References


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On a class of subalgebras of \( C(X) \)
with applications to \( \beta X \).

by

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W. Rudin has proved that, assuming the continuum hypothesis, \( \beta N \setminus N \) has a dense subset of 2\(^2 \) P-points. A similar theorem of J. J. Fine and L. Gillman states that, assuming the continuum hypothesis, \( \beta R \setminus R \) has a dense subset of remote points in \( R \). It is the purpose of this paper to unify these results by giving a more general method of finding such points.

Specifically, for a completely regular space \( X \), we define a class of subalgebras of \( C(X) \) called \( \beta \)-subalgebras. Examples of \( \beta \)-subalgebras include \( C(X) \) itself and \( C^*(X) \). With each \( \beta \)-subalgebra \( A \) of \( C(X) \) we associate a (possibly empty) set of points in \( \beta X \setminus X \) called \( A \)-points. We show that, under the continuum hypothesis and with reasonable restrictions on \( A \) and \( X \), \( \beta X \setminus X \) has a dense subset of 2\(^2 \) \( A \)-points. The Rudin theorem is then obtained by observing that the \( P \)-points of \( \beta N \setminus N \) are precisely the \( C^*(N) \)-points, and the Fine-Gillman theorem follows from the fact that the remote points in \( R \) are precisely the \( C(R) \)-points.

Our method considerably simplifies the Fine-Gillman proof of the existence of remote points in \( \beta R \) but does not have the power of their method. Using their method, we show the existence of remote points in \( \beta R \) which are not \( P \)-points of \( \beta R \). We conclude by investigating a \( \beta \)-subalgebra \( H \) of \( C(N) \) previously studied by R. M. Brooks. We correct Brooks's characterization of the maximal ideals in \( H \) and show that his characterization holds precisely for the ideals \( M p \) where \( p \) is a \( P \)-point of \( \beta N \setminus N \) (equivalently, where \( p \) is an \( H \)-point).

I. Preliminaries. The basic reference for this paper will be the Gillman and Jerison text [5]; the terminology and notation will, with only a few exceptions, be that of [5].

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The symbol $X$ will always denote a completely regular Hausdorff space. Specific spaces $X$ in which we shall be interested are the complex plane $C$ and its subspaces $R$ of real numbers, $Q$ of rational numbers, and $N$ of natural numbers.

In Sections 1 through 6, $C(X)$ will denote the collection of real-valued continuous functions on $X$, and $C^*(X)$ will denote the subcollection of bounded functions. The constant function on $X$ of value $r$ will be denoted by $r$. Under the pointwise operations, $C(X)$ and $C^*(X)$ are algebras over $R$.

A subalgebra of $C(X)$ will mean a subalgebra in the usual sense which contains the constant functions. By an ideal we shall mean a proper ideal.

In Section 7, the definition of subalgebras and ideals are changed slightly to accommodate complex-valued functions.

A subspace $Y$ of $X$ is said to be $C^*$-embedded if each function in $C(Y)$ is the restriction of some function in $C^*(X)$; the expression "$C^*$-embedded" is defined analogously. Given $X$, there is an essentially unique compact Hausdorff space $\beta X$ which contains $X$ as a dense $C^*$-embedded subspace (the extension of $f$ to $\beta X$ will be denoted by $f^\beta$). For notational simplicity, we write $X^* = \beta X \; X$. For additional properties of $\beta X$, the reader is referred to [5]. We mention one: if $f \in C(X)$ and $Mf$, denotes the one-point compactification of $R$, then there is a (unique) continuous $f^\beta: \beta X \to Mf$ which agrees with $f$ on $X$.

If $\tau$ is a function, then we let $\tau^{-}$ denote the inverse map of sets. If $f$ maps $\tau$ to $R$ or $\mathfrak{M}$, then $Z(f) = \tau^{-}(\emptyset)$ and $\cos(f) = \tau^{-}(X^* \cup Z(f))$. A zero-ideal of $X$ is a member of the family $Z(X) = (Z(f) : f \in C(X))$, and a cozero-ideal of $X$ is the complement in $X$ of some member of $Z(X)$.

If $S$ is a set, then $|S|$ will denote the cardinality of $S$. As is standard, we shall let $c$ denote the cardinality $c_0$ of the continuum. If $S \subset X$, then $cl_S S$, $int_S S$, and $\partial_S S$ will denote, respectively, the closure, interior, and boundary of $S$ in $X$ ($\partial_S S = cl_S S - int_S S$).

2. $\beta$-subalgebras. Recall the definition of the hull-kernel topology on a collection $A$ of prime ideals in a commutative ring $A$ with an identity. Define $S = \{ P : \emptyset \subset P \}$ to be the closure of the subset $\emptyset$ of $S$. It is easy to verify that the sets

$$E_P(a) = \{ P : a \subset P \}, \quad a \in A,$$

are closed and constitute a base for the closed sets in $S$. A detailed description of the hull-kernel topology is given in [4]. Let $\mathcal{M}_A$ denote the collection of maximal ideals in $A$ endowed with the hull-kernel topology.

Given a subalgebra $A$ of $C(X)$, we shall now introduce a family $\mathfrak{M}$ of prime ideals in $A$. The family $\mathfrak{M}$ will reduce to $\mathcal{M}_A$ in the case $A = C(X)$ and $A = C^*(X)$. To motivate our definition, we observe that the maximal ideals in $C = C(X)$ and $C^* = C^*(X)$ associated with the same point $p \in X$ can be characterized in the following parallel ways:

$$M^p_\mathcal{M} = \{ f \in C : (fg)^p = 0 \text{ for all } g \in C \}$$

$$M^p_\mathfrak{M} = \{ f \in C : (fg)^p = 0 \text{ for all } g \in C^* \}.$$

The first characterization was discussed by Gelfand and Kolmogoroff [6]; the second is elementary (see [5], 7.2). Gelfand and Kolmogoroff proved that the mappings $p \mapsto M^p_\mathcal{M}$ and $p \mapsto M^p_\mathfrak{M}$ are homeomorphisms of $\beta X$ onto the maximal-ideal spaces $\mathcal{M}_C$ and $\mathcal{M}_{C^*}$.

