Fundamental retracts and extensions of fundamental sequences

by

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In order to extend some standard notions of the homotopy theory onto arbitrary compacta \(X, Y\) lying in the Hilbert space \(H\), I introduced in [2] the notion of the fundamental sequence from \(X\) to \(Y\), defined as an ordered triple \(f = (f_k, X, Y)\) consisting of \(X, Y\) and of a sequence \(f_k\) of (continuous) maps of \(H\) into itself satisfying the following condition:

For every neighborhood \(V\) of \(Y\) (neighborhoods are understood here always in the space \(H\)) there exists a neighborhood \(U\) of \(X\) such that

\[ f_k(U) \simeq f_{k+1}(U) \text{ in } V \quad \text{for almost all } k. \]

The set \(X\) will be said to be the domain, and the set \(Y\)—the range of the fundamental sequence \(f\).

Setting \(i_k(x) = x\) for every point \(x \in H\), we immediately see that for every compactum \(X \subseteq H\) the triple \((i_k, X, X)\) is a fundamental sequence \(f_{X, X}\), called the fundamental identity sequence for \(X\).

If \(e\) is a point of a compactum \(X \subseteq H\), then setting \(e(x) = e\) for every point \(x \in H\), we get a fundamental sequence \(g_{X, e} = (e, X, X)\) called a constant fundamental sequence for \(X\).

Let us observe that if \(X\) is a closed subset of a compactum \(X \subseteq H\), and \(Y\) is a closed subset of a compactum \(Y \subseteq H\), and if \(f = (f_k, X, Y)\) is a fundamental sequence, then \(f = (f_{X, Y}, X, Y)\) is also a fundamental sequence.

Two fundamental sequences \(f = (f_k, X, Y)\) and \(g = (g_k, X, Y)\) are said to be homotopic (in symbols: \(f \simeq g\)) if for every neighborhood \(V\) of \(Y\) there exists a neighborhood \(U\) of \(X\) such that

\[ f_k(U) \simeq g_k(U) \text{ in } V \quad \text{for almost all } k. \]

The fundamental sequences from \(X\) to \(Y\) may be considered as a generalization of the maps of \(X\) into \(Y\), and the classes of all homotopic fundamental sequences from \(X\) to \(Y\) (called fundamental classes from \(X\) to \(Y\)) may be considered as a generalization of the homotopy classes of maps of \(X\) into \(Y\).
It is known ([3], p. 242) that every fundamental sequence \( f = (f_n, X, Y) \) induces a homomorphism

\[ f_*: H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z}), \]

where \( H_n(X, \mathbb{Z}) \) denotes the \( n \)th homology group (in the sense of Vietoris or of Čech) of \( X \) over the group of coefficients \( \mathbb{Z} \), and the homotopic fundamental sequences induce the same homomorphism.

If \( x_n \) is a point of a compactum \( X \subset H \) and \( y_0 \) is a point of a compactum \( Y \subset H \), then a sequence of maps \( f_n: (H, x_n) \to (H, y_n) \) is said to be a pointed sequence from \( (X, x_0) \) to \( (Y, y_0) \) if for every neighborhood \( V \) of \( Y \) there is a neighborhood \( U \) of \( X \) such that

\[ f_n(U, x_0) \subset f_{n+1}(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } n. \]

We denote this pointed sequence by \( (f_n, (X, x_0), (Y, y_0)) \) or simply by \( f_n \).

Two pointed sequences \( f = (f_n, (X, x_0), (Y, y_0)) \) and \( g = (g_n, (X, x_n), (Y, y_n)) \) are said to be homotopic (in symbols: \( f \simeq g \)) if for every neighborhood \( V \) of \( Y \) there exists a neighborhood \( U \) of \( X \) such that

\[ f_n(U, x_0) \subset g_n(U, y_0) \text{ in } (V, y_0) \quad \text{for almost all } n. \]

One proves ([2], p. 283) that each pointed sequence \( f = (f_n, (X, x_0), (Y, y_0)) \) induces a homomorphism

\[ f_*: \pi_n(X, x_0) \to \pi_n(Y, y_0) \quad \text{for } n = 1, 2, ..., \]

where \( \pi_n(X, x_0) \) denotes the \( n \)-th fundamental group of \( (X, x_0) \), which is an appropriate generalization of the \( n \)-th homotopy group (see [2], p. 251).

In the present note I consider the concept of the extension of a fundamental sequence, which permits the introduction of some generalizations of many notions and results of the theory of retracts.

The author acknowledges his gratitude to Dr. A. Lelek and to Dr. H. Patkowska, who read the manuscript and made several important suggestions.

1. Restriction and extension of fundamental and of pointed sequences. Let \( X, Y, X' \) be compacta in the real Hilbert space \( H \) and let \( X \subset X' \). Let us consider two fundamental sequences \( f = (f_n, X, Y) \) and \( f' = (f'_n, X', Y) \) (which in particular can be pointed sequences \( (f_n, (X, x_0), (Y, y_0)) \) and \( (f'_n, (X', x_0), (Y, y_0)) \)). We say that \( f' \) is a restriction of \( f' \) to \( X \), or that \( f' \) is an extension of \( f \) onto \( X' \) if

\[ f'(x) = f(x) \quad \text{for every point } x \in X \text{ and } k = 1, 2, ... \]

It is clear that for every fundamental sequence \( f' = (f'_n, X', Y) \) and for every compactum \( X \subset X' \) there exist restrictions of \( f' \) to \( X \). In fact, one of them is the fundamental sequence \( (f_n, X, Y) \). The question of the existence of an extension is more delicate. Let us prove that

(1.1) If two fundamental sequences \( f = (f_n, X, Y), g = (g_n, X, Y) \) satisfy the condition \( f(x) = g(x) \) for every point \( x \in X \) and for \( k = 1, 2, ... \), then \( f = g \).

Proof. Let \( V \) be an open neighborhood of \( Y \). Then there exist a neighborhood \( U \) of \( X \) and an index \( k_0 \) such that

\[ f_k(U, x_0) \subset g_k(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k \geq k_0. \]

This follows, in particular, that \( V \) is a neighborhood of the compact set \( f_k(X) = g_k(X) \). Hence \( f_k(x) = g_k(x) \) for every point \( x \in X \), and there exists a neighborhood \( U_k \subset U \) of \( X \) such that

\[ f_k(U, x_0) \subset g_k(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k \geq k_0. \]

The inclusion \( U_k \subset U \) and the homotopies (1.2) and (1.3) imply that

\[ f_k(U, x_0) \subset g_k(U, x_0) \subset U_k \subset U \quad \text{in } V \quad \text{for every } k \geq k_0, \]

and consequently \( f = g \).

Now let us consider a compactum \( X \subset X' \) contained in a compactum \( X \subset H \) and let \( j: X \to X' \) be the inclusion map. Moreover, let \( j = (j', X, Y) \) and \( f = (f_n, X, Y) \) be two fundamental sequences. Consider the homomorphism

\[ j_*: H_n(X, \mathbb{Z}) \to H_n(X', \mathbb{Z}), \]

induced by the map \( j \) and the homomorphisms

\[ f_*: H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z}) \quad \text{and} \quad j'_*: H_n(X', \mathbb{Z}) \to H_n(Y, \mathbb{Z}) \]

induced by the fundamental sequences \( f \) and \( f' \). Let us prove that

(1.4) If \( f' \) is an extension of \( f \) then \( f = f'j_* \).

Proof. Setting \( j_*(x) = x \) for every point \( x \in H \), we get a fundamental sequence \( j = (j_n, X, X') \) such that the composition \( j'_* f = (j'_* f_n, X, Y) \) satisfies the condition \( f_n(x) = f(x) \) for every point \( x \in X \). It follows by (1.1) that \( f = j'_* f \) and consequently (1.2), p. 242) \( f = f'j_* \). It remains to observe ([2], p. 242) that the homomorphism \( j_* \) coincides with the homomorphism \( j_* \) induced by the map \( j \).
Hence by (1.4) we obtain the following

(1.5) Theorem. Let $X, X', Y$ be compacta in $H$ such that $X \subset X'$. If a fundamental sequence $f$ from $X$ to $X'$ has an extension onto $X'$, then the kernel of the homomorphism $f_2: H_2(X, Y) \to H_2(X', Y)$ induced by $f$ contains the kernel of the homomorphism $f_2': H_2(X, Y) \to H_2(X', Y)$ induced by the inclusion map $j: X \to X'$.

By an analogous argument we infer that if $a_0 \in C \cap X$ and $y_0 \in Y$ and if $f = (f_s, (X, a_0), (Y, y_0))$ and $f' = (f_s', (X, a_0), (Y, y_0))$ are two pointed sequences, then

(1.6) If $f'$ is an extension of $f$, then the induced homomorphisms $f_2': H_2(X, a_0) \to H_2(Y, y_0)$ and $f_3': H_2(X, a_0) \to H_2(Y, y_0)$ satisfy the condition $f_3' = f_2'j_0$, where $j_0$ denotes the homomorphism of $H_2(X, a_0)$ into $H_2(Y, y_0)$ induced by the inclusion map $j: X \to X'$.

As an immediate consequence of (1.4) and (1.6), one gets the following

(1.7) Theorem. Let $X, X'$, $Y$ be compacta in $H$, $X \subset X'$, $a_0 \in X$, $y_0 \in Y$ and let $f = (f_s, (X, a_0), (Y, y_0))$ be a pointed sequence. If $f'$ has an extension $f' = (f_s', (X, a_0), (Y, y_0))$, then the kernels of the homomorphisms of the groups $H_2(X, a_0)$ and $H_2(X', a_0)$ into $H_2(Y, y_0)$ and $H_2(X', a_0)$, respectively, induced by the inclusion map $j: X \to X'$ are contained in the kernels of the homomorphisms of these groups induced by $f$.

2. Weak and fundamental retractions and retractions. Let $X$ be a closed subset of a compactum $X' \subset H$. A fundamental sequence $r = (r_s, X', X)$ is said to be a fundamental retraction of $X'$ to $X$ if $r_s(x) = x$ for every point $x \in X$. Thus the fundamental retractions of $X' \to X$ are the same as the fundamental sequences from $X'$ to $X$, being extensions of the fundamental identity sequence for $X$.

If $a_0 \in X$, then a pointed sequence $r = (r_s, (X, a_0), (X, a_0))$ is said to be a fundamental retraction of the pointed compactum $(X', a_0)$ to the pointed compactum $(X, a_0)$ if $r_s(x) = x$ for every point $x \in X$, that is if $r$ is an extension of the pointed identity sequence for $(X, a_0)$.

Let us observe that

(2.1) If $r = (r_s, X', X)$ is a fundamental retraction and $(n_k)$ is a sequence of indices with $\lim_{n_k} \infty$, then setting $r_k = r_{n_k}$ for $k = 1, 2, \ldots$, we get a fundamental retraction of $X'$ to $X$.

(2.2) If $r = (r_s, X', X)$ and $r' = (r'_s, X'', X)$ are fundamental retractions then $r' r = (r'_s r'_s, X'', X)$ is a fundamental retraction.

