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On a class of subalgebras of $C(X)$ with applications to $\beta X \setminus X$.

by

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W. Rudin has proved that, assuming the continuum hypothesis, $\beta\mathbf{N}\setminus\mathbf{N}$ has a dense subset of 2^c P-points. A similar theorem of N. J. Fine and L. Gillman states that, assuming the continuum hypothesis, $\beta\mathbf{R}\setminus\mathbf{R}$ has a dense subset of remote points in $\beta\mathbf{R}$. It is the purpose of this paper to unify these results by giving a more general method of finding such points.

Specifically, for a completely regular space X, we define a class of subalgebras of $C(X)$ called *β-subalgebras.* Examples of β-subalgebras include $C(X)$ itself and $C^*(X)$. With each β-subalgebra A of $C(X)$ we associate a (possibly empty) set of points in $\beta X\setminus X$ called *A-points.* We show that, under the continuum hypothesis and with reasonable restrictions on A and X, $\beta X\setminus X$ has a dense subset of 2^c A-points. The Rudin theorem is then obtained by observing that the P-points of $\beta\mathbf{N}\setminus\mathbf{N}$ are precisely the $C^*(\mathbf{N})$-points, and the Fine-Gillman theorem follows from the fact that the remote points in $\beta\mathbf{R}$ are precisely the $C(\mathbf{R})$-points.

Our method considerably simplifies the Fine–Gillman proof of the existence of remote points in $\beta\mathbf{R}$ but does not have the power of their method. Using their method, we show the existence of remote points in $\beta\mathbf{R}$ which are not P-points of $\beta\mathbf{R}\setminus\mathbf{R}$. We conclude by investigating a β-subalgebra H of $C(\mathbf{N})$ previously studied by R. M. Brooks. We correct Brooks's characterization of the maximal ideals in H and show that his characterization holds precisely for the ideals M^p where p is a P-point of $\beta\mathbf{N}\setminus\mathbf{N}$ (equivalently, where p is an H-point).

1. Preliminaries. The basic reference for this paper will be the Gillman and Jerison text [5]; the terminology and notation will, with only a few exceptions, be that of [5].

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The symbol X will always denote a completely regular Hausdorff space. Specific spaces X in which we shall be interested are the complex plane $\mathbf{C}$ and its subspaces $\mathbf{R}$ of real numbers, $\mathbf{Q}$ of rational numbers, and $\mathbf{N}$ of natural numbers.

In Sections 1 through 6, $C(X)$ will denote the collection of real-valued continuous functions on X, and $C^*(X)$ will denote the subcollection of bounded functions. The constant function on X of value r will be denoted by r. Under the pointwise operations, $C(X)$ and $C^*(X)$ are algebras over $\mathbf{R}$. A *subalgebra* of $C(X)$ will mean a subalgebra in the usual sense which contains the constant functions. By an *ideal* we shall mean a proper ideal. In Section 7, the definition of subalgebra and ideal are changed slightly to accommodate complex-valued functions.

A subspace Y of X is said to be *C^*-embedded* if each function in $C^*(Y)$ is the restriction of some function in $C^*(X)$; the expression "*C-embedded*" is defined analogously. Given X, there is an essentially unique compact Hausdorff space βX which contains X as a dense C^*-embedded subspace (the extension of f to βX will be denoted by f^β). For notational simplicity, we write $X^* = \beta X \setminus X$. For additional properties of βX, the reader is referred to [5]. We mention one: if $f \in C(X)$ and $a\mathbf{R}$ denotes the one-point compactification of $\mathbf{R}$, then there is a (unique) continuous $f^*\colon \beta X \to a\mathbf{R}$ which agrees with f on X.

If τ is a function, then we let $\tau^{\leftarrow}$ denote the inverse map (of sets). If f maps X to $\mathbf{R}$ or $a\mathbf{R}$, then $Z(f) = f^{\leftarrow}(0)$ and $\operatorname{Coz}(f) = X \setminus Z(f)$. A *zero-set* of X is a member of the family $Z(X) = \{Z(f)\colon f \in C(X)\}$, and a *cozero-set* of X is the complement in X of some member of $Z(X)$.

