On decreasing sequences of compact absolute retracts

by

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1. Introduction. It is known that every compact metric space can be written as the intersection of a decreasing sequence of compact ANR's. The purpose of this note is to characterize the compact metric spaces which can be written as the intersection of a decreasing sequence of compact AR's.

In [5], the author defined and studied the class of absolutely neighborhood contractible spaces. We will show that a compact metric space is the intersection of a decreasing sequence of compact AR's if and only if it is absolutely neighborhood contractible. Absolute neighborhood contractibility can be characterized in many other ways. Several characterizations are given in [5], and it is known that the class of compact absolutely neighborhood contractible spaces coincides with the class of fundamental absolute retracts recently introduced by Borsuk [2], [3]. We will summarize the known characterizations of compact absolute neighborhood contractible spaces in our main theorem.

2. Statement of the main theorem. By an ANR (or AR) we mean an ANR (or AR) for the class of all metrizable spaces. If B is a metric space, then $\mathrm{ANR}(B)$ denotes the class of all ANR's which contain B as a closed subset.

A subset B of a space Y is said to be *neighborhood contractible in* Y if B is contractible in every neighborhood of itself in Y. A metric space B is said to be *absolutely neighborhood contractible* if B is neighborhood contractible in every $Y \in \mathrm{ANR}(B)$ [5].

THEOREM. *The following statements concerning a compact metric space B are equivalent:*

(a) *B is absolutely neighborhood contractible.*

(b) *There exists a $Y \in \mathrm{ANR}(B)$ such that B is neighborhood contractible in Y.*

(c) *For every $Y \in \mathrm{ANR}(B)$, B is contractible in Y.*

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(d) *For every* $Y \in \mathrm{ANR}(B)$, *there exists a neighborhood* V *of* B *in* Y *such that for every metric pair* (X, A), *each map* (1) $f\colon A \to V$ *has an extension* $F\colon X \to Y$.

(e) *For every* $Y \in \mathrm{ANR}(B)$, *the natural projection* $p\colon Y \to Y/B$ *has a left homotopy inverse.*

(f) *For every* $Y \in \mathrm{ANR}(B)$, p *is a homotopy equivalence.*

(g) *For every* $Y \in \mathrm{ANR}$, *every map from* B *into* Y *is nullhomotopic.*

(h) B *is a weak proximate absolute retract in the sense of Yandl* [6].

(i) B *is a fundamental absolute retract in the sense of Borsuk* [2].

(j) B *is the intersection of a decreasing sequence of compact contractible metric spaces.*

(k) B *is the intersection of a decreasing sequence of compact absolute retracts.*

A proof of the equivalence of (a)-(h) can be found scattered throughout [5]. In [3], Borsuk proved that a closed subset of the Hilbert cube is a weak absolute retract if and only if it is contractible in each of its neighborhoods; hence (a) $\Rightarrow$ (i) $\Rightarrow$ (b). (k) $\Rightarrow$ (j) $\Rightarrow$ (g) is obvious; the rest of this paper is devoted to showing that (a) $\Rightarrow$ (k).

3. Some lemmas. In this section we will establish some notation and state three essentially known lemmas for future reference.

Let H^* denote real sequential Hilbert space with norm defined by

$$\|(a_n)\| = \Big(\sum_{n=1}^{\infty} a_n^2\Big)^{1/2}.$$

Let $H \subset H^*$ be the set

$$H = \{(a_n) \in H^* \mid 0 \leqslant a_n \leqslant 1/n\}.$$

H is homeomorphic to the Hilbert cube. $\Theta \in H$ is the point all of whose coordinates are zero. The closed interval $[0, 1]$ is denoted by I.

If (X, ϱ) is a metric space, then a deformation h_t on X is called an *ε-deformation*, $\varepsilon > 0$, if for each $x \in X$ the ϱ-diameter of the set $\{h_t(x) \mid t \in I\}$ is less than ε.

LEMMA 1. *A compact metric space* X *is an* ANR *if and only if for every* $\varepsilon > 0$ *there exists an* ε*-deformation* h_t *on* X *such that* $h_1(X)$ *is an* ANR.