(2.3) If $r = (r_s, X', X, a_0)$ is a fundamental retraction and $(n_k)$ is a sequence of indices with $\lim_{n_k} \infty$ then setting $r_k = r_{n_k}$ for $k = 1, 2, \ldots$, one gets a fundamental retraction of $(X', a_0)$ to $(X, a_0)$.

(2.4) If $r: X' \to X$, and $r': X'' \to X'$ (or $r: (X', a_0) \to (X, a_0)$ and $r': (X'', a_0) \to (X', a_0)$) are fundamental retractions, then $r' r$ is a fundamental retraction.

A fundamental sequence $f: X \to X'$ is said to be a $k$-fundamental sequence if there exists a fundamental sequence $g: Y \to X$ such that the composition $g f: Y \to Y$ is homotopic to the fundamental identity sequence $1_Y$. Replacing in this definition $X$ and $Y$ by pointed compacta $(X, a_0)$ and $(Y, y_0)$, one gets the notion of a pointed $k$-fundamental sequence. Let us observe that

(2.5) Every fundamental retraction is a $k$-fundamental sequence.

In fact, $r = (r_s, X', X)$ is a fundamental retraction then setting $g_s = i: H \to H$ for every $k = 1, 2, \ldots$, one gets a fundamental sequence $g = (g_s, X, X)$ such that $g r = (g r_s, X, X)$ is homotopic to the fundamental identity sequence $1_X$. Hence $r$ is a $k$-fundamental sequence. The same argument holds also in the case when compact $X$ and $X'$ are pointed.

If there exists a fundamental retraction of $X'$ to $X$ (or of $(X', a_0)$ to $(X, a_0)$), then $X$ is said to be a fundamental retract of $X'$ ($(X', a_0)$, respectively).

A closed subset $X_0$ of a compactum $X \subset H$ is said to be a fundamental neighborhood retract of $X$, if there exists a closed neighborhood $W$ of $X_0$ such that $X_0$ is a fundamental retraction of the set $W \cap X$.

If $r: X' \to X$ is a retraction of a compactum $X' \subset H$, then there exists a map $f: X \to H$ such that $f(x) = r(x)$ for every point $x \in X$. Setting $s = f(x)$ for every point $x \in X$, we get a sequence of maps $r_s: H \to H$ such that $r = (r_s, X', X)$ is a fundamental retraction of $X'$ to $X$. Hence

(2.6) Every retract of a compactum $X' \subset H$ is a fundamental retract of this compactum.

By an analogous argument one shows that

(2.7) Every neighborhood retract of a compactum $X' \subset H$ is a fundamental neighborhood retract of this compactum.

(2.8) Every fundamental retract of a fundamental retract of a compactum is a fundamental retract of this compactum.

Moreover,

(2.9) If $r = (r_s, X', X)$ is a fundamental retraction of $X'$ to $X$ and $X''$ is a compactum such that $X \subset C \subset X''$, then $r = (r_s, X'', X)$ is a fundamental retraction of $X'$ to $X$.

Let us show that the notion of the fundamental retract belong to topological invariants. More exactly, let us prove the following
(2.10) **Theorem.** Let $X$, $Y$ be two compacta in the Hilbert space $H$ and let $h$ be a homeomorphism mapping $X'$ to $Y$. Then the set $Y = h(X)$ is a fundamental retract of $Y'$ if $X$ is a fundamental retract of $X'$.  \\
Proof. Consider a map $a : H \rightarrow H$ such that $a(x) = h(x)$ for every point $x \in X$, and a map $\beta : H \rightarrow H$ such that $\beta(y) = h^{-1}(y)$ for every point $y \in Y$. Now let us assume that there exists a fundamental retraction $\tau = \{s_k, X', X\}$. Setting \[ r_k(y) = \alpha s_k \beta(y) \] for every point $y \in H$ and $k = 1, 2, \ldots$, we get a sequence of maps $r_k : H \rightarrow H$. Let us show that $\{r_k, Y', Y\}$ is a fundamental sequence.  \\
Consider a neighborhood $V$ of $X$. Since $a(X) = Y$, the set $U = a^{-1}(V)$ is a neighborhood of $X$. Since $\tau$ is a fundamental sequence from $X'$ to $X$, there exists a neighborhood $U'$ of $X'$ (in $H$) such that \[ r_k U' \supset r_{k+1} U' \] for almost all $k$.  \\
Since $\beta(Y') = X'$, the set $V' = \beta^{-1}(U')$ is a neighborhood of $Y'$ (in $H$). It follows by (2.11) that \[ r_k V' \supset r_{k+1} V' \] in $U$ for almost all $k$, whence also \[ \alpha \beta V' \supset \alpha \beta V' \] in $V$ for almost all $k$.  \\
Thus we have shown that $\tau = \{r_k, Y', Y\}$ is a fundamental sequence. Moreover, for every point $y \in Y$, we have \[ \beta(y) = h^{-1}(y) \in X', \] whence \[ r_k \beta(y) = \beta(y) = h^{-1}(y) \quad \text{and} \quad s_k \alpha \beta(y) = h \beta^{-1}(y) = h^{-1}(y) = y. \] It follows that $\tau$ is a fundamental retraction of $X'$ to $Y$.  \\
By an analogous argument, we get the following \[ r_k \beta \beta^{-1}(y) = \beta(y) = h^{-1}(y) \] for every point $y \in Y$.  \\
(2.12) **Theorem.** Let $X$, $X'$, $X''$, $Y$, $Y'$ be two pointed compacta in $H$ and let $h$ be a homeomorphism mapping $X'$ to $Y$. Then the set $Y = h(X)$ is a fundamental retract of $Y'$ if $X$ is a fundamental retract of $X'$.  \\
Proof. Since the fundamental identity sequence $\tilde{\tau} = \{r_k, X, X\}$ induces the identity homomorphism for every group $H_n(X', \mathbb{R})$, and since $\tau$ is an extension of $\tilde{\tau}$, we infer by (1.4) that the composition $r_k \beta h$ of the homomorphism $\tau_1$ and of the homomorphism $\tau_2$ induced by the inclusion map $j : X \rightarrow X'$ is the identity. Hence the homomorphism $\tau_2$ is right-inverse to the homomorphism $\tau_1$, and we infer that $\tau_1$ is an $r$-homomorphism.  \\
(2.14) **Corollary.** If $X$ is a fundamental retract of a compactum $X' \subset H$, then every homology group $H_n(X, \mathbb{R})$ is isomorphic to a factor of the group $H_n(X', \mathbb{R})$.  \\
By an analogous argument we infer by (1.6) \[ r_k \beta \beta^{-1}(y) = \beta(y) = h^{-1}(y) \] for every point $y \in Y$.  \\
(2.15) **Theorem.** If $\tau$ is a fundamental retraction of $X'$ to $Y$, then the composition $r_k \beta h$ of the homomorphism $\tau_1$ and the homomorphism $\tau_2$ induced by the inclusion map $j : X \rightarrow X'$ is the identity. Hence the homomorphism $\tau_2$ is right-inverse to the homomorphism $\tau_1$, and we infer that $\tau_1$ is an $r$-homomorphism.  \\
(2.16) **Corollary.** If $X$ is a fundamental retract of a compactum $X' \subset H$ and $a \in X$, then for $n > 1$ the group $H_n(X', \mathbb{R})$ is isomorphic to a factor of the group $H_n(X, \mathbb{R})$.  \\
(3.1) **Theorem.** Let $E^2$ denote the Euclidean plane which we consider as identical with the subset of the Hilbert space $H$ consisting of all points of the form $(x_1, x_2, 0, 0, \ldots)$, and let $p$ denote the projection of $H$ onto $E^2$ given by the formula \[ p(x_1, x_2, x_3, \ldots) = (x_1, x_2, 0, 0, \ldots). \] Let us prove the following \[ r_k \beta \beta^{-1}(y) = \beta(y) = h^{-1}(y) \] for every point $y \in Y$.  \\
(3.1) **Theorem.** Let $X$, $X'$ be two continua lying in $E^2$ such that $X \subset X'$. Then $X$ is a fundamental retract of $X'$ if and only if no component of the set $E^2 - X$ is contained in $X'$.  \\
Proof. First let us assume that there exists a fundamental retraction $\tau = \{r_k, X', X\}$ of $X'$ to $X$ and let $\tilde{\tau}$ be a bounded component of the set $E^2 - X$. Then there exists in the domain $D' = 0$ of $\tilde{\tau}$ a 1-dimensional true cycle $\gamma$ (over the group $\mathbb{R}$ of integers) homologous to zero in $D'$, but not homologous to zero in $X$. If we denote by $\langle \gamma \rangle$ the element of the group $H_1(X, \mathbb{R})$ with the representative $\gamma$, and by $\langle \gamma' \rangle$ the element of the group $H_1(X', \mathbb{R})$ with the representative $\gamma'$, then the homomorphism $j_1 : H_1(X, \mathbb{R}) \rightarrow H_1(X', \mathbb{R})$ induced by the inclusion map $j : X \rightarrow X'$, assigns $\langle \gamma' \rangle$ to $\gamma$. Since the homomorphism $j_2 : H_2(X, \mathbb{R}) \rightarrow H_2(X', \mathbb{R})$ induced by the fundamental identity sequence $\tilde{\tau}$ for $X$ is the identity and $\tau$ is an extension of $\tilde{\tau}$, we infer by (1.4) that $\langle \gamma' \rangle = j_2(\langle \gamma \rangle)$. It follows that $\gamma' = 0$ in $X'$. But $\gamma' = 0$ in $\tilde{\tau}$, whence $\tilde{\tau}$ is not contained in $X'$. It suffices to observe that also the unbounded component of $E^2 - X$ is not contained in $X'$, in order to obtain the first part of Theorem (3.1).
Now let us assume that no component of the set $E' - X$ is contained in $X'$. The collection of all components of the set $E' - X$ is finite or countable. We shall consider only the second case, because the proof in the first one is analogous, but simpler.

Let us arrange the components of $E' - X$ in a sequence $G_1, G_2, ...,$ where $G_i \neq G_j$ for $i \neq j$ and $G_b$ is the unbounded component. Since no $G_i$ is contained in $X'$, we can select a point

$$a_i \in G_i - X' \quad \text{for every } i = 1, 2, ...$$

It is clear that for every $k = 1, 2, ...$ there exists a sequence of disks $D_{k0}, D_{k1}, ...$ with interiors $D_{k0}, D_{k1}, ...$ satisfying the following conditions

1. $X \subset D_{k0}, a_i \in D_{k0} \subset D_{k1} \subset G_i$ for $i = 1, 2, ...$;
2. if $x \in G_0 \cap D_{k0}$, then $\varrho(x, X) < 1/k$
3. if $x \in G_i - D_{k0}$, then $\varrho(x, X) < 1/k$ for $i = 1, 2, ...$;
4. $D_{k0} \cap D_{k1} \subset D_{k1} \subset G_{k+1}$ for $i = 1, 2, ...$

By (1) we can assign to every $h, i = 1, 2, ...$, an open disk $U_{h,i}$ $\subset D_{k0} - X'$ such that

$$a_i \in U_{h,i} \quad \text{and} \quad U_{k,i} \supset U_{h+k,i} \quad \text{for } k, i = 1, 2, ...$$

Now let us consider a disk $D \subset E'$ such that

$$X' \cap D_{k0} \subset \hat{D}$$

and let us set

$$A_k = D - \bigcup_{i=1}^{k} U_{h,i}, \quad B_k = D_{k0} - \bigcup_{i=1}^{k} D_{k0}.$$

It is clear that $A_k, B_k$ are compacta (even curvilinear polyhedra) such that

$$X \subset B_k \subset A_k \subset E', \quad X' \subset A_k, \quad B_{k+1} \subset B_k \quad \text{for } k = 1, 2, ...,$$

and that for $k$ sufficiently large the distance of every point of $B_k$ from $X'$ is arbitrarily small.