The similarity of the expressions for $M^p_\mathcal{M}$ and $M^p_\mathfrak{M}$ suggests a generalization of these ideas to any subalgebra $A$ of $C(X)$. Thus, for $p \in \beta X$, let us define

$$M^p_\mathfrak{M} = \{ f \in A : (fg)^p = 0 \text{ for all } g \in A \}.$$

It is easy to see that, for $p \in \beta X$, $M^p_\mathfrak{M}$ is the fixed maximal ideal $\{ f \in A : (fg) = 0 \}$ in $A$, and we shall show next that, for $p \in \beta X$, $M^p_\mathfrak{M}$ is always a prime ideal. But the general correspondence $p \mapsto M^p_\mathfrak{M}$ need not be one-to-one, and, in spite of the notation, the ideal $M^p_\mathcal{M}$ need not be maximal.

For example, in the algebra $A$ of all real-valued polynomials on $R$, $M^0 \mathfrak{M}$ is the non-maximal ideal $0$ for all $p \in \beta X$.

Let us define $\mathfrak{M}_A = (M^p_\mathfrak{M} : p \in \beta X)$.

THEOREM 2.1. For each $p \in \beta X$, $M^p_\mathfrak{M}$ is a prime ideal in $A$; hence $\mathfrak{M}_A$ may be given the hull-kernel topology.

Proof. For $p \in \beta X$, $M^p_\mathcal{M} = \mathfrak{M}_A$, since $0 \in M^p_\mathcal{M}$ and $1 \in M^p_\mathfrak{M}$. Clearly $M^p_\mathcal{M}$ is an ideal in $A$. Next, $M^p_\mathcal{M}$ is prime since whenever $f, g \in A$ with $f \not\mid g$, there exist $h, k \in A$ such that $(fh)^p = 0$ and $(gk)^p = 0$; but then $(fghk)^p = 0$, whence $fg \not\in M^p_\mathcal{M}$.

Let us define $\tau_{\mathfrak{M}} : \beta X \to \mathfrak{M}_A$ by $\tau_{\mathfrak{M}}(p) = M^p_\mathfrak{M}$. For the special subalgebras $C(X)$ and $C^*(X)$, we have observed that $\tau_C$ and $\tau_{C^*}$ are homeomorphisms of $\beta X$ onto $\mathcal{M}_C$ and $\mathcal{M}_{C^*}$. Hence, $C$ and $C^*$ are $\beta$-subalgebras of $C(X)$ according to the following definition.

DEFINITION 2.2. A subalgebra $A$ of $C(X)$ is said to be a $\beta$-subalgebra of $C(X)$ if $\tau_A$ is a homeomorphism of $\beta X$ onto $\mathfrak{M}_A$.

For $f \in A$, write $S_A(f) = \tau_{\mathfrak{M}}(E_A(f)) = \{ p \in \beta X : f \in M^p_\mathfrak{M} \}$, a closed subset of $\beta X$. By [5], 7.3, 7.4, 7.2, it is immediate that

$$S_C(f) = cl_{\beta X} Z(f^0) \quad \text{for} \quad f \in C(X),$$

$$S_C(f) = Z(f^0) \quad \text{for} \quad f \in C^*(X).$$

Given $f, g \in A$, we have $S_A(f) \cup S_A(g) = S_A(fg)$ since each $M^p_\mathfrak{M}$ is prime, and $S_A(f) \cap S_A(g) = S_A(f^0 + g^0)$ by definition of $M^p_\mathfrak{M}$. 
When no confusion can arise, we shall abbreviate $\mathcal{M}_A$, $\mathcal{M}_A^*$, $\mathcal{B}_A$, $\mathcal{R}_A$, $\mathcal{D}_A$, $\mathcal{E}_A$, $\mathcal{F}_A$, $\mathcal{G}_A$, $\mathcal{H}_A$, and $\mathcal{S}_A$ to $\mathcal{M}_A$, $\mathcal{M}_A^*$, $\mathcal{B}_A$, $\mathcal{E}_A$, $\mathcal{F}_A$, $\mathcal{G}_A$, $\mathcal{H}_A$, $\mathcal{I}_A$, and $\mathcal{S}_A$, respectively.

**Proposition 2.4.** Let $A$ be a subalgebra of $C(X)$.

(a) $\tau_A: \beta X \to \mathcal{A}$ is continuous, whence $\mathcal{A}$ is compact.

(b) $\tau_A$ is a closed mapping if and only if $\mathcal{A}$ is a Hausdorff space.

**Proof.** (a) For the basic closed set $E(f) = \{x \in X : f(x) = 0\}$, we have $\tau_A^{-1}(E(f)) = \beta X, E(f) = \beta X, E(f) = \{x \in X : f(x) = 0\}$, a closed subset of $\beta X$.

(b) Since $\tau$ a continuous map of the compact Hausdorff space $\beta X$ onto $\mathcal{A}$, this is clear (cf. [5], p. 202).

In order to give a simple characterization of $\beta$-subalgebras $A$ of $C(X)$, we make the following definitions.

**Definition 2.5.** A subalgebra $A$ of $C(X)$ is said to be $\beta$-determining if $\{Z(f^*: f \in A) : f \in A\}$ is a base for the closed sets in $\beta X$; $A$ is said to be closed under bounded inversion if $f$ is a unit of $A$ whenever $f \in A$ with $f^* \geq 1$.

**Proposition 2.6.** The following are equivalent for a subalgebra $A$ of $C(X)$.

(a) $A$ is $\beta$-determining.

(b) $\mathcal{A}$ is Hausdorff, and $\tau$ is one-to-one.

(c) $\tau$ is a homeomorphism.

