loop theorem and Dehn's lemma to find a polyhedral disk $D \subset M$ such that $D \cap S_0 = \partial D$ and ∂D is not contractible on S_0. We then split $S_0 \cup D$ along D to obtain either one or two new 2-sided surfaces, neither of which is a 2-sphere in the second case, and at least one of which S_1 fails to separate M. We continue this process with S_1, if necessary, noting that

$\chi(S_0) < \chi(S_1) \leqslant 2$, so that the procedure must terminate after (say) k steps. We take $S = S_k$, the desired surface.

Since S is 2-sided and incompressible, the kernel of the inclusion-induced homomorphism $\pi_1(S) \to \pi_1(M)$ is trivial. Hence $\pi_1(S)$ is a free group, and in fact must be the trivial group. So S is a (necessarily 2-sided) 2-sphere which fails to separate M. The proof is now completed as in case (2) of the proof of [2], Theorem 32.1, by applying induction to the 3-manifold obtained by cutting M along S and then filling in the two resulting boundary 2-spheres with 3-cells.

COROLLARY. *Assume the hypotheses of lemma 2. Then $M = M' \# \Sigma$, where M' is a 3-sphere-with-handles and Σ is a homotopy 3-sphere.*

A special case of lemma 3 below appears in [1].

LEMMA 3. *Suppose that N is an orientable 3-manifold, not necessarily compact but with empty boundary, such that each orientable surface in N separates N (e.g. suppose that N is simply connected). If P_1, P_2 are punctured 3-cells (polyhedral and in general position) in N such that $P_1 \cap P_2 \neq \emptyset$ and $P_1 \cup P_2 \neq N$, then $P_1 \cup P_2$ is a special punctured-cube-with-handles.*

Proof. The proof follows by induction on the number, n, of components of $\partial P_1 \cap \partial P_2$.

If $n = 0$, then since $P_1 \cup P_2 \neq N$, some component S of $\partial P_1 \cup \partial P_2$, say $S \subset \partial P_1$, separates N into two components such that the closure of one of these contains $P_1 \cup P_2$. Now $\overline{P_2 - P_1}$ is the union of a finite collection of mutually exclusive punctured 3-cells each of which meets P_1 in exactly one 2-sphere in ∂P_1 and none of which intersects S. Thus it is easy to see, in this case, that $P_1 \cup P_2$ is a punctured 3-cell.

Now suppose $n > 0$. We choose a component J of $\partial P_1 \cap \partial P_2$ such that J bounds a disk $D \subset \partial P_1$ with $\operatorname{Int} D \cap \partial P_2 = \emptyset$. Thus either $P_2 \cap \operatorname{Int} D = \emptyset$ or $\operatorname{Int} D \subset \operatorname{Int} P_2$.

If $\operatorname{Int} D \cap P_2 = \emptyset$, we add to P_2 a 3-cell neighborhood of D in such a way to obtain a punctured 3-cell P_2^* such that $P_1 \cup P_2^*$ is homeomorphic to $P_1 \cup P_2$ and $\partial P_1 \cap \partial P_2^*$ has fewer than n components. By induction $P_1 \cup P_2^*$, hence $P_1 \cup P_2$, is a special punctured-cube-with-handles.

If $\operatorname{Int} D \subset \operatorname{Int} P_2$, then we choose a piecewise linear embedding $D \times [-1, 1] \subset P_2$ satisfying

$$D \times \{0\} = D\,,$$
$$\partial D \times [-1, 1] \subset \partial P_2\,,$$
$$(\operatorname{Int} D) \times [-1, 1] \subset \operatorname{Int} P_2\,,$$
$$D \times (0, 1] \cap P_1 = \emptyset\,, \text{ and}$$
$$D \times [-1, 0) \subset \operatorname{Int} P_1\,.$$