Moreover, there exist for $k = 1, 2, ...$ a retraction

$$\varrho_k: A_k \to B_k$$

such that

$$\varrho_k(D_{k0} - U_{h,i}) = D_{k0} - \hat{D}_{k0} \quad \text{for } i = 1, 2, ..., k,$$

and a map $\varrho_k: E' \to H$ such that

$$\varrho_k(x) = \varrho_k(a) \quad \text{for every } x \in A_k,$$

$$\varrho_k(B'' - D_{k0}) \subset D_{k0} - \hat{D}_{k0} \quad \text{and} \quad \varrho_k(U_{h,i}) \subset D_{k0} \quad \text{for } i = 1, 2, ..., k.$$}

One can easily see that there exists a homotopy $\varphi_k: A_k \times (0, 1) \to B_k$ such that

$$\varphi_k(x, 0) = \varrho_k(x), \quad \varphi_k(x, 1) = \varrho_{k+1}(x) \quad \text{for every } x \in A_k$$

and

$$\varphi_k(x, t) = \varrho_k \quad \text{for every } (x, t) \in X' \times (0, 1).$$

Let us show that setting

$$r_k = \varphi_k \quad \text{for } k = 1, 2, ...,$$

we get a fundamental retraction $r = (r_k, X', X)$. First let us observe that for every neighborhood $U$ of $X$ (in $H$) the inclusion $B_k \subset U$ holds for almost all $k$. Moreover, it is clear that the set

$$V_k = p^{-1}(A_k) \subset r_k^{-1}(B_k)$$

is a neighborhood of the set $X'$ (in $H$) for every $k = 1, 2, ...$. Setting

$$\varphi_0(x, t) = \varphi_0(p(x, t)) \quad \text{for every } (x, t) \in V_k \times (0, 1),$$

we get a homotopy

$$\varphi_0: V_k \times (0, 1) \to B_k$$

joining the map $r_k|_{V_k}$ with the map $r_{k+1}|_{V_k}$ and satisfying the condition

$$\varphi_0(x, t) = x \quad \text{for every } x \in X.$$

Hence $r$ is a fundamental retraction of $X'$ to $X$ and the proof of Theorem (3.1) is complete.

4. A special kind of fundamental retraction. Let us prove the following

4.1 Theorem. If $r$ is a fundamental retraction of a compactum $Z \subset H$, then for every sequence $\{V_x\}$ of neighborhoods of $Z$ such that each neighborhood of $Z$ contains $V_x$ for almost all $k$, there exists a fundamental retraction $r' = (r', Z, Y)$ satisfying the condition $r'(y) = y$ for every point $y \in V_x$, $k = 1, 2, ...$.

Proof. Let $r = (r_x, Z, Y)$ be a fundamental retraction of $Z$ to $Y$.

First let us prove that

4.2 There exists a sequence of indices $m_1 \leq m_2 \leq ...$ such that $\lim_{i \to \infty} m_i = \infty$

and that $f(m_i - 1) - g(y, m_i(y)) \leq 1$ for every point $y \in V_i$.

Since $r(y) = y$ for every point $y \in Y$ and since for each neighborhood $V$ of $Y$ the inclusion $V \subset V$ holds for almost all $k$, one can easily see that there exists a sequence of indices $1 < k_1 < k_2 < ...$ such that

4.3 $\varrho(y, m_{i+1}(y)) \leq \frac{1}{m}$ \quad if \quad $y \in V_i$ with $j > k_i$.
Setting
\[ n_i = 1 \quad \text{for} \quad i \leq k_1, \]
\[ n_i = m+1 \quad \text{for} \quad k_m < i < k_{m+1}, \quad m = 1, 2, \ldots, \]
we get a sequence of indices \( n_1 \leq n_2 \leq \ldots \) such that \( \lim n_k = \infty \). Moreover, the inequality
\[ (n_i - 1) \cdot \phi(y, r_{n_i}(y)) \leq 1 \]
is obvious for \( i \leq k_1 \), because then
\[ n_i = 1. \]
If \( k_m < i < k_{m+1} \), then \( n_i = m+1 \) and consequently (4.3) implies that for every point \( y \in V_k \), the relation
\[ (n_i - 1) \cdot \phi(y, r_{n_i}(y)) = m \cdot \phi(y, r_{m+1}(y)) \leq 1 \]
holds. Thus the proof of (4.2) is finished.

Now let us set
\[ \lambda_k(y) = y - r_{n_i}(y) \quad \text{for every point} \quad y \in V_k. \]
In particular, \( \lambda_k(y) = 0 \) for every point \( y \in Y \). Thus one gets a map \( \lambda_k : V_k \to H \) such that
\[ (n_k - 1) \cdot \lambda_k(y) \leq 1 \quad \text{for every point} \quad y \in V_k. \]
It is clear that \( \lambda_k \) can be extended to a map \( \lambda_k : H \to H \) satisfying the condition
\[ (n_k - 1) \cdot \lambda_k(y) \leq 1 \quad \text{for every point} \quad y \in H. \]
Setting
\[ r_k(y) = r_{n_k}(y) + \lambda_k(y) \quad \text{for every point} \quad y \in H, \]
one gets a sequence of maps \( r_k : H \to H \) such that
\[ r_k(y) = r_{n_k}(y) + \lambda_k(y) = y \quad \text{for every point} \quad y \in V_k. \]
Since \( \lim n_k = \infty \), we infer that \( \{r_{n_k}, Z, Y\} \) is a fundamental sequence homotopic to \( r \), actually, a fundamental retraction of \( Z \) to \( Y \). Moreover,
\[ \phi(r_{n_k}(y), r_{n_k}(y)) = \lambda_k(y) \leq \frac{1}{n_k - 1} \quad \text{for every point} \quad y \in H \quad \text{and} \quad k > n_k. \]
It follows that the sequence \( r_k \) is obtained from the sequence \( r_{n_k} \) by an infinitely small translation. Hence
\[ r' = (r_k, Z, Y) \simeq (r_{n_k}, Z, Y) \simeq r. \]
Thus we have shown that \( r' \) is a fundamental retraction of \( Z \) to \( Y \) satisfying by (4.4) the condition of Theorem (4.1).

5. Fundamental retracts and extension of fundamental sequences. Now let us prove the following

(5.1) THEOREM. If \( X \) is a fundamental retract of \( X' \), then for every fundamental sequence \( f = (f_k, X, Y) \) there exists a fundamental sequence \( f' \) from \( X' \) to \( Y \) which is an extension of \( f \).

Proof. Let \( r = \{r_k, X', X\} \) be a fundamental retraction of \( X' \) to \( X \). It suffices to set \( f' = f \circ r \) in order to obtain the required extension of \( f \).

(5.2) THEOREM. Let \( X, X' \) be compacta such that \( X \subset X' \subset H \), and let \( f' \) be a fundamental retraction of a compactum \( Y' \subset H \). If for a fundamental sequence \( f = (f_k, X, Y) \) there exists a fundamental sequence \( f' = (f'_k, X', Y') \) such that \( f'_k(x) = f_k(x) \) for every point \( x \in X \), then there exists also a fundamental sequence \( f' \) from \( X' \) to \( Y \) which is an extension of \( f \).

Proof. It is clear that there is a sequence \( \{V_k\} \) of neighborhoods of \( Y \) satisfying both of the following conditions:

1. If \( V \) is a neighborhood of \( Y \), then \( V \subset V_k \) for almost all \( k \).
2. \( f_k(X) \subset V_k \) for every \( k = 1, 2, \ldots \)

By Theorem (4.1), there exists a fundamental retraction \( r' = (r'_k, X', Y) \) such that
\[ r'_k(y) = y \quad \text{for every point} \quad y \in V_k, \quad k = 1, 2, \ldots \]
Setting \( f' = f'_k \circ f_k, X', Y \), we get a fundamental sequence such that
\[ r'_k f_k(a) = r'_k f_k(a) = f_k(a) \quad \text{for every point} \quad x \in X, \]
because \( f'_k(x) = f_k(x) \subset f_k(X) \subset V_k \). Hence \( f' \) is an extension of \( f \).

6. Fundamental absolute retracts and absolute neighborhood retracts. A compactum \( X \subset H \) is said to be a fundamental absolute retract (shortly \( X \subset \operatorname{FAB} \)) if it is a fundamental retract of every compactum \( X \subset H \) containing \( X \). If for every compactum \( X' \) such that \( X \subset X' \subset H \) the set \( X \) is a fundamental retraction of \( X' \), then \( X \) is said to be a fundamental absolute neighborhood retract (shortly \( X \subset \operatorname{FANR} \)). Evidently every \( \operatorname{FAR} \) is a \( \operatorname{FANR} \), and every \( \operatorname{ANR} \) is a \( \operatorname{FANR} \).

It is clear that

(6.1) Every \( \operatorname{AR} \)-set lying in \( H \) is an \( \operatorname{FAR} \) and every \( \operatorname{ANR} \)-set lying in \( H \) is an \( \operatorname{FANR} \).

Now let us prove the following

(6.2) THEOREM. Every fundamental retract of an \( \operatorname{FAR} \)-set is an \( \operatorname{FAR} \)-set.

Proof. If \( Y \subset \operatorname{FAR} \) is a fundamental retraction of \( X \subset \operatorname{FAR} \), then there exists a fundamental retraction \( r = \{r_k, X, Y\} \). Let \( X' \) be a compactum such that \( Y \subset X' \subset H \). Since \( X \subset \operatorname{FAR} \), there exists a fundamental retraction \( r' = \{r_k, Y \cup X', Y\} \). Setting \( f' = r' \circ r = \{r_k, Y \cup X', Y\} \), we get by (2.2)
a fundamental retraction of the set \( Y \cap Y' \) to \( Y_\delta \). It follows by (2.9) that \((r_\delta, r_\delta, Y_\delta)\) is a fundamental retraction, whence \( Y_\delta \in \text{FAR} \).

(6.3) **Corollary.** FAR-sets are the same as fundamental retracts of the \( \text{AB-sets} \) lying in \( H \).

In order to obtain (6.3) from (6.2), it suffices to observe that

(6.4) For every compactum \( X \subset H \) there exists an \( \text{AB-sets} \) \( X' \) such that \( X \subset X' \subset H \).

In fact, the convex hull \( X' \) of \( X \) is a compactum ([4], p. 7) containing \( X \). The convexity of \( X' \) implies ([5], p. 308; also [1], p. 85) that \( X' \in \text{AB} \).