Proof. This is an immediate consequence of [4], IV, 5.3.

Recall that if $f\colon X \to Y$ is a map, then the mapping cylinder C_f of f is the space obtained from the free union of $X \times I$ and Y by identifying each point $(x, 1) \in X \times \{1\}$ with $f(x) \in Y$.

(1) A map is a continuous function. A metric pair (X, A) is a metric space X together with a closed subset A.

LEMMA 2. *If* X *and* Y *are compact* ANR's *and if* $f\colon X \to Y$ *is a map, then* C_f *is an* ANR.

Proof. This is an immediate consequence of [4], VI, 1.1 and 1.2.

The *cone* over a space X is the quotient $X \times I / X \times \{1\}$. The cone is homeomorphic to C_f, where f is the unique map from X to a point. Consequently, by Lemma 2, the cone over a compact ANR is an ANR.

LEMMA 3. *If* $X \subset H$ *is a compact absolutely neighborhood contractible space, then there exists a decreasing sequence of compact* ANR's $\{X_1, X_2, \ldots\}$ *such that*

(a) $X_1 = H$,

(b) *each* X_n *is contractible in* X_{n-1}, $n > 1$, *and*

(c) $X = \bigcap_{n=1}^{\infty} X_n$.

Proof. Let $\{Y_k \mid k \geqslant 1\}$ be a decreasing sequence of compact ANR's such that $X = \bigcap_{k=1}^{\infty} Y_k$ (2). We may and do assume that $Y_1 = H$. Take $X_1 = Y_1$ and recursively define $X_{n+1} = Y_{k(n)}$, where $k(n)$ is the smallest index greater than n such that $Y_{k(n)}$ is contractible in X_n (3). $\{X_n \mid n \geqslant 1\}$ is the desired sequence.

4. Proof of (a) $\Rightarrow$ (k). Let X be a compact absolutely neighborhood contractible space. Embed X in H, and let $\{X_1, X_2, \ldots\}$ be a decreasing sequence of compact ANR's (satisfying the conclusions of Lemma 3).

Consider the set $Z = H \times H \times I \times I$. Let α, β, γ and δ be the projections of Z onto its factor spaces H, H, I and I, respectively. Thus every $z \in Z$ can be written uniquely in the form $(\alpha(z), \beta(z), \gamma(z), \delta(z))$. Metrize Z by the rule

$$\varrho(z, z') = \max\{\|\alpha(z) - \alpha(z')\|,\ \|\beta(z) - \beta(z')\|,\ |\gamma(z) - \gamma(z')|,\ |\delta(z) - \delta(z')|\}.$$

Recalling that $X_n \subset X_1 = H$, and writing $\tau_n = 1/n$ for each positive integer n, define

$$Y_n = \{(x, \tau_n x, 0, \tau_n) \in Z \mid x \in X_n\}, \quad n \geqslant 1.$$

Since X_n is contractible in X_{n-1}, $n > 1$, there exists such a map $f_n\colon X_n \times [0, \tau_n] \to X_{n-1}$ that $f_n(x, 0) = x$ for all $x \in X_n$ and such that $f_n|\, X_n \times \{\tau_n\}$ is constant. Define a map $\Psi_n\colon X_n \times [0, \tau_n] \to Z$ by

$$\Psi_n(x, t) = \big((\tau_n - t)x + (1 - \tau_n + t) f_n(x, t),\ \tau_n f_n(x, t),\ t,\ \tau_n\big) \tag{1}$$

(2) The existence of the sequence $\{Y_k\}$ follows from the methods of [1].

(3) $Y_{k(n)}$ exists: for by the methods of [5], 3.5, some neighborhood V of X in X_n is contractible in X_n. Since X is the intersection of $\{Y_k\}$, some Y_k, $k > n$, lies in V; hence Y_k is contractible in X_n.

