A homotopy extension theorem for fundamental sequences

by

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The notion of the fundamental sequence has been introduced by K. Borsuk ([1], p. 225) in order to study the homotopy properties of compacta lying in the Hilbert space H. It is defined as a triple $\underline{f} = \{f_k, X, Y\}$ consisting of a sequence $\{f_k\}$ of (continuous) maps $f_k\colon H \to H$ and of two compacta $X, Y \subset H$ satisfying the following condition

For every neighborhood V of Y (neighborhoods are always understood in the space H) *there exists a neighborhood U of X such that the homotopy*

$$f_k/U \simeq f_{k+1}/U \text{ in } V$$

holds for almost all k.

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are said to be *homotopic* (written $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq g_k/U \text{ in } V \quad \text{for almost all } k .$$

Let X' be a compactum such that $X \subset X' \subset H$. A fundamental sequence $\underline{f}' = \{f'_k, X', Y\}$ is said to be an *extension* of the fundamental sequence $\underline{f} = \{f_k, X, Y\}$ if $f'_k/X = f_k/X$ for every $k = 1, 2, \ldots$

In the present note we shall prove (answering a problem posed by K. Borsuk) the following

THEOREM. *Let X, X', Y be compacta in H such that $X \subset X'$, and let $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, X, Y\}$ be two homotopic fundamental sequences. If $\underline{f}$ has an extension $\underline{f}' = \{f'_k, X', Y\}$ then $\underline{g}$ has an extension $\underline{g}' = \{g'_k, X', Y\}$ homotopic to $\underline{f}'$.*

Proof. Let $V_1, V_2, \ldots$ be a decreasing sequence of open neighborhoods of Y such that

(1) *For every neighborhood V of Y the inclusion $V_k \subset V$ holds for almost all k.*

The homotopy $f \simeq g$ implies that for every $k = 1, 2, \ldots$ there is an index m_k and a closed neighborhood U'_k of X' such that

$$(2) \qquad f_i/X \simeq g_i/X \text{ in } V_k \quad \text{for every } i \geqslant m_k,$$

$$(3) \qquad f'_i(U'_k) \subset V_k \quad \text{for every } i \geqslant m_k.$$

It is clear that we can assume that

$$(4) \qquad 1 < m_1 < m_2 < \ldots \quad \text{and} \quad U'_1 \supset U'_2 \supset \ldots$$

In order to finish the proof, it suffices to construct a sequence of maps g'_i: $H \to H$, $i = 1, 2, \ldots$ satisfying the following conditions:

$$(5) \qquad g'_i/U'_k \simeq f'_i/U'_k \text{ in } V_k \quad \text{for} \quad i \geqslant m_k,$$

$$(6) \qquad g'_i/X = g_i/X \quad \text{for} \quad i = 1, 2, \ldots$$

We define g'_i as follows: If $i < m_1$ then $g'_i = g_i$. Then (5) is immaterial. Now let us assume that $k \geqslant 1$ and that for every $i < m_k$ a map g'_i: $H \to H$ satisfying (5) and (6) is already defined. Consider now an index i such that $m_k \leqslant i < m_{k+1}$. In order to define the map g'_i, first let us construct for every $j = 1, 2, \ldots, k$ a map $\hat{g}_{i,j}$: $U'_j \to V_j$ and a homotopy $\varphi_{i,j}$: $U'_j \times \langle 0, 1\rangle \to V_j$ such that

(7) $\varphi_{i,j}(x, 0) = f'_i(x)$ *and* $\varphi_{i,j}(x, 1) = \hat{g}_{i,j}(x)$ *for every point* $x \in U'_j$.

(8) *If* $j < k$, *then* $\varphi_{i,j}/(U'_{j+1} \times \langle 0, 1\rangle) = \varphi_{i,j+1}$.

$$(9) \qquad \hat{g}_{i,j}/X = g_i/X.$$

The construction of $\hat{g}_{i,j}$ and of $\varphi_{i,j}$ will be inductive (from j to $j-1$). Since V_k, as an open subset of H, is an ANR($\mathfrak{M}$) ([3], p. 391; also [2], p. 85), we infer by theorem on extension of a homotopy ([2], p. 94) and by (2) and (3) that there exists a homotopy $\varphi_{i,k}$: $U'_k \times \langle 0, 1\rangle \to V_k$ such that

$$\varphi_{i,k}(x, 0) = f'_i(x) \quad \text{for every point } x \in U'_k$$

and

$$\varphi_{i,k}(x, 1) = g_i(x) \quad \text{for every point } x \in X.$$

Setting

$$\hat{g}_{i,k}(x) = \varphi_{i,k}(x, 1) \quad \text{for every point } x \in U'_k,$$

we get $\hat{g}_{i,k}$ and $\varphi_{i,k}$ satisfying the conditions (7), (8) (trivially) and (9).

Now let us assume that $1 \leqslant j < k$ and that $\hat{g}_{i,j+1}$ and $\varphi_{i,j+1}$ have been already defined. The values of the homotopy $\varphi_{i,j+1}$ belong to $V_{j+1} \subset \subset V_j \in$ ANR($\mathfrak{M}$). Since $\varphi_{i,j+1}$ is defined on the closed subset $U'_{j+1} \times \langle 0, 1\rangle$ of $U'_j \times \langle 0, 1\rangle$ and since $\varphi_{i,j+1}(x, 0) = f'_i(x)$ for every point $x \in U'_{j+1}$ and $f'_i(U'_j) \subset V_j \in$ ANR($\mathfrak{M}$) (because $i \geqslant m_k > m_j$), we can apply again the theorem on extension of a homotopy and thus we obtain a homotopy $\varphi_{i,j}$: $U'_j \times \langle 0, 1\rangle \to V_j$ such that

$$\varphi_{i,j}(x, 0) = f'_i(x) \quad \text{for every point } x \in U'_j$$

and

$$\varphi_{i,j}/(U'_{j+1} \times \langle 0, 1\rangle) = \varphi_{i,j+1}.$$

Setting

$$\hat{g}_{i,j}(x) = \varphi_{i,j}(x, 1) \quad \text{for every point } x \in U'_j,$$

we see that $\hat{g}_{i,j}$ and $\varphi_{i,j}$ satisfy conditions (7), (8) and (9).

Now we define the map g'_i as an arbitrary map of H into itself satisfying the condition

$$g'_i(x) = \hat{g}_{i,1}(x) \quad \text{for every point } x \in U'_1.$$

It follows by (7) and (8) that

$$g'_i(x) = \hat{g}'_{i,j}(x) \quad \text{for every point } x \in U'_j,\ j = 1, 2, \ldots, k.$$

We infer by (7) and (9) that the map g'_i satisfies both conditions (5) and (6). Thus the proof of Theorem is concluded.

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223-254.

[2] — *Theory of Retracts*, Monografie Matematyczne 44, Warszawa 1967.

[3] O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Math. 1 (1951), pp. 389-408.

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