If we recall that a disk is an \( \text{AB-set} \), we get from (6.2) and (3.1) the following

(6.5) **Corollary.** Every non-empty continuum \( C \subset \mathbb{E}^2 \) which does not decompose \( \mathbb{E}^2 \) is an \( \text{FAR-set} \).

The following proposition is useful for the sequel:

(6.6) **Every compactum \( X \subset H \) is the intersection of a decreasing sequence of \( \text{ANR-sets} \) lying in the convex hull \( X' \) of \( X \).**

**Proof.** Let \( K(x, \epsilon) \) denote, for every point \( x \in X \) and every \( \epsilon > 0 \), the set of all points \( y \in H \) such that \( d(x, y) < \epsilon \). Since \( X \) is compact, there exists for every \( n = 1, 2, \ldots, \) a finite system of points \( a_{n,1}, a_{n,2}, \ldots \) \( a_{n,k_n} \in X \) such that \( X \) is contained in the set

\[
X_n = \bigcup_{j=1}^{k_n} K\left(a_{n,j}, \frac{1}{n}\right) \cap X).
\]

Setting

\[
X_\delta = X_1 \cap X_2 \cap \cdots \cap X_\delta,
\]

we obtain a decreasing sequence of compacta \( X_1, X_2, \ldots \) lying in \( X' \) such that \( X = \bigcap_{n=1}^{\infty} X_n \). It is evident that \( X_\delta \) may be represented as a finite union of sets which are common parts of the set \( X' \) and of a finite system of closed balls in \( H \). We infer, by easy induction, that each \( X_\delta \) is an \( \text{ANR-set} \). Thus the proof of (6.6) is finished.

(6.7) **Theorem.** Every fundamental retract of an \( \text{FANR-set} \) is an \( \text{FANR-set} \).

**Proof.** If \( Y_\delta \) is a fundamental retract of \( Y \in \text{FANR} \), then there exists a fundamental retraction \( r = (r_\delta, Y_\delta) \). Let \( Y' \) be a compactum such that \( Y_\delta \subset Y' \subset H \). Since \( Y \in \text{FANR} \), there exists a closed neighborhood \( V \) of \( Y \) (whence also of \( Y_\delta \)) such that \( Y \) is a fundamental retract of the set \( V \cap (Y \cap Y') \). Since \( Y_\delta \) is a fundamental retract of \( Y \), we infer by (2.2) that \( Y_\delta \) is a fundamental retract of the set \( V \cap (Y \cap Y') \), whence also by (2.9) a fundamental retract of the set \( V \cap Y' \), which is a neighborhood of \( Y_\delta \) in the space \( Y' \). Thus the proof of the first part of Theorem (6.7) is finished.

Combining Theorem (6.7) with proposition (6.6), we get:

(6.8) **Corollary.** FANR-sets are the same as fundamental retracts of \( \text{ANR-sets} \) lying in \( H \).

(6.9) **Corollary.** For every \( X \in \text{FANR} \) there exists a polyhedron \( P \) such that every group \( H_n(X, \mathbb{M}) \) is an r-image of the group \( H_n(P, \mathbb{M}) \).

**Proof.** By (6.8), there exists an ANR-set \( Z \) such that \( X \) is a fundamental retract of \( Z \). By Theorem (2.13), each group \( H_n(X, \mathbb{M}) \) is an r-image of the group \( H_n(Z, \mathbb{M}) \). Moreover, \( Z \in \text{ANR} \) implies ([1], p. 106) that there is a polyhedron \( P \) such that every group \( H_n(Z, \mathbb{M}) \) is an r-image of the group \( H_n(P, \mathbb{M}) \). It remains to recall that every r-image of an r-image of a group is an r-image of this group.

Applying Theorem (2.15) and (6.8), we get by an analogous argument the following

(6.10) **Corollary.** For every \( X \in \text{FANR} \) and for every point \( x_0 \in X \), there exist a polyhedron \( P \) and a point \( x_0 \in P \) such that every group \( H_n(P, x_0) \) is an r-image of the group \( H_n(P, x_0) \).

(6.11) **Corollary.** All Betti groups of an \( \text{FANR-set} \) are finitely generated and almost all are trivial.

(6.12) **Corollary.** All fundamental groups of an \( \text{FANR-set} \) are finitely generated.

It follows by Corollary (6.11) that every connected \( \text{FANR-set} \) lying in the plane \( \mathbb{E}^2 \subset H \) decomposes \( \mathbb{E}^2 \) into a finite number of regions.

On the other hand, it is clear that every continuum \( X \) lying in \( \mathbb{E}^2 \) and decomposing \( \mathbb{E}^2 \) into a finite number of regions is contained in a polyhedron \( P \subset \mathbb{E}^2 \) such that no component of the set \( \mathbb{E}^2 - X \) is contained in \( P \). Hence, by Theorem (3.1), we obtain the following

(6.13) **Corollary.** A continuum \( X \subset \mathbb{E}^2 \) is an \( \text{FANR-set} \), if and only if \( \mathbb{E}^2 - X \) has a finite number of components.

(6.14) **Theorem.** If \( Y \) is an \( \text{ANR-set} \) lying in the \( \text{Hilbert space} \), then every fundamental neighborhood retract of \( Y \) is an \( \text{FANR-set} \).

**Proof.** Since \( Y \in \text{ANR} \), there exists a neighborhood \( V \) of \( Y \) and a map \( \varphi \): \( H \rightarrow H \) such that

\[
\varphi(V) = Y \quad \text{and} \quad \varphi(y) = y \quad \text{for every point} \ y \in Y.
\]
Let $Y_0$ be a fundamental neighborhood retract of $Y$. Then there are a closed neighborhood $V_0$ of $Y_0$ in $H$ and a fundamental retraction $r = (r_k, V_k \cap Y, Y_k)$. To this neighborhood $V_0$ there exists a closed neighborhood $V_1 \subset V$ of $Y_0$ such that the set $\phi(Y_1)$ lies in the interior of $V_0$.

Consider now a compactum $Y'$ such that $Y_0 \subset Y' \subset H$, and let us set $r_k = r_k : H \to H$ for $k = 1, 2, ...$. In order to finish the proof of (6.14), it remains to show that

$$r' = (r_k, V_k \cap Y', Y_k)$$

is a fundamental retraction.

Since $Y_0 \subset Y'$, the equality $\phi(y) = y$ holds for every point $y \in Y_0$. We have $r_0(y) = y$ for every point $y \in Y_0$ and $k = 1, 2, ...$. In order to show that $r'$ is a fundamental retraction of $V_1 \cap Y'$ into $Y_0$, consider an arbitrary neighborhood $W$ of $Y_0$. Since $Y$ is a fundamental retraction, there exist a neighborhood $U$ of the set $V_1 \cap Y$ and a homotopy $\phi : U \times (0, 1) \to W$ such that $\phi_0(x, 0) = r_0(x)$, $\phi_0(x, 1) = r_0(x)$ for every point $x \in U$.

Since $\phi_0(Y_1)$ lies in the interior of $V_0$, there exists a neighborhood $U_1$ of the set $V_1 \cap Y'$ such that

$$\phi(U_1) \subset V_0 \cap Y \subset U.$$

It follows that the formula

$$\phi_t(x, t) = \phi_t([x, 0])$$

for $(x, t) \in U_1 \times (0, 1)$

defines a homotopy $\phi_t : U \times (0, 1) \to W$ joining $r_0$ with $r_1$.

Thus we have shown that $r'$ is a fundamental retraction and the proof of Theorem (6.14) is finished.

(6.15) Remark. Let us observe that not every fundamental neighborhood retract of an FANR-set is an FANR-set. In fact, let $A$ be the set consisting of points $a_n = (1/n, 0, 0, ...)$ with $n = 1, 2, ...$ and of the point $a_0 = (0, 0, ...)$. Let $L_n$ denote the segment in $H$ with endpoints $a_n$ and $b = (0, 1, 0, ...)$ for $n = 0, 1, 2, ...$. One can easily see (by (3.1)) that the set $X = \bigcup_{n=0}^\infty L_n$ is an FANR-set and $A$ is its fundamental neighborhood retract (even a neighborhood retraction). However, $A$ is not an FANR-set, because its $0$-dimensional Betti number is infinite.

7. FANR-sets and FANR-sets and extension of fundamental sequences. By Theorem (1.5), in general not every fundamental sequence from $X$ to $Y$ has an extension onto a given compactum $X' \subset H$ containing $X$. Now let us consider some cases in which the existence of an extension is ensured.

It follows by Theorem (5.1) that

(7.1) If $X \subset X'$ is a compactum such that $X \subset X' \subset H$, then every fundamental sequence from $X$ to $Y$ can be extended to a fundamental sequence from $X'$ to $Y$.

and

(7.2) If $X \subset X'$ is a compactum such that $X \subset X' \subset H$, then there is a closed neighborhood $Z$ of $X$ such that every fundamental sequence from $X$ to $Y$ can be extended to a fundamental sequence from the set $Z \setminus X'$ to $Y$.

Now let us prove the following proposition:

(7.3) If $X$ and $X' \supset X$ are compacta in $H$ and if $Y \subset X'$, then every fundamental sequence from $X$ to $Y$ can be extended to a fundamental sequence from $X'$ to $Y$.

Proof. Let $f = (f_k, X, Y)$ be a fundamental sequence. It is clear that the set

$$Z = Y \cap \bigcup_{k=1}^\infty f_k(x)$$

is a compactum lying in $H$. By (6.4) there exists in $H$ and AR-set $Y' \supset Z$. Then there is a map $\phi : H \to H$ such that

$$\phi(H) \subset Y' \quad \text{and} \quad \phi(y) = y \quad \text{for every point } y \in Y'.$$

Setting $f_k = \phi f_k$ for $k = 1, 2, ..., we get a sequence of maps $f_k : H \to H$ with values in $Y'$. Since $Y' \subset H$, all these maps are homotopic to $Y'$, and we infer that $f' = (f_k, X', Y')$ is a fundamental sequence. Moreover, if $x \in X$, then $f_k(x) \in f_k(x') \subset Z \subset Y'$ and consequently $f_k(x) = f_k(x') = f_k(x)$.

Since $X$, as an FANR-set lying in $Y'$, is a fundamental retract of $Y'$, we see that all the hypotheses of Theorem (5.2) are satisfied and we infer that there is a fundamental sequence $f$ from $X'$ to $Y$ which is an extension of $f'$. Moreover, let us show that

(7.4) If $X$ and $X' \supset X$ are compacta in $H$ and if $X \subset X'$, then for every fundamental sequence $f = (f_k, X, Y)$ there is a closed neighborhood $M$ of $X$ such that $f$ can be extended to a fundamental sequence from $M \setminus X'$ to $Y$.