for all $x \in X_n$, $0 \leqslant t \leqslant \tau_n$; and let

$$R_n = \Psi_n(X_n \times [0, \tau_n]), \quad n > 1.$$

Inspection of (1) shows that $\Psi_n | X_n \times \{\tau_n\}$ is constant, but if $(x, t) \neq (x', t')$ and if either $t \neq \tau_n$ or $t' \neq \tau_n$, then $\Psi_n(x, t) \neq \Psi_n(x', t')$. It follows that R_n is homeomorphic to the cone on X_n; in particular, R_n is an ANR by the remarks following Lemma 2. Taking $t = 0$ in (1), we have

$$(1_0) \qquad \Psi_n(x, 0) = (x, \tau_n x, 0, \tau_n).$$

Consequently

$$(2) \qquad Y_n = \{z \in R_n |\ \gamma(z) = 0\}, \quad n > 1.$$

Since f_n is constant on $X_n \times \{\tau_n\}$, it follows that the assignment

$$(3) \qquad (x, t) \to \big(f_n(x, t), \tau_{n-1} f_n(x, t), 0, \tau_{n-1}\big), \quad x \in X_n, \quad 0 \leqslant t \leqslant \tau_n,$$

induces a (single-valued) map

$$\Phi_n\colon R_n \to Y_{n-1}, \quad n > 1.$$

Since Z is convex, we may consider the segment $\sigma(z)$ joining $z \in R_n$ to $\Phi_n(z)$. Let

$$C_n = Y_{n-1} \cup \bigcup \{\sigma(z) |\ z \in R_n\}, \quad n > 1.$$

From (1) and (3) a straightforward calculation shows that if z and z' are distinct points of R_n, then the segments $\sigma(z)$ and $\sigma(z')$ have no points in common except possibly an endpoint, and they have an endpoint in common if and only if $\Phi_n(z) = \Phi_n(z')$. From this it follows that C_n is homeomorphic to the mapping cylinder of Φ_n. Since R_n and $Y_{n-1} \cong X_{n-1}$ are compact ANR's, C_n is an ANR by Lemma 2.

Identify X with $X \times \{\Theta\} \times \{0\} \times \{0\} \subset Z$, and let

$$D_n = X \cup R_n \cup \bigcup_{k=n+1}^{\infty} C_k, \quad n > 1;$$

$$E_{mn} = R_n \cup \bigcup_{k=n+1}^{m} C_k, \quad m > n > 1;$$

$$E_{nn} = R_n, \quad n > 1.$$

From (1), (2), (3) and the definitions of the sets involved, we have

$$(4) \qquad C_n = \{z \in D_2 |\ \tau_n \leqslant \delta(z) < \tau_{n-1}\} \cup \{z \in D_2 |\ \delta(z) = \tau_{n-1} \text{ and } \gamma(z) = 0\}, \quad n > 1;$$

$$(5) \qquad R_n = \{z \in D_2 |\ \delta(z) = \tau_n\}, \quad n > 1;$$

$$(6) \qquad D_n = \{z \in D_2 |\ 0 \leqslant \delta(z) \leqslant \tau_n\}, \quad n > 1;$$

$$(7) \qquad E_{mn} = \{z \in D_2 |\ \tau_m \leqslant \delta(z) \leqslant \tau_n\}, \quad m \geqslant n > 1;$$

$$(8) \qquad Y_n = \{z \in D_2 |\ \delta(z) = \tau_n \text{ and } \gamma(z) = 0\}, \quad n > 1;$$

$$(9) \qquad X = \{z \in D_2 |\ \delta(z) = 0\}.$$

By (6) and (9),

$$X = \bigcap_{n=2}^{\infty} D_n.$$

Therefore if we can show that D_n is a compact AR, $n > 1$, then we will have established (k).

We show that D_n is compact by showing that if $\{z_i |\ i \geqslant 1\}$ is an infinite subset of D_n, then $\{z_i\}$ has an accumulation point in D_n. Since X, R_n and C_k, $k > n$, are compact, we need only show that if $z_i \in C_{k(i)}$, where $\lim_{i\to\infty} k(i) = \infty$, then $\{z_i\}$ accumulates at some point in X. By (1) and (3), z_i is of the form (a_i, b_i, c_i, d_i), where [4]