If S is a set, then $|S|$ will denote the cardinality of S. As is standard, we shall let c denote the cardinality $2^{\aleph_0}$ of the continuum. If $S \subset X$, then $\mathrm{cl}_X S$, $\mathrm{int}_X S$, and $\partial_X S$ will denote, respectively, the closure, interior, and boundary of S in X ($\partial_X S = \mathrm{cl}_X S \setminus \mathrm{int}_X S$).

2. β-subalgebras. Recall the definition of the hull-kernel topology on a collection $\mathcal{P}$ of prime ideals in a commutative ring A with an identity. Define $\bar{\mathcal{S}} = \{P \in \mathcal{P}\colon \bigcap \mathcal{S} \subset P\}$ to be the closure of the subset $\mathcal{S}$ of $\mathcal{P}$. It is easy to verify that the sets

$$E_{\mathcal{P}}(a) = \{P \in \mathcal{P}\colon a \in P\}, \quad a \in A,$$

are closed and constitute a base for the closed sets in $\mathcal{P}$. A detailed description of the hull-kernel topology is given in [4]. Let $\mathcal{M}_A$ denote the collection of maximal ideals in A endowed with the hull-kernel topology.

Given a subalgebra A of $C(X)$, we shall now introduce a family $\mathcal{G}_A$ of prime ideals in A. The family $\mathcal{G}_A$ will reduce to $\mathcal{M}_A$ in the cases $A = C(X)$ and $A = C^*(X)$. To motivate our definition, we observe that the maximal ideals in $C = C(X)$ and $C^* = C^*(X)$ associated with the same point $p \in \beta X$ can be characterized in the following parallel ways

$$M_C^p = \{f \in C\colon (fg)^*(p) = 0 \text{ for all } g \in C\};$$

$$M_{C^*}^p = \{f \in C^*\colon (fg)^*(p) = 0 \text{ for all } g \in C^*\}.$$

The first characterization was discussed by Gelfand and Kolmogoroff [6]; the second is elementary (see [5], 7.2). Gelfand and Kolmogoroff proved that the mappings $p \to M_C^p$ and $p \to M_{C^*}^p$ are homeomorphisms of βX onto the maximal-ideal spaces $\mathcal{M}_C$ and $\mathcal{M}_{C^*}$.

The similarity of the expressions for M_C^p and $M_{C^*}^p$ suggests a generalization of these ideals to any subalgebra A of $C(X)$. Thus, for $p \in \beta X$, let us define

$$M_A^p = \{f \in A\colon (fg)^*(p) = 0 \text{ for all } g \in A\}.$$

It is easy to see that, for $p \in X$, M_A^p is the fixed maximal ideal $\{f \in A\colon f(p) = 0\}$ in A, and we shall show next that, for $p \in \beta X$, M_A^p is always a prime ideal. But the general correspondence $p \to M_A^p$ need not be one-to-one, and, in spite of the notation, the ideal M_A^p need not be maximal. For example, in the algebra A of all real-valued polynomials on $\mathbf{R}$, M_A^p is the non-maximal ideal (0) for all $p \in \beta\mathbf{R} \setminus \mathbf{R}$.

Let us define $\mathcal{G}_A = \{M_A^p\colon p \in \beta X\}$.

THEOREM 2.1. *For each $p \in \beta X$, M_A^p is a prime ideal in A; hence $\mathcal{G}_A$ may be given the hull-kernel topology.*

Proof. For $p \in \beta X$, $\emptyset \neq M_A^p \neq A$, since $0 \in M_A^p$ and $1 \notin M_A^p$. Clearly M_A^p is an ideal in A. Next, M_A^p is prime since whenever $f, g \in A$ with $f \notin M_A^p$ and $g \notin M_A^p$, there exist $h, k \in A$ such that $(fh)^*(p) \neq 0$ and $(gk)^*(p) \neq 0$; but then $(fghk)^*(p) \neq 0$, whence $fg \notin M_A^p$.

Let us define $\tau_A\colon \beta X \to \mathcal{G}_A$ by $\tau_A(p) = M_A^p$. For the special subalgebras $C(X)$ and $C^*(X)$, we have observed that τ_C and τ_{C^*} are homeomorphisms of βX onto $\mathcal{M}_C$ and $\mathcal{M}_{C^*}$. Hence, C and C^* are β-subalgebras of $C(X)$ according to the following definition.

DEFINITION 2.2. A subalgebra A of $C(X)$ is said to be a *β-subalgebra* of $C(X)$ if τ_A is a homeomorphism of βX onto $\mathcal{M}_A$.