Proof. Let $Y'$ denote the convex hull of the set $Z = Y \cup \bigcup_{k=1}^\infty f_k(x)$. Since $X \subset X'$, there is a closed neighborhood $N$ of $Y$ such that $Y$ is a fundamental retract of the set $N \setminus Y'$. Evidently $N$ contains a neighborhood $N_x$ of $X$ which is the union of a finite number of balls (in the
space \( H \)). Since \( Y' \) is a convex compactum, we can easily see that the set \( Y' = N_\varepsilon \cap X' \) is an ANR-set which is a neighborhood of \( Y \) in the space \( Y' \). It follows that there exists an index \( k_0 \) such that
\[
f_k(X) \subset Y'' \quad \text{for every } k > k_0.
\]
Since \( Y' \in \text{ANR} \), there exist a neighborhood \( V_\varepsilon \) of \( Y'' \) and a map \( \varphi: H \to H \) such that
\[
\varphi(V_\varepsilon) \subset Y'' \quad \text{and} \quad \varphi(y) = y \quad \text{for every point } y \in Y''.
\]

Let us show that setting
\[
f_k = f_k \quad \text{for } k = 1, 2, \ldots, k_0 \quad \text{and} \quad f_k = \varphi f_k \quad \text{for } k > k_0,
\]
one gets a sequence of maps \( f_k: H \to H \) such that there is a closed neighborhood \( M \) of \( X \) such that \( \{f_k, M \cap X', Y'\} \) is a fundamental sequence. In fact, since \( V_\varepsilon \) is a neighborhood of \( Y'' \cap X \) and since \( \{f_k, X, Y\} \) is a fundamental sequence, there exists a closed neighborhood \( M_k \) of \( X \) such that
\[
f_k/M_k \subset f_{k+1}/M_k \subset V_\varepsilon \quad \text{for almost all } k.
\]
If we recall that \( \varphi(V_\varepsilon) \subset Y'' \), we infer that \( f_k/M_k \subset f_{k+1}/M_k \subset Y'' \) for almost all \( k \), and consequently the homotopy
\[
f_k/M_k \simeq f_{k+1}/M_k \text{ in } V_\varepsilon \quad \text{for almost all } k.
\]
holds for every neighborhood \( V' \) of \( Y'' \) for almost all \( k \). It follows that if \( M \) is a closed neighborhood of \( X \) contained in the interior of \( M_k \), then \( \{f_k, M \cap X', Y'\} \) is a fundamental sequence. Moreover, if \( x \in X \) then
\[
f_k(x) = f_k(x) \quad \text{for every point } x \in X \quad \text{and} \quad k = 1, 2, \ldots,
\]
because \( f_k = f_k \) for \( k \leq k_0 \) and \( f_k \neq f_k(X) \subset Y'' \), whence \( f_k(x) = \varphi f_k(x) = f_k(x) \) for \( k > k_0 \).

The inclusion \( N_\varepsilon \subset N \) implies that \( Y' \subset N \cap Y' \). As we have already shown, \( \{f_k, M \cap X', Y'\} \) is a fundamental sequence, and consequently \( \{f_k, M \cap X', N \cap Y'\} \) is also a fundamental sequence, being an extension of the fundamental sequence \( \{f_k, X, N \cap Y'\} \). Since \( Y \) is a fundamental retract of \( N \cap Y' \), we infer by Theorem (5.5) that there exists a fundamental sequence \( f \) from \( M \cap X' \) to \( Y' \) which is an extension of \( f \). Thus the proof of (7.4) is achieved.

8. Topological invariance of FAR and of FANR. Let us prove the following

(8.1) THEOREM. If \( X \) is an FAR-set, then every set \( Y \subset X \) homeomorphic to \( X \) is also an FAR-set.

**Proof.** Let \( h: X \to Y \) be a homeomorphism. It is clear that there exist two maps \( f: H \to H \) and \( g: H \to H \) such that
\[
f(x) = h(a) \quad \text{for every point } a \in X \quad \text{and} \quad g(y) = h^{-1}(y) \quad \text{for every point } y \in Y.
\]
Consider a set \( \tilde{Y} \subset X \) and let us denote the set \( \tilde{X} = g(\tilde{Y}) \) by \( \tilde{X} \). Since \( X \subset \text{FAR} \), there exists a fundamental retraction \( r = \{r_k, \tilde{X}, X\} \). Let us show that \( \{f \circ r, \tilde{Y}, \tilde{X}\} \) is a fundamental retraction.

Let \( V \) be a neighborhood of \( X \). Then \( V = f^{-1}(V) \) is a neighborhood of \( X \) and since \( r \) is a fundamental retraction, there exists a neighborhood \( \tilde{U} \) of the set \( \tilde{X} \) such that for almost all \( k \) there is a homotopy
\[
\varphi_k: \tilde{U} \times (0, 1) \to \tilde{U}
\]
satisfying the condition:
\[
\varphi_k(x, 0) = r_k(x), \quad \varphi_k(x, 1) = r_{k+1}(x) \quad \text{for every point } x \in \tilde{U}.
\]
Since \( g(\tilde{Y}) \subset \tilde{X} \), we infer that \( \tilde{V} = g^{-1}(\tilde{U}) \) is a neighborhood of the set \( \tilde{Y} \). It follows that setting
\[
\varphi_k(y, t) = \varphi_k(g(y), t) \quad \text{for every } (y, t) \in \tilde{V} \times (0, 1),
\]
we get a homotopy \( \varphi_k: \tilde{V} \times (0, 1) \to \tilde{U} \) satisfying the condition
\[
\varphi_k(y, 0) = r_k(y), \quad \varphi_k(y, 1) = r_{k+1}(y) \quad \text{for every point } y \in \tilde{V}.
\]
Since \( f(\tilde{U}) \subset \tilde{V} \), we infer that \( f\varphi_k: \tilde{V} \times (0, 1) \to \tilde{V} \) is a homotopy satisfying the condition
\[
f\varphi_k(y, 0) = f\varphi_k(g(y)), \quad f\varphi_k(y, 1) = f\varphi_{k+1}(g(y)) \quad \text{for every point } y \in \tilde{V}.
\]
Thus we have shown that \( \{f \circ g, \tilde{Y}, \tilde{X} \} \) is a fundamental retraction. Hence \( Y \) is a fundamental retract of the set \( \tilde{Y} \subset X \), and we infer by Theorem (5.5) that \( Y \subset \text{FAR} \).

(8.2) THEOREM. If \( X \) is an FANR-set, then every set \( Y \subset X \) homeomorphic to \( X \) is also an FANR-set.

**Proof.** Let \( \tilde{X}, \tilde{Y}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{a} \) be as in the proof of Theorem (8.1). Since \( X \subset \text{FANR} \), there exist a closed neighborhood \( M \) of \( X \) and a fundamental retraction \( r = \{r_k, M \cap X, X\} \). Let \( V \) be a neighborhood of \( X \). Then \( V = f^{-1}(V) \) is a neighborhood of \( X \) and since \( r \) is a fundamental retraction, there exists a neighborhood \( \tilde{U} \) of the set \( \tilde{X} = M \cap X \) such that for almost all \( k \) there is a homotopy
\[
\varphi_k: \tilde{U} \times (0, 1) \to \tilde{U}
\]
satisfying the following condition
\[
\varphi_k(x, 0) = r_k(x), \quad \varphi_k(x, 1) = r_{k+1}(x) \quad \text{for every point } x \in \tilde{U}.
\]
Now let us set \( Y' = g^{-1}(M) \cap \tilde{X} \). Evidently \( Y' \) is a neighborhood of \( X \) in the space \( \tilde{X} \). Since \( g(\tilde{Y}) \subset \tilde{X} \), we infer that \( g(Y') \subset X' \) and con-
sequently the set \( V' = \varphi^{-1}(U) \) is a neighborhood of the set \( Y' \). It follows that

\[
\varphi(y, t) = \varphi(y, 0) \quad \text{for every} \quad (y, t) \in V' \times (0, 1),
\]

we get a homotopy \( \varphi : V' \times (0, 1) \rightarrow U \). By an analogous argument to that used in the proof of (8.1) we show that \( \varphi : V' \times (0, 1) \rightarrow V \) is a homotopy joining the maps \( f_{\varphi}(y') \rightarrow f_{\varphi}(y) \).

Hence \( (f_{\varphi}(y'), Y', X) \) is a fundamental retraction. Since \( Y' \) is a neighborhood of \( Y \), the space \( X \) ε \( AB \), we infer by Theorem (6.14) that \( Y \in \text{FANR} \).

Remark. Theorems (8.1) and (8.2), allow to generalize the concepts of the FAB- and FANR-sets as follows:

A compactum \( X \) (not necessarily lying in \( H \)) is an FAB (or an FANR, respectively) if there exists a subset \( Y \) of \( X \) homeomorphic to \( X \) and being an FAB (or an FANR, respectively) in the previous sense.

9. Two conditions characterizing FAB-sets. The first of these conditions appears in the following

(9.1) Theorem. A compactum \( X \subset H \) is an FAB-set if and only if every neighborhood \( U \) of \( X \) contains a neighborhood \( U' \) of \( X \) which is contractible in \( U \).

Proof. If \( X \in \text{FAB} \), then there exists, by (6.4), an \( AB \)-set \( \tilde{X} \) such that \( \tilde{X} \subset X \subset H \). Consider a map \( \varphi : H \rightarrow H \) such that

\[
\varphi(H) \subset \tilde{X} \quad \text{and} \quad \varphi(x) = x \quad \text{for every point} \ x \in \tilde{X}.
\]

Let \( \{V_k\} \) be a sequence of neighborhoods of \( X \) such that for each neighborhood \( V \) of \( X \) the inclusion \( V \subset \tilde{X} \) holds for all \( k \). By Theorem (4.1) there exists a fundamental retraction \( r = (r_k, \tilde{X}, X) \) such that \( r_k(x) = x \) for every point \( x \in V_k \).

Consider now a neighborhood \( U \) of \( X \). It is clear that there exists an index \( k \_\), such that

\[
V_k \subset U \quad \text{and} \quad r_k(X) \subset U.
\]

For this index \( k \_\), there exists an index \( k_0 \) such that for every point \( x \in V_{k_0} \) the segment (in \( H \)) with endpoints \( x \) and \( \varphi(x) \) lies in \( V_{k_0} \). Now let us set

\[
(9.2) \quad f(x, t) = 2t \cdot \varphi(x) + (1 - 2t) \cdot x \quad \text{for every point} \ x \in V_{k_0} \quad \text{and} \quad 0 \leq t \leq \frac{1}{2},
\]

\[
(9.3) \quad \varphi(x, t) = r_k(2t \cdot \varphi(x)) \quad \text{for every point} \ x \in V_{k_0} \quad \text{and} \quad \frac{1}{2} < t \leq 1.
\]

The formulas (9.2) and (9.3) are compatible, because for \( t = \frac{1}{2} \) the value of \( f(x, t) \) given by (9.2) is \( \varphi(x) \), and (9.3) gives \( f(x, \frac{1}{2}) = r_k(2 \cdot \varphi(x)) = \varphi(x) \), since \( \varphi \) satisfies the condition \( \varphi \circ \varphi = \varphi(x) \) for every point \( x \in X \), and \( \varphi(x) \) belongs, for \( x \in V_{k_0} \), to \( V_{k_0} \), whence \( r_k(\varphi(x)) = \varphi(x) \).