**Proof.** (a) implies (b). Suppose that $A$ is $\beta$-determining, and let $p, q \in \beta X$ with $p \neq q$. By [5], 6.5(b), there exist $Z_1, Z_2 \subseteq Z(X)$ such that $p \in \beta Z_1, q \in \beta Z_2, Z_1 \cap Z_2 = X$. Choose $f, g \in A$ such that $p \in Z(f^*), q \in Z(g^*)$. Then $f = 0$, $f \not\in M^*$, and $g \not\in M^*$. It follows that $\mathcal{A}$ is Hausdorff and $\tau$ is one-to-one.

(c) implies (a). Let $f$ be a closed set in $\beta X$ with $p \in \beta f, f \not\in f$. If $\tau$ is a homeomorphism, then $\{S(f) : f \in A\}$ is a base for the closed sets in $\beta X$, so there exists $f \in A$ such that $f \subseteq S(f), p \not\in S(f)$. But then $(f^*)\in(p^*) \neq 0$ for some $g \in A$, and $f \subseteq S(f) \subseteq Z(f^*)$.

An ideal $I$ in $A$ is said to be absolutely convergent if $f \in I$ whenever $f \in A$ and $g \in I$ satisfy $|f| \leq |g|$.

**Proposition 2.7.** The following are equivalent for a subalgebra $A$ of $C(X)$.

(a) $A$ is closed under bounded inversion.

(b) If $I$ is an ideal in $A$, then $\bigwedge I Z(f^*) \cap \mathcal{A}$.

(c) Every ideal in $A$ is contained in some $M^*$.

(d) $\mathcal{M}_A \subseteq \mathcal{A}$.

(e) Every $\mathcal{M} \in \mathcal{M}_A$ is absolutely convex.

**Proof.** (a) implies (b). Assume (a), and let $I$ be an ideal in $A$. Define $\beta = \{Z(f^*), f \in I\}$ to prove (b), it is clearly sufficient to show that $I$ has the finite intersection property. Thus, let $f_1, f_2, \ldots, f_n \in I$ defining $g = \sum f_1 + \sum f_2 + \ldots + \sum f_n \in I$, we have $Z(g) = \bigcap Z(f_i)$. If $Z(g) = \emptyset$, then there exists $x \in X$, $x > 0$, such that $g > x$; but then $g$ is a unit of $A$, contradicting the fact that $g$ belongs to an ideal in $A$. So $Z(g) \neq \emptyset$; hence $I$ has the finite intersection property.

(b) implies (c). Let $I$ be an ideal in $A$. By (b), choose some $p \in \beta X$ such that $p^*(p) = 0$ for all $g \in I$. But then, for $f \in I$, $fg \in I$ for all $g \in A$, whence $f \in M^*$.

(c) implies (d). Obvious.

(d) implies (e). Each $M^*$ is absolutely convex.

(e) implies (a). Since no maximal ideal contains 1, every $f \in A$ with $f \not\in 1$ is a unit of $A$.

We now classify the $\beta$-subalgebra $A$ of $C(X)$, as promised.

**Theorem 2.8.** The following are equivalent for a subalgebra $A$ of $C(X)$.

(a) $A$ is a $\beta$-subalgebra of $C(X)$.

(b) $A$ is $\beta$-determining and closed under bounded inversion.

**Proof.** (a) is a homomorphism. Let $p, q \in \beta X$ with $p \neq q$. By [5], 6.5(b), there exist $Z_1, Z_2 \subseteq Z(X)$ such that $p \in \beta Z_1, q \in \beta Z_2, Z_1 \cap Z_2 = X$. Choose $f, g \in A$ such that $p \in Z(f^*), q \in Z(g^*)$. Then $f = 0$, $f \not\in M^*$, and $g \not\in M^*$. It follows that $\mathcal{A}$ is Hausdorff and $\tau$ is one-to-one.

(c) implies (a). Suppose that $A$ is $\beta$-determining, then $A$ is $\beta$-determining, and closed under bounded inversion. By 2.6, $\tau$ is a homeomorphism of $\beta X$ onto $\mathcal{A}$, and by 2.7, $\mathcal{M}_A \subseteq \mathcal{A}$. Since $\mathcal{A}$ is $T_1$, no two ideals of $\mathcal{A}$ are comparable. Clearly then $\mathcal{M}_A = \mathcal{A}$.

The topology of uniform convergence, or $u$-topology, is defined on $C(X)$ by taking as a neighborhood base for $g \in U$ the $s$-neighborhoods $U_s(g) = \{f \in U : |f - g| < s\}$. A discussion of the $u$-topology may be found in [5]. We now give a simple characterization of the $u$-closed $\beta$-subalgebras of $C(X)$; this characterization clearly provides a large class of examples of $\beta$-subalgebras.

**Theorem 2.9.** A subalgebra $A$ of $C(X)$ is a $u$-closed $\beta$-subalgebra if and only if $C^*(X) \subseteq A$.

**Proof.** Assume that $A$ is a $u$-closed $\beta$-subalgebra, and let $A^* = A \cap C^*$. Clearly $A^*$ is a $u$-closed subalgebra of $C^*$. Next, $A^*$ separates points in $\beta X$. For, let $p, q \in \beta X$ with $p \neq q$. Since $A$ is $\beta$-determining, there exists $f \in A$ such that $f^*(p) = 0, f^*(q) \neq 0$. Since $A$ is closed under bounded inversion, $g = (1 - f^*)^* : A^*; clearly \delta g = 1, \delta g = 1$. By the Stone-Weierstrass Theorem, $A^* = C^*$, whence $C^* \subseteq A$. \


Suppose, conversely, that $C \subset A$. Now, $A$ is $u$-closed; for let $f \in C$ be in the $u$-closure of $A$. Then there exists $g \in A$ such that $|f - g| < 1$, which means that $f = (f - g) + g \in C + A \subset A$. Since $C$ is $\beta$-determining, $A$ is also. Clearly $A$ is closed under bounded inversion.