In ∂P_2 there are 2-cells E_1 and E_2 such that $E_1 \cap E_2 = \partial E_1 = \partial E_2 = \partial D$ and such that $\partial D \times [0, 1] \subset E_1$ ($E_1 \cup E_2$ is the component of ∂P_2 containing ∂D). If $P_1 \cap \operatorname{Int} E_1 = \emptyset$, we can use the argument of the preceding case (with E_1 taking the role of D) to conclude that $P_1 \cup P_2$ is a special punctured-cube-with-handles. Thus we assume that $P_1 \cap \operatorname{Int} E_1 \neq \emptyset$. Now $P_2 - D \times (-1, 1)$ is the union of two disjoint punctured 3-cells P_{21} and P_{22} with

$$E_2 - \partial D \times [-1, 0] \subset \partial P_{22} \quad \text{and} \quad E_1 - \partial D \times [0, 1] \subset \partial P_{21}\,.$$

Since P_1 meets both $E_1 - \partial D \times [0, 1]$ and $E_2 - \partial D \times [-1, 0]$, there is a polyhedral arc A in $\operatorname{Int} P_1$ such that

$$\operatorname{Int} A \cap P_2 = \emptyset\,,$$
one end point of A is in ∂P_{21}, and
the other end point of A is in ∂P_{22}.

We add to $P_{21} \cup P_{22}$ a small neighborhood of A to obtain a punctured 3-cell P_2^* satisfying

$$P_1 \cup P_2^* \subset P_1 \cup P_2\,,$$
$$(P_1 \cup P_2) - (P_1 \cup P_2^*) = D \times (0, 1)\,, \text{ and}$$
$\partial P_1 \cap \partial P_2^*$ *has fewer than n components*.

By induction $P_1 \cup P_2^*$ is a special punctured-cube-with-handles. Since $P_1 \cup P_2$ is obtained from $P_1 \cup P_2^*$ by attaching the handle $D \times [0, 1]$ and since N is orientable, the proof will be complete if we show that $D \times \{0\}$ and $D \times \{1\}$ lie in the same component of $\partial(P_1 \cup P_2^*)$. If this is not the case, there is a closed, orientable 2-manifold S in $\operatorname{Int}(P_1 \cup P_2^*)$ and an arc B in $P_1 \cup P_2^*$ with one end point in $D \times \{0\}$, the other in $D \times \{1\}$ and with $B \cap S$ consisting of exactly one point at which B pierces S. We join the end points of B by an arc in $D \times [0, 1]$ to obtain a simple closed curve which meets S in exactly one point and pierces S at this point. This leads to the contradiction that S fails to separate N.

LEMMA 4. *Suppose that M is a closed 3-manifold and F and F' are compact, polyhedral contractible 3-manifolds-with-boundary in M with $F \subset \operatorname{Int} F'$. If there are punctured 3-cells P_1 and P_2 in M such that $F \subset \operatorname{Int} P_1 \cup \operatorname{Int} P_2$, then there are punctured 3-cells P_1^* and P_2^* in M (polyhedral and in general position) such that $F \subset \operatorname{Int} P_1^* \cup \operatorname{Int} P_2^*$ and $P_1^* \cup P_2^* \subset F'$.*

Proof. We assume that P_1 and P_2 are polyhedral and that $\partial P_1 \cup \cup \partial P_2 \cup \partial F'$ is in general position. We let $n =$ (number of components of $\partial P_1 \cap \partial F'$) + (number of components of $\partial P_2 \cap \partial F'$).

If $n = 0$ we let $P_i^* = P_i \cap F'$ $(i = 1, 2)$.

If $n > 0$ then there is a disk $D \subset \partial F'$ and a value of i such that $\partial D \subset \partial P_i \cap \partial F'$ and $\operatorname{Int} D \cap \partial P_i = \emptyset$. Suppose $i = 1$.

If $\operatorname{Int} D \cap P_1 = \emptyset$, we add to P_1 a 3-cell neighborhood of D to obtain a punctured 3-cell P_1' such that $F \subset \operatorname{Int} P_1' \cup \operatorname{Int} P_2$ and $\partial P_1' \cap \partial F'$ has fewer components than does $\partial P_1 \cap \partial F'$. The conclusion then follows by induction.