Thus formulas (9.2) and (9.3) both define a map of \( V_{k_0} \times (0, 1) \) into \( H \). It follows by (9.2) that for \( 0 \leq t \leq \frac{1}{2} \) the point \( f(x, t) \) belongs to the segment with endpoints \( x \) and \( \varphi(x) \), whence \( f(x, t) \in V_{k_0} \subset U \) for \( (x, t) \in V_{k_0} \times (0, \frac{1}{2}) \). On the other hand, the formula (9.3) implies that for \( (x, t) \in V_{k_0} \times (\frac{1}{2}, 1) \) the point \( f(x, t) \) lies in the set \( r_k(X) \subset U \in U \).

Moreover, \( f(x, 0) = x \) and \( f(x, 1) = r_k(\varphi(0)) = \varphi(x) \). Hence \( f \) is a homotopy contracting the set \( U = V_{k_0} \) in the set \( U \) to the point \( 
we, \varphi(0) \). Thus the necessity of the condition is proved.

Now let us assume that every neighborhood \( U \) of a compactum \( X \subset H \) contains a neighborhood \( U' \) contractible to a point in \( U \). As before, let us consider an \( AB \)-set \( \tilde{X} \) such that \( X \subset \tilde{X} \subset H \). To prove that \( X \in \text{FAB} \), it suffices to show by Theorem (6.2)—that there exists a fundamental retraction of \( X \) to \( X \).

Consider a decreasing sequence \( \{V_k\} \) of open neighborhoods of \( X \) such that for each neighborhood \( V \) of \( X \) the inclusion \( V \subset \tilde{X} \) holds for almost all \( k \). By our hypothesis, there exists a sequence \( \{W_k\} \) of closed neighborhoods of \( X \) such that \( W_k \) is contractible in \( V_k \) to a point \( a \in X \). Since \( V_k \), as an open subset of \( H \), is an absolute neighborhood retract for metric spaces, we infer by the theorem on the extension of a homotopy ([1], p. 94) that there is a map \( r_k : H \rightarrow H \) such that \( r_k(H) \subset V_k \) and \( r_k(x) = x \) for every point \( x \in V_k \), and \( r_k \) is homotopic in \( V_k \) to the constant map \( e \). Let us show that \( (r_k, \tilde{X}, X) \) is a fundamental retraction. In fact, since \( X \subset W_k \), we have \( r_k(x) = x \) for every point \( x \in X \). Moreover, if \( U \) is a neighborhood of \( X \), then there exists an index \( k_0 \) such that \( V_{k_0} \subset U \) for every \( k \geq k_0 \).

Hence \( r_k \simeq a \) in \( V_k \subset U \) for every \( k \geq k_0 \) and consequently \( r_k \simeq r_{k+1} \) in \( V_k \subset U \) for every \( k \geq k_0 \). Thus we have shown that \( (r_k, \tilde{X}, X) \) is a fundamental retraction and the proof of Theorem (9.1) is finished.

(9.4) Corollary. The intersection of every decreasing sequence of FAB-sets is an FAB-set.

In fact, if \( X_1 \supset X_2 \supset \ldots \) is a sequence of FAB-sets and \( X = \bigcap_{k=1}^{\infty} X_k \), then for every neighborhood \( U \) of \( X \) there exists an index \( k_0 \) such that \( U \) is a neighborhood of \( X_{k_0} \). By Theorem (9.1), there is a neighborhood \( V_{k_0} \) of \( X_{k_0} \) contractible in \( U \). Since \( U_{k_0} \) is also a neighborhood of \( X \), the set \( X \) satisfies the condition characterizing FAB-sets.

(9.5) Corollary. A compactum \( X \subset H \) is an FAB-set if and only if it is contractible in each of its neighborhoods (in \( H \)).

Proof. The necessity of the condition is an immediate consequence of Theorem (9.1). In order to prove its sufficiency, consider an open neighborhood \( U \) of \( X \) (in \( H \)). Then there are a point \( a \in X \) and a map...
Fundamental retract

Proof. Let \( \tau = \{ r_2, \mathcal{X}, Y \} \) be a fundamental retraction. Setting \( \tau' = \{ r_2, \mathcal{X}, Y \} \), we get a fundamental sequence \( \gamma' \). Since \( r_2(g) = \gamma(g) \) for every point \( y \in Y \), we infer by (1.1) that

\[
(9.11) \quad \gamma' \simeq \gamma.
\]

Now let \( V \) be a neighborhood of \( Y \). Since \( \tau \) is a fundamental sequence, there exists a neighborhood \( U \) of \( \mathcal{X} \) and an index \( k_0 \) such that

\[
(9.12) \quad r_2(U) \subset V \quad \text{for every } k \geq k_0.
\]

Since \( \xi \simeq \gamma \), there exists a neighborhood \( U_\xi \) of \( X \) such that

\[
(9.13) \quad \gamma \simeq \gamma \quad \text{in } U_\xi.
\]

It follows by (9.12) and (9.13) that

\[
r_2/\xi U_\xi = r_2/\xi U_\xi \simeq r_2/\xi \gamma \gamma \quad \text{in } V \quad \text{for every } k \geq k_0.
\]

Since \( Y \subset X \), the set \( U_\xi \) is a neighborhood of \( Y \). Thus we have shown that for every neighborhood \( V \) of \( Y \) there exists a neighborhood \( U_\xi \) of \( Y \) and an index \( k_0 \) such that

\[
r_2/\xi U_\xi \simeq \gamma \quad \text{in } V \quad \text{for every } k \geq k_0.
\]

Hence \( \gamma' \simeq \gamma \). It follows by (9.11) that \( \gamma' \simeq \gamma \), and the proof of Lemma (9.10) is finished.

Proof of Theorem (9.8). If \( X \simeq \mathcal{X} \), then (6.3) implies that \( X \) is a fundamental retract of an AR-set \( \mathcal{X} \). Since \( X \) is contractible in itself to every point \( \xi \in \mathcal{X} \), we infer by (9.9) and (9.10) that \( \xi \simeq \gamma \simeq \gamma \). On the other hand, if \( X \subset \mathcal{X} \) is a compactum such that \( \xi \simeq \gamma \) is \( \gamma \mathcal{X} \), where \( \gamma \) is a point of \( \mathcal{X} \), then for every neighborhood \( U \) of \( \mathcal{X} \) such that \( i/\xi U_\xi \simeq \gamma \) \( U_\xi \) in \( U \). This means that \( U_\xi \) is contractible in \( U \) and we infer by Theorem (9.1) that \( X \simeq \mathcal{X} \).

10. A property of FANR-sets. The following theorem gives a condition for FANR-set which is to some extent similar to the condition for AR-seats in Theorem (9.1):

(10.1) Theorem. Every FANR-set \( X \) satisfies the following condition:

\begin{itemize}
  \item [(\ast)] \quad For every neighborhood \( U \subset X \) there is a neighborhood \( U_\xi \subset \mathcal{X} \) such that \( \xi \simeq \gamma \) \( U \subset U_\xi \) and \( \xi \simeq \gamma \) \( \mathcal{X} \subset \mathcal{X} \).
\end{itemize}

Proof. By (6.4) there exists an AR-set \( \mathcal{X} \). Let \( \gamma : \mathcal{X} \to \mathcal{X} \) be a retraction. Consider a sequence \( V_i : \mathcal{X} \to \mathcal{X} \) of neighborhoods of \( \mathcal{X} \) such that every neighborhood of \( \mathcal{X} \) contains \( V_i \) for almost all \( k \). Since \( X \simeq \mathcal{X} \),
we infer by Theorem (4.1) that there exist a closed neighborhood $W$ of $X$ and a fundamental retraction $r = \langle r_k, W \cap \hat{X}, X \rangle$ such that

$$r_k \cap r_{k+1} = i \cap r_k$$

for every $k = 1, 2, ...$

Now let $U$ be a neighborhood of $X$. Then there is an index $k_0$ such that

$$V_{k_0} \subset U \cap \hat{X},$$

$$r_k (W \cap \hat{X}) \cong r_{k+m} (W \cap \hat{X})$$

for every $m = 1, 2, ...$

Moreover, there exists a neighborhood $U_k$ of $X$ such that

$$r_k (W \cap \hat{X}) \cong r_{k+m} (W \cap \hat{X})$$

for every $m = 1, 2, ...$

For every point $x \in U_k$ the segment $(0, 1)$ with endpoints $x$ and $a(x)$ lies in $V_{k_0}$.

It follows by (10.3) and (10.5) that

$$a(U_k) \subset V_{k_0} \cap X \subset W \cap \hat{X}.$$  

Moreover, (10.5) implies that

$$a(U_k) \subset V_{k_0} \cap X \subset W \cap \hat{X}.$$  

Hence, for every neighborhood $V$ of $X$ the inclusion

$$r_{k+m} (U_{k+m} \cap V) \subset U$$

holds for almost all $m$. Moreover, if $x \in X$ then $a(x) = x$ and $r_k(x) = x$, whence $r_{k+m} (a(x)) = x$. It follows by (10.10) that condition (*) is satisfied.

11. A sufficient condition for a set to be an ANR. Let us prove the following

Theorem. If $X_1, X_2, ...$ are ANR-sets lying in $H$ and if $X_{k+1}$ is a deformation retract of $X_k$ for every $k = 1, 2, ...$, then the set

$$X = \bigcap_{k=1}^\infty X_k$$

is a ANR-set.

Proof. Since $X_1 \in \text{ANR}$, there exist a neighborhood $U_1$ of $X_1$ (in $H$) and a map $a_1 : H \to H$ such that $a_1 (U_1) = X_1$ and $a_1(x) = x$ for every point $x \in X_1$.

Since $X_{k+1}$ is a deformation retract of the set $X_k \in \text{ANR}$, there exists (I55, p. 448) a homotopy

$$\psi_k : X_k \times [0, 1) \to X_1$$

such that

$$\psi_k (x, 0) = x$$

for every point $x \in X_k$,

$$\psi_k(x, t) = x$$

for every $(x, t) \in X_{k+1} \times (0, 1)$,

$$\psi_k (X_k, 1) = X_{k+1}.$$  

Hence the formula $r_k(x) = \psi_k(x, 1)$ defines a retraction

$$r_k : X_k \to X_{k+1}.$$  

Consider now a map $a_2 : H \to H$ such that $a_2(x) = r_k(x)$ for every point $x \in X_k$. Setting

$$f_k = a_2 a_1 = \cdots a_k a_1,$$

we get a map $f_k : H \to H$ such that the formula

$$r_k(x) = f_k(x)$$

defines a retraction $r_k : X_k \to X_{k+1}$.