$$(10) \qquad \begin{aligned} a_i &= s_i(\tau_{k(i)} - t_i)x_i + s_i(1 - \tau_{k(i)} + t_i) f_{k(i)}(x_i, t_i) + (1 - s_i) f_{k(i)}(x_i, t_i), \\ b_i &= s_i \tau_{k(i)} f_{k(i)}(x_i, t_i) + (1 - s_i)\tau_{k(i)-1} f_{k(i)}(x_i, t_i), \\ c_i &= s_i t_i, \\ d_i &= s_i \tau_{k(i)} + (1 - s_i)\tau_{k(i)-1} \end{aligned}$$

for some $x_i \in X_{k(i)}$, $0 \leqslant t_i \leqslant \tau_{k(i)}$, $0 \leqslant s_i \leqslant 1$. Since $f_{k(i)}(x_i, t_i) \in X_{k(i)-1}$ and since $\bigcap_{k=1}^{\infty} X_k = X$, it follows that the sequence $\{f_{k(i)}(x_i, t_i)\}$ accumulates at some $x \in X$. Since $\lim_{i\to\infty} \tau_i = 0$, it follows that $\{a_i\}$ accumulates at x, and $\{b_i\} \to \Theta$, $\{c_i\} \to 0$, $\{d_i\} \to 0$. Therefore $\{z_i\}$ accumulates at $(x, \Theta, 0, 0) \equiv x \in X$. This completes the proof that D_n is compact.

It remains to prove that D_n is an AR, $n > 1$. Let n be fixed, and let $\varepsilon > 0$ be given. Consider the following statements:

(11) *For every $m \geqslant n$, E_{mn} is an ANR.*

(12) *For every $m \geqslant n$, E_{mn} is contractible.*

(13) *For some $m \geqslant n$, there exists an ε-deformation of D_n onto E_{mn}.*

Combining (11) and (13) with Lemma 1, we see that D_n is an ANR; combining (12) and (13) with the fact that every contractible ANR is an AR ([4], p. 96), we see that D_n is an AR. We proceed to establish (11)-(13).

Proof of (11). We have observed that $R_n = E_{nn}$ is an ANR. Assume inductively that E_{mn} is an ANR, $m \geqslant n$, and write $E_{m+1,n} = C_{m+1} \cup E_{mn}$. By (4), (7) and (8), $C_{m+1} \cap E_{mn} = Y_m$. We have already observed that Y_m and C_{m+1} are ANR's, and by the induction hypothesis, E_{mn}

[4] If $z_i \in Y_{k(i)-1}$, then z_i will not necessarily have the form (10). However, in this case $z_i \in C_{k(i)-1}$; therefore z_i will have the form (10) with $k(i)$ replaced by $k(i)-1$. Consequently there is no loss in generality in assuming that z_i has the form (10).

is an ANR. Therefore $E_{m+1,n}$ is an ANR [5]. This completes the induction and establishes (11).

For each $i > 1$, define a deformation g_i on C_i by

$$g_i(z,s) = \begin{cases} \big(1+s\delta'(z)-s-\delta'(z)\big)r+\big(s+\delta'(z)-s\delta'(z)\big)\Phi_i(r), & \text{if} \quad z\in\sigma(r),\ r\in R_i, \\ z & \text{if} \quad z\in Y_{i-1}, \end{cases}$$

where $\delta'(z) = \big(\delta(z)-\tau_i\big)/(\tau_{i-1}-\tau_i)$. g_i is a strong deformation retraction of C_i onto Y_{i-1}; g_i slides each segment $\sigma(r)$ to its endpoint in Y_{i-1}. Since the diameter of a segment in H or I is the distance between its endpoints, the same is true of Z (under the metric ϱ). By (1) and (3), we see that if $r = \Psi_i(x,t)\in R_i$ and if $z\in\sigma(r)$, then

$$\begin{aligned}\operatorname{diam} g_i(z\times I) &\leqslant \varrho\big(r,\Phi_i(r)\big) \\ &\leqslant \max\big\{(\tau_i-t)\big(\|x\|+\|f_i(x,t)\|\big),\ (\tau_{i-1}-\tau_i)\|f_i(x,t)\|,\ t,\ \tau_{i-1}-\tau_i\big\}.\end{aligned}$$

Since $\|\ \|$ is bounded on H by $\sum_{n=1}^{\infty}(1/n^2) < 2$, it follows that g_i is a $4\tau_i$-deformation.