If $\operatorname{Int} D \subset \operatorname{Int} P_1$, we remove (as in lemma 3) a small product neighborhood of D from P_1 to obtain two disjoint punctured 3-cells P_{11} and P_{12}. We require that the neighborhood we remove is small enough that $F \subset \operatorname{Int} P_{11} \cup \operatorname{Int} P_{12} \cup \operatorname{Int} P_2$. If one of P_{1j} (say P_{12}) does not meet F' we let $P_1' = P_{11}$. If both meet F' we take an arc A in $\operatorname{Int} F'$ such that $\operatorname{Int} A \cap (P_{11} \cup P_{12}) = \emptyset$ and one end point of A is in ∂P_{11} and the other is in ∂P_{12}. In this case we let $P_1' = P_{11} \cup P_{12} \cup$ (small neighborhood of A). In either case the construction can be made so that P_1' is a punctured 3-cell such that $F \subset \operatorname{Int} P_1' \cup \operatorname{Int} P_2$ and $\partial P_1' \cap \partial F'$ has fewer components than does $\partial P_1 \cap \partial F'$. Again the conclusion follows by induction.

LEMMA 5. *Suppose that M is a closed 3-manifold and F is a (polyhedral, compact) contractible 3-manifold-with-boundary in M. If there exist punctured 3-cells P_1 and P_2 (polyhedral and in general position) in M such that $F \subset \operatorname{Int} P_1 \cup \operatorname{Int} P_2$ then F is a (combinatorial) 3-cell.*

Proof. We add a collar to F to obtain F' such that $F \subset \operatorname{Int} F'$ and $\overline{F'-F} = \partial F \times [0, 1]$. By lemma 4 we assume that $P_1 \cup P_2 \subset \operatorname{Int} F'$.

Then by lemma 3 (with $N = \operatorname{Int} F'$), $P_1 \cup P_2$ is a special punctured-cube-with-handles. Thus F can be piecewise-linearly embedded in S^3 and hence is a 3-cell.

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f-closure algebras

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1. Introduction. In the theory of closure algebras (cf. [2] and the references given there) several elementary algebraic problems can only be solved if appropriate completeness conditions hold for the algebras involved. The following are examples of this situation. A denotes a closure algebra, B a Boolean algebra.

1.1. Consider the set $\mathfrak{C}$ of all closure operators on B, ordered as usual. Whereas in topology the corresponding set of all topologies is a complete lattice, this will in general be no longer true for $\mathfrak{C}$. If B is complete, however, $\mathfrak{C}$ is a complete lattice.

1.2. Consider a Boolean epimorphism $A \to B$. In his paper [2], Sikorski solved the problem of defining a suitable closure operator on B as an analogue to the quotient topology. His construction makes use of a basis and of several assumptions about completeness properties of A.

1.3. The inverse problem of lifting a closure operator has also been investigated by Sikorski. Given a Boolean epimorphism $B \to A$, can a suitable closure operator be defined on B similar to the topology induced by a mapping on its domain? Sikorski (cf. [2], [3]) constructed a closure operator on B in such a way that its quotient operator on A coincides with the given one. Again he assumed the existence of a basis and σ-completeness.

1.4. If $A \to B$ or $B \to A$ are Boolean monomorphisms instead of epimorphisms, one is confronted with the problems of extension and of contraction of closure operators. Both problems have been investigated, though only incidentally, by McKinsey and Tarski (cf. [1]), using complete algebras.

1.5. Given a family A_i $(i \in I)$ of closure algebras, one might be interested in their product and in their coproduct. While the construction of the product is trivial, the coproduct will in general not exist. Let Q denote the Boolean coproduct of the Boolean algebras A_i (called "product" in [4]). A suitable closure operator can be defined on Q using the methods mentioned in 1.1 and 1.4, provided Q is complete.