Let us show that $(f_k, X_k, X)$ is a fundamental retraction. Consider a neighborhood $V$ of $X$ (in the space $H$). Then there exists an index $k_0$ such that the set $U_{k_0} \cap V$ is a neighborhood of $X_{k_0}$, whence also a neighborhood of $X_k$ for every $k \geq k_0$. But

$$f_k (X_{k_0}) = a_k a_{k-1} \cdots a_1 (X_{k_0}) \subset a_k a_{k-1} \cdots a_1 (U_{k_0}) = X_{k+1}.$$  

It follows that there exists a neighborhood $U$ of $X_k$ (in $H$) such that $U \subset U_{k_0} \cap V$ and that $f_k (U) \subset U \cap V$. Then, for every $k \geq k_0$, the set $U$ is a neighborhood of the set $X_k$ such that

$$f_k (U) \subset a_k a_{k-1} \cdots a_1 (U_{k_0}) = X_{k+1} \subset U,$$

and

$$f_k (x) = x$$

for every point $x \in X_k$.

Since the values of the homotopy $\psi_{k+1}$ belong to the set $X_{k+1} \subset V$, we infer that setting

$$\psi_k (x, t) = \psi_{k+1} (f_k (x), t)$$

for every $(x, t) \in U \times (0, 1)$,
we get a homotopy $\varphi_k: U \times (0, 1) \to V$ satisfying the following conditions:

$\varphi_0(x, 0) = f_0(x)$ for every point $x \in U$,

$\varphi_0(x, 1) = r_{k+1}f_0(x) = s_{k+1}f_0(x) = f_{k+1}(x)$ for every point $x \in U$,

$\varphi_0(x, t) = x$ for every $(x, t) \in X \times (0, 1)$,

because $x \in X$ implies that $f_0(x) = x \in X \subset X_{k+1}$.

Thus we have shown that the homotopy $\varphi_k: U \times (0, 1) \to V$ joins the map $f_k/U$ with the map $s_{k+1}/U$ and satisfies the condition $\varphi_0(x, t) = x$ for every $(x, t) \in X \times (0, 1)$. Hence $(f_k, X_k, X)$ is a fundamental retraction. It follows that

$$X$$

is a fundamental retraction of $X_k$.

Since $X_k \in \mathrm{ANR}$, we infer by Theorem (6.14) that $X \in \mathrm{FANR}$.

As a consequence of (11.2) and of Theorem (6.2) we get the following

(11.3) **Corollary.** The intersection of a decreasing sequence of AR-sets lying in $U$ is an FAR-set.

In fact, it suffices to observe that for $Y, Z \in \mathrm{AR}$ the inclusion $Z \subset Y$ implies that $Z$ is a deformation retract of $Y$.

(11.4) **Problem.** Does Theorem (11.1) remain true if we replace the hypothesis that $X_{k+1}$ is a deformation retract of $X_k$ by a weaker one, namely, that $X_{k+1}$ is a retract of $X_k$?

(11.5) **Problem.** Is it true that for every sequence $A_1, A_2, \ldots$ of ANR-sets such that $A_{k+1}$ is a retract of $A_k$ for $k = 1, 2, \ldots$, there exists an index $k_0$ such that $A_{k_0}$ is a deformation retract of $A_k$ for every $k > k_0$?

12. **Cartesian product of FAR-sets.** First let us prove the following

(12.1) **Lemma.** Let $A$ be an AR-set in $H$ and let $X$ and $X'$ be closed subsets of $A$ such that $X \subset X'$. In order that $X$ be a fundamental retract of $X'$ it is necessary and sufficient that there exists a sequence $(a_k)$ of maps of $A$ into itself such that for every neighborhood $U$ of $X$ in the space $A$ there exists a neighborhood $U'$ of $X'$ in $A$ such that for almost all $k$ there is a homotopy $\varphi_k: U' \times (0, 1) \to U$ such that

$$\varphi_k(x, 0) = a_k(x), \quad \varphi_k(x, 1) = a_k(x)$$

for every point $x \in U'$.

**Proof.** Since $A \in \mathrm{AR}$, there exists a map $s: H \to H$ such that

$s(H) = A, \quad s(x) = x$ for every point $x \in A$.

If $X$ is a fundamental retract of $X'$, then there exists a fundamental retraction

$$r = (r_k, X', X).$$

Setting $a_k(x) = s_k(x)$ for every point $x \in A$ and $k = 1, 2, \ldots$, we get a sequence $(a_k)$ of maps of $A$ into itself. Moreover, if $U$ is a neighborhood of $X$ in the space $A$, then $V = s^*(U)$ is a neighborhood of $X$ in the space $H$. Since $r$ is a fundamental retraction, there is a neighborhood $V'$ of $X'$ (in $H$) such that for almost all $k$ there exists a homotopy

$$\varphi_k: V' \times (0, 1) \to V$$

such that

$$\varphi_k(x, 0) = r_k(x), \quad \varphi_k(x, 1) = a_k(x)$$

for every point $x \in V'$.

It suffices to set $U' = A \cap V'$ and

$$\varphi_k(x, 0) = a_k(x), \quad \varphi_k(x, 1) = a_k(x)$$

for every $(x, 0) \in U' \times (0, 1)$

for almost all $k$, in order to obtain a neighborhood $U'$ of $X'$ in $A$ and a homotopy $\varphi_k: U' \times (0, 1) \to U$ satisfying (12.2).

Now let us assume that there is a sequence $(a_k)$ of maps of $A$ into itself with the required properties. Setting

$$r_k(x) = a_k(x)$$

for every point $x \in H$,

one gets a sequence of maps $r_k: H \to H$. Let us prove that $(r_k, X', X)$ is a fundamental retraction.

Let $V$ be a neighborhood of $X$ in the space $H$. Then the set $U = A \cap V$ is a neighborhood of $X$ in $A$ and we infer that there exists a neighborhood $U'$ of $X'$ in $A$ and a homotopy $\varphi_k$ satisfying (12.2). Setting $V_k = s^*(U')$,

we get a neighborhood $V_k$ of $X'$ in the space $H$. It is clear that for almost all $k$ the formula

$$\varphi_k(x, t) = \varphi_k(x, t)$$

defines a homotopy $\varphi_k: V_k \times (0, 1) \to V$ such that

$$\varphi_k(x, 0) = a_k(x) = r_k(x), \quad \varphi_k(x, 1) = a_k(x)$$

for every point $x \in V_k$.

Hence $(r_1, X', X)$ is a fundamental retraction and the proof of Lemma (12.1) is finished.

Now let us prove the following

(12.3) **Theorem.** Let $X$ be a compactum in $H$ homeomorphic to the Cartesian product $\prod_{i=1}^\infty X_i$, where $X_i \subset H$ for $i = 1, 2, \ldots$. Then $X \in \mathrm{FAR}$ if and only if $X_i \in \mathrm{FAR}$ for every $i = 1, 2, \ldots$. 
Proof. Let us order all natural numbers in a double sequence $m_{ij}$ such that
\[ m_{ij} = m_{i',j'} \text{ implies } i = i' \text{ and } j = j', \]
\[ m_{ij} < m_{i'+1,j} \text{ for every } i, j. \]
Then the Hilbert cube $Q^n$ can be represented as a product
\[ Q^n = \prod_{i=1}^{m} Q_i^n, \]
where $Q_i^n$ denotes the set of all points $x = (x_1, x_2, \ldots)$ such that for $n = m_{i,j}$, the coordinate $x_n$ runs through the interval $(0,1/m_{i,j})$, and for $n \neq m_{i,j}$, with $j = 1, 2, \ldots$, $x_n = 0$. It is obvious that $Q_i^n$ is homeomorphic to $Q^n$ for $i = 1, 2, \ldots$. Since $X$ is homeomorphic to a retract of $X$, we infer by Theorem (8.1), and since every compactum is homeomorphic to a subset of $Q^n$, we can assume that $X_i \subset Q_i^n$ for $i = 1, 2, \ldots$ and that $X = \prod_{i=1}^{m} X_i \subset Q^n$.

By formula (12.4), every point $x \in Q^n$ can be represented in the form $x = (a_1, a_2, \ldots)$, with $a_i \in Q_i^n$. In particular, $x \in X$ if and only if $a_i \in X_i$ for $i = 1, 2, \ldots$. Since $X$ is homeomorphic to a retract of $X$ we infer by Theorems (6.2) and (8.1) that $X \in \text{FANR}$ implies $X_i \in \text{FANR}$ for $i = 1, 2,$...

Now let us assume that $X_i \in \text{FANR}$ for every $i = 1, 2, \ldots$. By Lemma (12.1), there exists a sequence of maps $a_i^n: Q_i^n \to Q_i^n$ such that for every neighborhood $U_i$ of $X_i$ in $Q_i^n$ there is an index $k_i$ such that for every $k > k_i$ there is a homeomorphism $\varphi_i: Q_i^n \times (0, 1) \to U_i$ satisfying the condition
\[ \varphi_i(x, 0) = a_i^n(x), \quad \varphi_i(x, 1) = a_{i+1}(x) \text{ for every point } x \in Q_i^n. \]

Let us notice that in the case $U_i = Q_i^n$ the homomorphism $\varphi_i$ satisfying this condition exists for every $k = 1, 2, \ldots$. Hence in the case $U_i = Q_i^n$, we can set $k_i = 0$.

Setting
\[ a_i^n = [a_i^n, a_{i+1}^n, \ldots] \text{ for every point } x = (a_1, a_2, \ldots) \in Q^n, \]
we get a sequence of maps $a_i^n: Q^n \to Q^n$.

Consider now a neighborhood $U$ of $X$ in $Q^n$. Then there exists for every $i = 1, 2, \ldots$ a neighborhood $U_i$ of $X_i$ in $Q_i^n$ and an index $i_0$ such that
\[ (1) \quad U_i = Q_i^n \text{ for every } i > i_0, \]
\[ (2) \quad \prod_{i=1}^{m} U_i \subset U. \]

It is clear that $U_i = \prod_{i=1}^{m} U_i$ is a neighborhood of $X$ in $Q^n$.

Consider the homotopies $\varphi_{i_0}, \varphi_{i_1}, \ldots$ and the indices $k_1, k_2, \ldots$ as defined above. Then $k_i = 0$ for $i > i_0$ and we infer that there is an index $k_0$ such that (12.5) holds for every $k > k_0$.

Setting
\[ \varphi_i(x, t) = [\varphi_i(x_1, t), \varphi_i(x_2, t), \ldots] \]
for every $x = (a_1, a_2, \ldots) \in Q^n$ and $0 < t < 1$, we get a homotopy
\[ \varphi: Q^n \times (0, 1) \to U \]
such that
\[ \varphi_i(x, 0) = a_0(x), \quad \varphi_i(x, 1) = a_{i+1}(x) \quad \text{for every point } x \in Q. \]

It follows by Lemma (12.1) that $X$ is a fundamental retract of $Q^n$, and consequently (by Theorem (8.3)), $X \in \text{FANR}$.

13. Cartesian product of FANR-spaces. We now pass to the proof of the following theorem.

(13.1) Theorem. Let $X$ be a compactum lying in $H$, homeomorphic to the Cartesian product $\prod X_i$, where $X_i \subset H$. Then $X \in \text{FANR}$ if and only if $X_i \in \text{FANR}$ for $i = 1, 2, \ldots$ and $X_i \in \text{FANR}$ for almost all $i$.

Proof. First let us assume that $X \in \text{FANR}$. Then $X$ is homeomorphic to a retract of $X$ and we infer by (6.7) and (8.2) that $X \in \text{FANR}$ for $i = 1, 2, \ldots$.