As a corollary, $O(X)$ and $O(X \setminus A)$ are $u$-closed $\beta$-subalgebras of $O(X)$. We remark that a $u$-closed subalgebra of $O(X)$ need not be $\beta$-determining or closed under bounded inversion. An example is the algebra of all real-valued polynomials on $R$.

3. The $A$-points of $\beta X \setminus X$. Let $A$ be a $\beta$-subalgebra of $O(X)$. We shall now associate with $A$ a set of points in $X = \beta X \setminus X$, called the $A$-points of $X$. Three examples of $\beta$-subalgebras $A$ and their $A$-points will be examined separately in Sections 4, 5, and 7. First, we introduce some notation. By 2.2, the collection $\{S(f) \subseteq A \mid f \in A\}$ is a base for the closed sets in $\beta X$. For $f \in A$, define $S(f) = S(f) \cap X$; then the collection $\{S(f) \subseteq X \mid f \in A\}$ is clearly a base for the closed sets in $X$—a natural base associated with $A$. When no confusion can arise, we shall write $S(f)$ for $S(f)$. Since most of our topological considerations will take place in $X$, let us agree that the symbols $\overline{a}$, $\overline{\mathbb{N}}$, and $\overline{\mathbb{N}}^*$, without subscripts, refer to the topology of $X$.

DEFINITION 3.1. Let $A$ be a $\beta$-subalgebra of $O(X)$. A point $p \in X^*$ is called an $A$-point of $X$ if, for all $f \in A$, $p \notin S(f)$. Clearly a point $p \in X^*$ is an $A$-point if and only if $S(f) = X$ is a neighborhood of $p$ whenever $f \in A$ and $p \notin S(f)$. The set of $A$-points is precisely the set $\bigcap_{A} (X^* \setminus S(f))$, an intersection of a family of $|A|$ dense open subsets of $X^*$.

Let us now prove an existence theorem for $A$-points. A space $X$ is said to have the $G_\delta$-property if every nonvoid $G_\delta$-subset of $X$ has a nonvoid interior; equivalently, if every nonvoid zero-set in $X$ is a nonvoid interior (16, 3.11(b)). The following analogue of the Baire category theorem is essentially proved in (11), 4.2.

PROPOSITION 3.2. Let $X$ be a nonvoid locally compact Hausdorff space with the $G_\delta$-property. If $D$ is a family of at most $\kappa$ dense open subsets of $X$, then $\cap D$ is dense in $X$. If, in addition, $X$ has no isolated points, then $\cap D$ is empty.

Proof. We may write $D = (U_\alpha; \alpha < \kappa)$. Suppose that $G$ is an arbitrary nonvoid open set in $Y$; we shall show that $(\cap D) \cap G = \emptyset$. Let $\alpha < \kappa$, and suppose that there is a collection $(V_\beta; \beta < \alpha)$ of nonvoid open sets in $G$ satisfying the three conditions

(a) $V_\beta \subset U_\alpha$ for $\beta < \alpha$,
(b) $V_\beta \subset U_\alpha$ for $\beta < \alpha$, and
(c) $\bigcap_{\alpha < \kappa} V_\beta = \emptyset$.

Now $\cap V_\beta$ is a $G_\delta$-subset of $X$, and therefore has a nonvoid interior which must meet the dense open set $U_\alpha$. By local compactness, there is a nonvoid open set $V_\alpha$ in $X$ such that $\text{cl} V_\alpha$ is compact and $\text{cl} V_\alpha \subset U_\alpha \cap \cap (\cap V_\beta) \subset U_\alpha \cap G$; in fact, if $X$ has no isolated points, there are two such $V_\alpha$'s with disjoint closures. Thus, $(\cap V_\beta; \alpha < \kappa)$ is defined inductively in such a way that $\text{cl} V_\alpha; \alpha < \kappa)$ is a collection of compact subsets with the finite intersection property satisfying $\text{cl} V_\alpha \subset U_\alpha \cap G$ for all $\alpha < \kappa$. So $(\cap D) \cap G \cap \cap V_\beta = \emptyset$. If $X$ has no isolated points, at each stage of the construction, there are two choices of $V_\alpha$ with disjoint closures; hence $(\cap D) \cap G \cap \cap V_\beta \cap \cap V_\alpha = \emptyset$.

Let us agree to use the symbol $\{\text{CEH}\}$ to indicate that we are assuming the continuum hypothesis ($c = \kappa$). A space $X$ is said to be realcompact if, for every $p \in X^*$, there is a $Z \subset Z(X)$ such that $p \in Z \subset Z^*$.

THEOREM 3.3. [CEH]. Let $X$ be locally compact and realcompact but not compact. If $A$ is a $\beta$-subalgebra of $O(X)$ with $|A| = \kappa$, then $X^*$ has a dense subset of $\kappa$ $A$-points.

Proof. Clearly $X^*$ is a nonvoid compact set. In (2), 3.1, it is shown that, if $X$ is locally compact and realcompact, then $X^*$ has the $G_\delta$-property. The realcompactness of $X$ prevents isolated points in $X^*$. For suppose that $p$ were isolated in $X^*$. Then there would be a zero-set neighborhood $Z_p$ of $p$ in $\beta X$ such that $Z_p \cap X^* = \{p\}$, and by realcompactness, there would be a $Z_p \subset Z(X)$ such that $p \in Z_p \subset X^*$. But then we would have $|p| = Z_p \cap \cap Z_p \subset Z(X^*)$, which by (3), 3.6, would be impossible.

Let $D = (X^* \setminus S(f); f \in A)$, a family of $\kappa$ dense open subsets of $X^*$. Letting $X^*$ play the role of $Y$ in 3.2, we conclude that $\cap D$ is a dense subset of $X^*$ with cardinality at least $\kappa$. But, since $A$ is a $\beta$-subalgebra of $O(X)$, $|X^*| \leq \omega^{\omega} = 2^\kappa$, so that $|\cap D| = 2^\kappa$. As we have pointed out, $(\cap D)$ is the set of $A$-points of $X^*$.