Proof of (12). Since the cone over every space is contractible $E_{nn} = R_n$ is contractible. Assume inductively that E_{mn}, $m\geqslant n$, is con, tractible, and define a deformation η on $E_{m+1,n}$ by

$$\eta(z,s) = \begin{cases} g_{m+1}(z,s) & \text{if} \quad z\in C_{m+1},\ 0\leqslant s\leqslant 1, \\ z & \text{if} \quad z\in E_{mn},\ 0\leqslant s\leqslant 1. \end{cases}$$

η deforms $E_{m+1,n}$ onto the contractible set E_{mn}. Therefore $E_{m+1,n}$ is contractible, and (12) follows.

Proof of (13). Let

$$L_n = \{z\in D_n|\ \gamma(z) = 0\}.$$

For each $z\in L_n$ and for each $s\in I$ such that $\delta(z)+s\leqslant\tau_n$, there exists a unique point $z' = \lambda(z,s)\in L_n$ such that $\alpha(z') = \alpha(z)$ and such that $\delta(z') = \delta(z)+s$; in fact we can determine $\beta(z')$—and consequently z' itself—as follows: Taking $t = 0$ in (3), we have

$$(3_0) \qquad (x,0)\to(x,\tau_{n-1}x,0,\tau_{n-1}).$$

If $\delta(z)+s > 0$, then (1_0) and (3_0) yield

$$(14) \qquad \beta(z') = \delta(z')x = \big(\delta(z)+s\big)\alpha(z).$$

If $\delta(z)+s = 0$ then $s = 0$ and $\delta(z) = 0$. $s = 0$ implies that $z = z'$, and $\delta(z) = 0$ implies that $z\in X$, which in turn implies that $\beta(z) = \Theta$. Therefore (14) holds in all cases. Since $\|\alpha(z)\| < 2$, it follows that if $\delta(z) < \tau_i$, $i\geqslant n$, then

$$(15) \qquad \operatorname{diam}\lambda\big(\{z\}\times[0,\tau_i-\delta(z)]\big) < 2\tau_i.$$

[5] Recall that if A, B and $A\cap B$ are compact ANR's, then $A\cup B$ is an ANR ([4], p. 49).

From (14) and from the definition of λ we see that λ is continuous on its domain $(=\{(z,s)\in L_n\times I|\ \delta(z)+s\leqslant\tau_n\})$.

For each $m\geqslant n$, define a function $h_{mn}\colon D_n\times I\to D_n$ by

$$h_{mn}(z,s) = \begin{cases} z & \text{if} \quad z\in C_i,\ i\leqslant m,\ 0\leqslant s\leqslant 1; \\ z & \text{if} \quad z\in C_i,\ i>m,\ 0\leqslant s\leqslant\delta(z); \\ g_i\Big(z,\big(s-\delta(z)\big)/\big(\tau_{i-1}-\delta(z)\big)\Big) & \text{if} \quad z\in C_i,\ i>m,\ \delta(z)\leqslant s\leqslant\tau_{i-1},\ \delta(z)\neq\tau_{i-1}; \\ \lambda\big(g_i(z,1),\min\{s-\tau_{i-1},\tau_m-\tau_{i-1}\}\big) & \text{if} \quad z\in C_i,\ i>m,\ \tau_{i-1}\leqslant s\leqslant 1; \\ \lambda(z,\min\{s,\tau_m\}) & \text{if} \quad z\in X,\ 0\leqslant s\leqslant 1. \end{cases}$$

If $z\in C_i\cap C_{i+1} = Y_i$, then $h_{mn}(z,s)$ may be defined in more than one way. However, by (1_0), (3_0) and by the definitions of g_i and λ, it is straightforward to show that h_{mn} is single-valued. The continuity of h_{mn} is obvious except possibly on $X\times I$; on this set the continuity of h_{mn} follows from (15) and from the fact that g_i is a $4\tau_i$-deformation. Applying (15) and this property of g_i again, we see that if $m > 6/\varepsilon$, then h_{mn} is an ε-deformation. Since h_{mn} deforms D_n onto E_{mn}, (13) is established.

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