In order to prove that $X_i \in \text{FANR}$ for almost all $i$, let us consider a decomposition of $Q^n$ into the Cartesian product $\prod Q_i^n$ with factors $Q_i^n$ homeomorphic to $Q^n$ and let us assume that $X_i \subset Q_i^n$ for $i = 1, 2, \ldots$ and that $X = \prod X_i$. Thus every point $x \in Q^n$ can be represented in the form
\[ x = (a_1, a_2, \ldots) \text{ with } a_i \in Q_i^n \text{ for } i = 1, 2, \ldots. \]

Let us select a point $a = (a_1, a_2, \ldots) \in X$. Since $X \in \text{FANR}$, there is a closed neighborhood $V$ of $X$ in $Q^n$ such that $X$ is a fundamental retract of $V$. It follows by Lemma (12.1) that there exists a sequence of maps
\[ a_i: Q^n \to Q^n \]
such that for every neighborhood $U$ of $X$ in $Q^n$ there exists an index $k(U)$ such that for every $k > k(U)$ there exists a homotopy
\[ \varphi_k: V \times (0, 1) \to U. \]
satisfying the condition

\[(13.2) \quad \varphi(x, 0) = a^x(0), \quad \varphi(x, 1) = a^x(1) \quad \text{for every point } x \in V.
\]

Since \( V \) is a neighborhood of the compactum \( X \), there is an index \( i_0 \) such that \( V \) is a neighborhood of the set

\[ V = X_1 \times X_2 \times \ldots \times X_n \times \Omega_{i+1} \times \Omega_{i+2} \times \ldots \]

It is evident that the maps \( a_0 \) and \( \varphi_0 \) can be represented in the form

\[ a_0 = [a_1, a_2, \ldots], \quad a_1 : Q^0 \to Q^1, \]

\[ \varphi_0 = [\varphi_1, \varphi_2, \ldots], \quad \varphi_1 : V \times (0, 1) \to Q^0.
\]

Consider now an index \( i > i_0 \) and an arbitrary neighborhood \( U_0 \) of \( x_i \) in \( Q_i^0 \). Then the set

\[ U = Q_i^0 \times Q_i^2 \times \ldots \times Q_i^{n-1} \times U_i \times Q_i^{n+1} \times \ldots \]

is a neighborhood of \( x \) in \( X \) and thus \((13.2)\) holds for every \( k > k(U) \). Now let us set, for every \( i > i_0 \):

\[(13.3) \quad \beta_i(x_i) = a_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots]) \quad \text{for every } x \in Q_i^0,
\]

and

\[(13.4) \quad \psi_i(x_i, t) = \psi_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots], t) \quad \text{for every } (x, t) \in Q_i^0 \times (0, 1).
\]

It follows by \((13.2), (13.3)\) and \((13.4)\) that the maps \( \beta_i : Q_i^0 \to Q_i^0 \) and the homotopy \( \psi_i : Q_i^0 \times (0, 1) \to U_0^0 \) satisfy the following conditions:

\[ \psi_i(x_i, 0) = \psi_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots], 0), \]

\[ \psi_i(x_i, 0) = a_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots]), \]

\[ \psi_i(x_i, 1) = \psi_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots], 1), \]

\[ \psi_i(x_i, 1) = a_i([a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots]). \]

It follows by Lemma \((12.1)\) that \( X_i \) is a fundamental retract of \( Q_i^0 \).

We infer by Theorem \((6.3)\) that \( X_i \) is FAR for every \( i > i_0 \).

Before we prove the converse, let us show that

\[(13.5) \quad \text{Let } X \text{ be a compactum in } H \text{ homeomorphic to the Cartesian product of two FANR-sets } X', X'' \text{. Then } X \text{ is FANR.}
\]

Proof. Let us represent \( H \) as the Cartesian product \( H' \times H'' \), where \( H', H'' \) are homeomorphic to \( H \). We can assume that \( X' \subset H', X'' \subset H'' \) and \( X = X' \times X'' \). By \((8.8)\) there exist two ANR-sets \( A' \subset H' \) and \( A'' \subset H'' \) and two fundamental retracts \( r' = (r'_i, A', X') \) and \( r'' = (r''_i, A'', X'') \). Since \( H = H' \times H'' \), every point \( x \in H \) is of the form \( x = (x', x'') \), where \( x' \in H', \ x'' \in H''. \)

We get a sequence of maps \( r_k : H \to H \) such that for almost all \( k \)

\[(13.8) \quad \varphi(x', 0) = r_0(x'), \quad \varphi(x', 1) = r_k(x'), \quad \beta(x') = \varphi(x', t) \quad \text{for every point } x \in U', \]

\[(13.9) \quad \varphi(x'', 0) = r_0(x''), \quad \varphi(x'', 1) = r_k(x''), \quad \beta(x'') = \varphi(x'', t) \quad \text{for every point } x \in U''.
\]

Then \( U = U' \times U'' \) is a neighborhood in \( H \) of the set \( A = A' \times A'' \) being an ANR-set and we infer by \((13.6), (13.7), (13.8), \) and \((13.9)\) that the formula

\[ \varphi(x, t) = [\varphi(x', t), \varphi(x'', t)] \quad \text{for every } (x, t) \in U', \]

defines a homotopy \( \varphi : U \times (0, 1) \to V \) satisfying the condition

\[ \varphi(x, 0) = r_0(x), \quad \varphi(x, 1) = r_k(x) \quad \text{for every point } x \in U.
\]

It follows that \( r = (r_k, A, X) \) is a fundamental retraction of the set \( A \times X \) being an ANR to \( X \). Hence \( X \) is FANR and proposition \((13.5)\) is proved.

In order to finish the proof of Theorem \((13.1)\), let us assume that \( X = P X_i \), where \( X_i \in \text{FANR} \) for \( i = 1, 2, \ldots \) and that there exists an index \( i_0 \) such that \( X_i \in \text{FANR} \) for every \( i > i_0 \). Then \( X \) is homeomorphic to \( X' \times X'' \), where \( X' = P X_i \) for every \( i > i_0 \). Then \( X \) is homeomorphic to \( X' \times X'' \), where \( X' = \bigcap_{i=1}^{i_0} X_i \) and \( X'' = \bigcap_{i=i_0+1}^{i} X_i \). It follows by \((13.5)\) that \( X' \times X'' \) is FANR, and by \((13.4)\)—that \( X' \times X'' \) is FANR. Again applying \((13.5)\), we infer that \( X \in \text{FANR} \).
14. Union of two FAR-sets. We now pass to the proof of a theorem on FAR-sets, analogous to a well-known theorem on AR-sets:

(14.1) Theorem. If \( X_1, X_2 \) and \( X_3 = X_1 \cap X_2 \) are FAR-sets, then the set \( X = X_1 \cup X_3 \) is also an FAR-set.

First let us establish the following

(14.2) Lemma. Let \( A_2 \) be a closed subset of a metric space \( A \) and let \( a \) be a point of a space \( M \) which is an absolute neighborhood retract for metric spaces. Suppose \( f: A \to M \) and \( \varphi: A_2 \times (0,1) \to M \) be two maps such that \( \varphi(x,0) = f(x) \), \( \varphi(x,1) = a \) for every point \( x \in A_2 \).

If the set \( f(A) \) is contractible in \( M \) to \( a \), then there exists a homotopy

\[ \psi: A \times (0,1) \to M \]

such that \( \psi(x,0) = f(x) \), \( \psi(x,1) = a \) for every point \( x \in A \) and \( \psi(x,t) = \varphi(x,t) \) for every \( (x,t) \in A_2 \times (0,1) \).

Proof. Consider the set

\[ Z = [A \times (0,1)] \cup [A_2 \times (0,1)] \cup [A \times (1)], \]

and the map \( g: Z \to M \) given by the formula

\[ g(x,0) = f(x), \quad g(x,1) = a \quad \text{for every point } x \in A, \]

\[ g(x,t) = \varphi(x,t) \quad \text{for every } (x,t) \in A_2 \times (0,1). \]

Setting \( \varphi(x,t,u) = \varphi(x,t(1-u)) \) for every \( (x,t) \in Z \) and \( 0 \leq u \leq 1 \), we get a homotopy \( \varphi: Z \times (0,1) \to M \) joining the map \( g \) with a map having all values in \( f(A) \). Since the set \( f(A) \) is contractible in \( M \), we infer that \( g \) is homotopic in \( M \) to the constant map \( a \). If we observe that the set \( Z \) is closed in \( A \times (0,1) \) and recall that \( M \) is an absolute neighborhood retract for metric spaces, we infer by the homotopy extension theorem (11), p. 94) that \( g \) can be extended to a map \( \psi: A \times (0,1) \to M \) satisfying the lemma.

Proof of Theorem (14.1). By Theorem (9.1), for every open neighborhood \( U \) of \( X \) there exists a closed neighborhood \( U_i \) of \( X_i \) (\( i = 1, 2 \)) contractible in \( U \). Since the set \( U \cap U_i \) is a neighborhood of \( X_3 \), we infer by (9.1) that there is a closed neighborhood \( U_3 \) of \( X_3 \) contractible in \( U \cap U_i \). Let \( \varphi: U_3 \times (0,1) \to U \cap U_i \) be a homotopy contracting \( U_3 \) to a point \( a \in U_i \). Consider, for \( i = 1, 2 \), a closed neighborhood \( V_i \) of the set \( X_i \) such that \( V_i \cap U_i \) and that \( V_3 = V_i \cap V_j \cap U_j \).

Now let us consider the map \( f_i: V_i \to U \) given by the formula

\[ f_i(x) = x \quad \text{for every point } x \in V_i. \]

Since \( U \) (as an open subset of \( H \)) is an absolute neighborhood retract for metric spaces (I1), p. 96), we infer by Lemma (14.2) (where we set \( A_3 = V_3, A = V_i, M = U \), \( f = f_i \)) that there exists a homotopy \( \psi: V_i \times (0,1) \to U \) such that

\[ \psi(x,0) = x, \quad \psi(x,1) = a \quad \text{for every point } x \in V_i, \]

\[ \psi(x,t) = f_i(x,t) \quad \text{for every } (x,t) \in V_i \times (0,1). \]

It remains to set

\[ \psi_i(x,t) = \psi(x,t) \quad \text{for every } (x,t) \in V_i \times (0,1), \]

in order to obtain a homotopy \( \psi \) contracting the neighborhood \( V_i \) and \( V_3 \) of the set \( X \) in the neighborhood \( U \). It follows by Theorem (9.1) that \( X \) is FAR.

(14.3) Problem. Let \( X \) denote the union and \( X_3 \) the common part of two compacta \( X_1, X_2 \). Is it true that \( X_3 \times X_3 \times X_3 \times X_3 \) implies \( X \) FAR?

(14.4) Problem. Is it true that for every two compacta \( X_1, X_2 \subset H \) such that the set \( X_1 \times X_2 \) is a fundamental retract of \( X_1 \), the set \( X_1 \) is a fundamental retract of \( X_1 \times X_1 \)?

References


Reçu la Conception le 15. 8. 1967.