Suppose that $(A_\alpha; \alpha \in A)$ is a family of $\beta$-subalgebras of $O(X)$. The set of points in $X^*$ that are simultaneously $A_\alpha$-points for all $\alpha \in A$ is given by

$$\bigcap_{\alpha \in A} \cap_{\beta \in A} (X^* \setminus S(f))$$

An obvious modification of the proof of 3.3 gives the following generalization.

THEOREM 3.4. [CEH]. Let $X$ be locally compact and realcompact but not compact. If $(A_\alpha; \alpha \in A)$ is a family of $\beta$-subalgebras of $O(X)$ with $|A_\alpha| = \kappa$ for each $\alpha \in A$ and with $|A| \leq \kappa$, then $X^*$ has a dense subset of $\kappa$ points which are simultaneously $A$-points for all $\alpha \in A$.

If $X$ is separable and $A$ is a $\beta$-subalgebra of $O(X)$, then obviously...
5. \( \mathcal{C} \)-points. In this section, we shall turn our attention to the \( \mathcal{C} \)-points of \( X \); thus, we shall consider \( \mathcal{C}(X) \) as a \( \beta \)-subalgebra of itself. We shall relate the concept of \( \mathcal{C} \)-point with that of remote point, defined by Fine and Gillman.

**Proposition 5.1.** If \( X \) is completely uniformizable, in particular if \( X \) is realcompact or metrisable, then \( \mathcal{I}^\mathcal{S}(f) = \mathcal{I}^\mathcal{S}(g) \) for all \( f, g \in \mathcal{O}(X) \).

**Proof.** Obviously, \( \mathcal{I}^\mathcal{S}(f) \cap X^* \subseteq \mathcal{I}^\mathcal{S}(g) \cap X^* \). Let \( p \in \mathcal{I}^\mathcal{S}(f) \); then there exists \( g \in \mathcal{C} \) such that \( p \in X^* \setminus \mathcal{I}^\mathcal{S}(g) \subseteq \mathcal{S}(f) \). But then, \( g \in \mathcal{M}^\mathcal{P} \) and \( f_g \in \mathcal{C}_{\mathcal{P}}(\mathcal{M}^\mathcal{P}) \). In [10] it is shown that, if \( X \) is completely uniformizable, then \( \mathcal{C}_{\mathcal{P}} \) consists of all \( \mathcal{P} \in \mathcal{C}(X) \) with compact support. Thus, \( p \in \mathcal{C}_{\mathcal{P}} X^* \cap \mathcal{I}^\mathcal{S}(g) \subseteq \mathcal{I}^\mathcal{S}(f) \). Hence, \( p \in \mathcal{I}^\mathcal{S}(g) \) if \( p \in \mathcal{I}^\mathcal{S}(f) \), so that \( p \in \mathcal{I}^\mathcal{S}(g) \).

**Definition 5.2.** A point \( p \in \beta X \) is called a remote point in \( \beta X \) if \( p \in \beta X \) is not in the \( \mathcal{C} \)-closure of any discrete subset of \( X \).

A remote point in \( \beta X \) necessarily lies in \( X^* \). Following [5], we associate with each maximal ideal \( \mathcal{M}^\mathcal{P} \) in \( \mathcal{C}(X) \) the \( \mathcal{E} \)-ultrafilter

\[ \mathcal{A}_{\mathcal{P}} = \{ f \in \mathcal{M}^\mathcal{P} \mid f(\mathcal{E}) = \mathcal{E}(Z) : p \in \mathcal{I}^\mathcal{S}(Z) \} \]  

(see 2.3).

**Theorem 5.3.** Let \( p \in X^* \) where \( X \) is a metric space, and consider the following four conditions.

(a) \( p \) is a \( \mathcal{C} \)-point of \( X^* \).

(b) \( A^p \) has no member which is nowhere dense.

(c) \( M^p = \mathcal{O}^p \).

(d) \( \mathcal{P} \) is a remote point in \( \beta X \).

Conditions (a), (b), and (c) are mutually equivalent and are implied by (d). All four conditions are equivalent if \( X \) has no isolated points.

**Proof.** (a) implies (b). Suppose that \( p \) is a \( \mathcal{C} \)-point, and let \( Z \in A^p \). Then \( p \in \mathcal{I}^\mathcal{S}(Z X) \), and by Proposition 5.1, \( p \in \mathcal{I}^\mathcal{S}(Z X) \). Thus, \( \mathcal{S} = \mathcal{O} \setminus X \subseteq Z \), and \( Z \) is nowhere dense.

(b) implies (c). Assume (b), and let \( f \in \mathcal{M}^p \). Since \( X \) is a metric space, we may find \( g \in \mathcal{C}(X) \) such that \( Z(g) = \mathcal{O}^p \). But then, \( p \in \mathcal{I}^\mathcal{S}(Z(g)) \subseteq \mathcal{I}^\mathcal{S}(f) \). Now, if \( p \in \mathcal{C}_{\mathcal{P}} X^* \), then \( p \in \mathcal{I}^\mathcal{S}(Z^*(f)) \subseteq \mathcal{I}^\mathcal{S}(Z(g)) \), contradicting our hypothesis, since \( \mathcal{S} \subseteq Z(g) \) is nowhere dense. Hence, \( p \in \mathcal{I}^\mathcal{S}(Z(f)) \subseteq \mathcal{I}^\mathcal{S}(Z(g)) \), so that \( p \in \mathcal{O}^p \).

(c) implies (a). This follows from 3.5.

(d) implies (b). Suppose that \( A^p \) has a nowhere dense member \( Z \).

It is shown in [7], p.388 (VIII), that, if \( Z \) is a closed nowhere dense set in the metric space \( X \), then there is a discrete subset \( D \) of \( X \) such that \( D \setminus Z = \mathcal{C}X \). Thus, \( p \in \mathcal{C}_{\mathcal{P}} X \subseteq \mathcal{C}X \), so that \( p \) is not a remote point.

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Assume that $X$ has no isolated points; we shall prove that (b) implies (d). Suppose then that $p$ is not a remote point; then there is a discrete subset $D$ of $X$ such that $p \in \text{cl}_X D$. Since any point common to $D$ and $\text{int}_X \text{cl}_X D$ would be isolated, one easily sees that $Z = \text{cl}_X D$ is nowhere dense; clearly $Z \subset A'$.

The equivalence of (b) and (d) appears in [3] for $X = \mathbb{R}$; we wish to thank Mark Mandelker for communicating (b) implies (c).

**Theorem 5.4.** [CH] If $X$ is a separable, locally compact, noncompact metric space without isolated points, then $\beta X$ has a collection of $2^\omega$ remote points which forms a dense subset of $X^\omega$.

**Proof.** Since $X$ is a separable metric space, it is clear that $X$ is realcompact and $\mathcal{C}(X) = c$. (In fact, [CH] for a metric space $X$, the separability of $X$ is equivalent to the condition $\mathcal{C}(X) = c$.) By 3.3, $X^\omega$ has a dense subset of $2^\omega$ $\mathcal{C}$-points, and by 5.3, the $\mathcal{C}$-points are precisely the remote points in $\beta X$.

An obvious corollary to 5.4 is that [CH] $\beta \mathbb{R}$ has a collection of remote points which is dense in $\mathbb{R}$. This result was proved by Fine and Gillman in [3] by another method. Our proof appears to be simpler than the Fine-Gillman proof, but their methods have wider application; they show that [CH] $\beta \mathbb{Q}$ has remote points, whereas our method fails in this case ($\mathbb{Q}^\omega$ does not have the $\mathcal{C}$-property). Using the methods of [3], we now extend 5.4 to include the case $X = Q$ by removing the local compactness from the hypotheses.

**Theorem 5.5.** [CH] If $X$ is a separable, noncompact metric space without isolated points, then $\beta X$ has a collection of $2^\omega$ remote points which forms a dense subset of $X^\omega$.

**Proof.** Let $V$ be a closed neighborhood in $\beta X$ of any point in $X^\omega$. Since $X$ is a separable metric space, $X$ is realcompact and has no more than $\aleph_1 (= c)$ dense open subsets. By [3, 2.3], there is a family $\mathcal{F}$ of nonempty sets of $X$ such that $\mathcal{F}$ has the finite-intersection property, $\cap \mathcal{F} = G$, and every dense open subset of $X$ contains a member of $\mathcal{F}$. Since $X$ is realcompact, we may construct $\mathcal{F}$ so that each of its members is contained in $V$ (see [3, 2.5]). Now let $A = \{ p \in \beta X : \mathcal{F} \subset A \} = \bigcap \text{cl}_X Z$, a nonvoid compact subset of $V \cap X^\omega$. A simple modification of the proof of [3, 2.3] guarantees that $A$ is infinite; hence, by [5, 9.11], we have $|A| \geq 2^\omega$. As in the proof of 3.3, $|X^\omega| \leq 2^\omega$, whence $|A| = 2^\omega$. Now, for $p \in A$, $A^p$ contains no member which is nowhere dense; each such $p$ is remote by 5.3.

Thus, [CH] $\mathbb{Q}^\omega$ has $\mathcal{C}$-points but no $\mathcal{C}$*-points (see [5], 6.05). We remark that 5.3 and 5.5 remain true if we assume only that the set of isolated points in $X$ has compact closure.

6. Remote points in $\beta \mathbb{R}$ vs. $P$-points in $\beta \mathbb{R} \setminus \mathbb{R}$. We now concentrate on the case $X = \mathbb{R}$. Let $P$ denote the set of $P$-points of $\mathbb{R}$, $\mathbb{R}$ denote the set of remote points in $\beta \mathbb{R}$, $\mathbb{R}^* = \mathbb{R} \setminus P$, and $\mathbb{R}^* = \mathbb{R} \setminus R$. We shall now show that no inclusions hold between the sets $P$, $R$, $\mathbb{R}^*$, and $\mathbb{R}$. First we prove a preliminary result. We call $X$ an $F$-space if every cozero-set in $X$ is $\mathcal{C}$*-embedded in $X$. Every $\mathcal{C}$*-embedded subset of an $F$-space is an $F$-space ([5], 14.26), $N\ast$ and $\mathbb{R}^\ast$ are compact $F$-spaces ([5], 14.27), and every countable subset of an $F$-space is $\mathcal{C}$*-embedded ([5], 14.35).

**Proposition 6.1.** If $X$ is an infinite compact $F$-space, then $X$ contains at least $2^\omega$ non-$P$-points.

**Proof.** Let $X$ be an infinite compact $F$-space. Then, by [5], 6.13, $X$ contains a countable discrete set $D = \{ p_n : n \in \mathbb{N} \}$. As a countable set, $D$ is $\mathcal{C}$*-embedded in $X$, whence $\text{cl}_X D = \beta D$ ([5], 6.9 (a)). Define $f : \mathcal{C}(X)$ by letting $f(p_n) = n^{-1}$ for $n \in \mathbb{N}$ and extending over $X$. Then, for every $p \in D^* = \text{cl}_X D^* = D$, $p \not\in f(f)$, but $Z(f)$ is not a neighborhood of $p$. Thus, every one of the $2^\omega$ points in $D^*$ is a non-$P$-point of $X$.

As a corollary, $N\ast$ and $\mathbb{R}^\ast$ each have $2^\omega$ non-$P$-points.

**Theorem 6.2.** [CH] The sets $P \cap \mathbb{R}$, $P \cap \mathbb{R}^\ast$, $\mathbb{R} \cap \mathbb{R}^\ast$, and $\mathbb{R} \cap \mathbb{R}^\ast$ are each dense subsets of $\mathbb{R}^\ast$ of cardinality $2^\omega$.

**Proof.** ($P \cap \mathbb{R}$). Apply 3.4 to the family $(\mathcal{C}(\mathbb{R}), \mathcal{C}(\mathbb{R}))$ of $\beta$-subalgebras of $\mathcal{C}(\mathbb{R})$.

($P \cap \mathbb{R}^\ast$ and $\mathbb{R} \cap \mathbb{R}^\ast$). Let $V$ be a closed neighborhood in $\beta \mathbb{R}$ of any point in $\mathbb{R}^\ast$. Then $V \cap \mathbb{R}$ is a nonpseudocompact and is $\mathcal{C}$-embedded in $\mathbb{R}$ ([5], 14.4); hence $V \cap \mathbb{R}$ contains a copy of $D$ of $N\ast$ which is $\mathcal{C}$-embedded in $\mathbb{R}$ ([5], 13.29). Then $D^\ast = \text{cl}_X D^* = \mathcal{C}(\mathbb{R}) \cap \mathbb{R}^\ast$, since $D$ is closed and $\mathcal{C}$*-embedded in $\mathbb{R}$. A point in $D^*$ is a $P$-point of $D^*$ if and only if it is a $P$-point of $\mathbb{R}^\ast$ ([5], 4.12, 9.32). But $D^*$ is homeomorphic with $N\ast$, so that $D^\ast$ has $2^\omega$ non-$P$-points by 6.1 and [CH] $2^\omega$ $P$-points by 4.2. Clearly, no point of $D^\ast$ is a remote point in $\beta \mathbb{R}$. ($P \cap \mathbb{R}^\ast$). Let $V$ be a closed neighborhood in $\beta \mathbb{R}$ of any point in $\mathbb{R}^\ast$. As in the proof of 5.5, construct an infinite compact set $\Delta$ of remote points in $\beta \mathbb{R}$. Since $\mathbb{R}^\ast$ is an $F$-space, the $\mathcal{C}$*-embedded subset $\Delta$ is also an $F$-space. Then, by 6.1, $\Delta$ has $2^\omega$ non-$P$-points, and each of these is a non-$P$-point of $\mathbb{R}^\ast$. Thus, $V \cap \mathbb{R}^\ast$ has $2^\omega$ points which are non-$P$-points of $\mathbb{R}^\ast$ and remote points in $\beta \mathbb{R}$.

7. The algebra $H$. In this section, we shall let $C(X)$ denote the algebra (over the complex numbers $C$) of complex-valued continuous functions on $X$ and $C^\omega(X)$ the subalgebra of bounded functions. A subalgebra of $C(X)$ will mean a subalgebra in the usual sense which contains the constant functions and which is self-adjoint (closed under the formation
of complex conjugates). By an ideal we shall mean a proper self-adjoint ideal. With these conventions, it is not difficult to see that all the results that we have obtained for subalgebras of $O(X)$ in the real case are true in the complex case as well.

Following R. M. Brooks [1], let us define

$$H = \{ f \in C(N) : \limsup_{n \to \infty} |f(n)| = 1 \}$$

where $f(n) = |f(n)|^m$ for $n \in N$. It is shown in [1] that $H$ is a subalgebra of $C(N)$ containing $C^0(N)$, so by 2.1, $H$ is a $\alpha$-closed $\beta$-subalgebra of $C(N)$. Thus, $\pi_H$ is homeomorphic with $\pi_N$ ([1], 2.4).

**Proposition 7.1.** $H = \{ f \in C(N) : jf \leq 1 \}$ on $N^*$. A function $f \in H$ is a unit of $H$ if and only if $Z(f) = \emptyset$ and $j^f = 1$ on $N^*$.

**Proof.** The first fact follows by observing that $\limsup_{n \to \infty} |f(p)| = \limsup_{n \to \infty} |f(n)| = 0$ for any real-valued $f \in C(N)$. The second part is clear since $j^f = \frac{f}{j^f}$ for $f, g \in H$.

Following Brooks, let us define, for $p \in N^*$, the collection $J^p = \{ f \in H : j^f(p) < 1 \}$ of non-units of $H$.

**Proposition 7.2.** For $p \in N^*$, $J^p$ is a prime ideal in $H$ contained in $M^p$, whence $O^p \subset J^p \subset M^p$.

**Proof.** We first note that $f \in J^p$ implies $f^p(p) = 0$. For suppose that $j^f(p) < 1$. Then there exists $\delta < 1$ and a neighborhood $V$ of $p$ in $\pi_N$ such that $|f(n)|^m \leq \delta$ whenever $n \in V \cap N$; that is, $|f(n)| \leq \delta^m$ whenever $n \in V \cap N$. If $U$ is a neighborhood of $p$ in $\pi_N$, then $U \cap V$ contains arbitrarily large $n \in N$ yielding arbitrarily small positive values of $|f(n)|$; hence $f^p(p) = 0$.

$J^p$ is easily seen to be an ideal (see [1], 2.3.4, 2.3.5) and is clearly prime, since $fg = tf^p$. Suppose $f \in J^p$, whence $fg \in J^p$ for all $g \in H$; then $(fg)^p(p) = 0$ for all $g \in H$, whereby $f \in M^p$. Since $J^p \subset M^p$, it follows from [4], 3.4, that $O^p \subset J^p$.

By considering $H$ as a topological ring, it is shown in [1], 4.9, that $H$ has at least one nonmaximal prime ideal. We can now improve on this result.

**Proposition 7.3.** $H$ has $2^n$ nonmaximal prime ideals.

**Proof.** Since $|H| = \omega$, $H$ has no more than $2^n$ nonmaximal prime ideals. By [4], 2.6, 3.4, it suffices to prove that $M^p \neq O^p$ for $p \in N^*$. Thus, define $f(n) = n^m$ for $n \in N$, and let $p \in N^*$ be arbitrary. Since $f(n) = n^m$, clearly $f \in J^p \subset M^p$. It is easy to see that $O^p = O_\omega \cap H$. Therefore $f \notin O^p$, hence $Z(f) = \emptyset$.

Let us now give a simple characterization of the basic closed set $S^0(f)$ for $f \in H$ (cf. 2.3). First we state a lemma.

**Lemma 7.4.** Let $p \in N^*$ and $f \in H$. If $j^f = 1$ on some $N^*$-neighborhood of $p$, then $f \in M^p$.

**Proof.** Suppose that $j^f = 1$ on some $N^*$-neighborhood $V$ of $p$. We may assume that $V = cl_N E \cap B$ for some subset $B$ of $N$ and that $f(n) \geq \delta$ for $n \in V$. Define $g \in C(N)$ by letting $g(n) = f(n)^{-1}$ for $n \in E$ and $g(n) = 1$ for $n \notin E$. Then $\lim g(n) = 1$, so that $g \in H$. Furthermore, $(fg)^p(p) = 1$, so that $f \notin M^p$.

**Proposition 7.5.** For $f \in H$, $S^0(f)$ is a regular closed subset of $N^*$; moreover, $S^0(f) = \cl\{ g \in N^* : j^f(g) = 1 \}$ and $\text{int} S^0(f) = \{ g \in N^* : j^f(g) < 1 \}$.

**Proof.** By 7.2, it is clear that $\cl\{ g \in N^* : j^f(g) < 1 \} \subset S^0(f)$. Suppose that $p \in S^0(f)$. By 7.4, in every $N^*$-neighborhood of $p$, there is a point $q$ such that $j^f(q) < 1$; that is, $p \in \cl\{ g \in N^* : j^f(g) < 1 \}$.

By Proposition 7.3, we have $\{ g \in N^* : j^f(g) < 1 \} \subset \text{int} S^0(f)$. Suppose that $p \in \text{int} S^0(f)$ and $j^f(p) = 1$; we shall deduce a contradiction. Let $(a_k)_{k \in N}$ be an increasing sequence in $N$ such that $\lim j^f(a_k) = 1$. Letting $E = \{ a_k : k \in N \}$, we may assume that $cl_N E \subset \text{int} S^0(f)$. Then $j^f = 1$ on the nonvoid open subset $cl_N E \subset S^0(f)$, and this contradicts 7.4.

In [1], it is stated that $M^p = j^p$, for all $p \in N^*$. We now show that this is false; in fact, the equality holds precisely when $p$ is a $P$-point of $N^*$.

**Theorem 7.6.** The following are equivalent for a point $p \in N^*$.

(a) $j^p = M^p$.
(b) $p$ is an $H$-point of $N^*$.
(c) $p$ is a $P$-point of $N^*$.

**Proof.** (a) implies (b). Suppose that $j^p = M^p$. If $p \in S^0(f)$, then $p \in cl_N \{ g \in N^* : j^f(g) < 1 \} \subset \text{int} S^0(f)$. Hence, $p$ is an $H$-point of $N^*$.

(b) implies (c). Let $p$ be a non-$P$-point of $N^*$, and let $g \in \pi_N$ be a real-valued function which is nonconstant on every $N^*$-neighborhood of $p$; we may assume that $0 \leq g \leq 1$ when $g(p) = 1$. Let $f(n) = g(n)^{m}$ for $n \in N$; then $f \in H$, and by 7.5, $p \in \text{int} S^0(f)$. Now, in every $N^*$-neighborhood of $p$, there is a point $q$ such that $j^f(q) < 1$, by the construction of $f$. So $p \in S^0(f)$, by 7.5. Hence, $p$ is not an $H$-point of $N^*$.

(c) implies (a). Suppose that $j^p = M^p$ and $j^p \notin J^p$. Then $j^f(p) = 1$, but by 7.4, $j^f(p) = 1$ is not identically 1 on any $N^*$-neighborhood of $p$. Clearly then, $p$ is not a $P$-point of $N^*$.

**References**


Fundamental retracts and extensions of fundamental sequences

by

Karol Borsuk (Warszawa)

In order to extend some standard notions of the homotopy theory onto arbitrary compacta $X$, $Y$ lying in the Hilbert space $H$, I introduced in [2] the notion of the fundamental sequence from $X$ to $Y$, defined as an ordered triple $\mathbf{f} = (f_k, X, Y)$ consisting of $X$, $Y$ and of a sequence $(f_k)$ of (continuous) maps of $H$ into itself satisfying the following condition:

For every neighborhood $V$ of $Y$ (neighborhoods are understood here always in the space $H$) there exists a neighborhood $U$ of $X$ such that $f_k/U \simeq f_{k+1}/U$ in $V$ for almost all $k$.

The set $X$ will be said to be the domain, and the set $Y$—the range of the fundamental sequence $\mathbf{f}$.

Setting $i(x) = x$ for every point $x \in H$, we immediately see that for every compactum $X \subset H$ the triple $(i, X, X)$ is a fundamental sequence called the fundamental identity sequence for $X$.

If $\mathbf{e}$ is a point of a compactum $X \subset H$, then setting $e(x) = x$ for every point $x \in H$, we get a fundamental sequence $\mathbf{e}_X = (e, X, X)$ called a constant fundamental sequence for $X$.

Let us observe that if $\hat{X}$ is a closed subset of a compactum $X \subset H$, and $\hat{Y}$ is a closed subset of a compactum $Y \subset H$, and if $\mathbf{f} = (f_k, X, Y)$ is a fundamental sequence, then $\mathbf{f} = (f, \hat{X}, \hat{Y})$ is also a fundamental sequence.

Two fundamental sequences $\mathbf{f} = (f_k, X, Y)$ and $\mathbf{g} = (g_k, X, Y)$ are said to be homotopic (in symbols: $\mathbf{f} \simeq \mathbf{g}$) if for every neighborhood $V$ of $Y$ there exists a neighborhood $U$ of $X$ such that $f_k/U \simeq g_k/U$ in $V$ for almost all $k$.

The fundamental sequences from $X$ to $Y$ may be considered as a generalization of the maps of $X$ into $Y$, and the classes of all homotopic fundamental sequences from $X$ to $Y$ (called fundamental classes from $X$ to $Y$) may be considered as a generalization of the homotopy classes of maps of $X$ into $Y$. 