For $f \in A$, write $S_A(f) = \tau_A^{\leftarrow}[E_{\mathcal{G}_A}(f)] = \{p \in \beta X\colon f \in M_A^p\} = \bigcap_{g \in A} Z((fg)^*)$, a closed subset of βX. By [5], 7.3, 7D, 7.2, it is immediate that

$$S_C(f) = \mathrm{cl}_{\beta X} Z(f) \quad \text{for} \quad f \in C(X),$$
$$(2.3) \qquad S_{C^*}(f) = Z(f^\beta) \quad \text{for} \quad f \in C^*(X).$$

Given $f, g \in A$, we have $S_A(f) \cup S_A(g) = S_A(fg)$ since each M_A^p is prime, and $S_A(f) \cap S_A(g) = S_A(f^2+g^2)$ by the definition of M_A^p.

When no confusion can arise, we shall abbreviate $\mathcal{M}_A$, M^p_A, $\mathcal{G}_A$, $E_{\mathcal{G}_A}$, τ_A and S_A to $\mathcal{M}$, M^p, $\mathcal{G}$, E, τ and S, respectively.

PROPOSITION 2.4. *Let A be a subalgebra of $C(X)$.*

(a) *$\tau_A\colon \beta X \to \mathcal{G}_A$ is continuous, whence $\mathcal{G}_A$ is compact.*

(b) *τ_A is a closed mapping if and only if $\mathcal{G}_A$ is a Hausdorff space.*

Proof. (a) For the basic closed set $E(f)$, $f \in A$, we have $\tau^{\leftarrow}[E(f)] = S(f)$, a closed subset of βX.

(b) Since τ a continuous map of the compact Hausdorff space βX onto $\mathcal{G}$, this is clear (cf. [9], p. 252).

In order to give a simple characterization of β-subalgebras of $C(X)$, we make the following definitions.

DEFINITION 2.5. A subalgebra A of $C(X)$ is said to be *β-determining* if $\{Z(f^*)\colon f \in A\}$ is a base for the closed sets in βX; A is said to be *closed under bounded inversion* if f is a unit of A whenever $f \in A$ with $f \geqslant 1$.

PROPOSITION 2.6. *The following are equivalent for a subalgebra A of $C(X)$.*

(a) *A is β-determining.*

(b) *$\mathcal{G}_A$ is Hausdorff, and τ is one-to-one.*

(c) *τ is a homeomorphism.*

Proof. (a) *implies* (b). Suppose that A is β-determining, and let $p, q \in \beta X$ with $p \neq q$. By [5], 6.5(b), there exist $Z_1, Z_2 \in Z(X)$ such that $p \notin \mathrm{cl}_{\beta X} Z_1$, $q \notin \mathrm{cl}_{\beta X} Z_2$ and $Z_1 \cup Z_2 = X$. Choose $f, g \in A$ such that $p \notin Z(f^*) \supset \mathrm{cl}_{\beta X} Z_1$ and $q \notin Z(g^*) \supset \mathrm{cl}_{\beta X} Z_2$; then $fg = 0$, $f \notin M^p$ and $g \notin M^q$. It follows that $\mathcal{G}$ is Hausdorff and τ is one-to-one.

(b) *implies* (c). If $\mathcal{G}$ is Hausdorff, then τ is a closed mapping, by 2.4. If, in addition, τ is one-to-one, then it is a homeomorphism.

(c) *implies* (a). Let F be a closed set in βX with $p \in \beta X$, $p \notin F$. If τ is a homeomorphism, then $\{S(f)\colon f \in A\}$ is a base for the closed sets in βX, so there exists $f \in A$ such that $F \subset S(f)$, $p \notin S(f)$. But then $(fg)^*(p) \neq 0$ for some $g \in A$, and $F \subset S(f) \subset Z\big((fg)^*\big)$.

An ideal I in A is said to be *absolutely convex* if $f \in I$ whenever $f \in A$ and $g \in I$ satisfy $|f| \leqslant |g|$.

PROPOSITION 2.7. *The following are equivalent for a subalgebra A of $C(X)$.*

(a) *A is closed under bounded inversion.*

(b) *If I is an ideal in A, then $\bigcap_{f \in I} Z(f^*) \neq \emptyset$.*

(c) *Every ideal in A is contained in some M^p.*

(d) *$\mathcal{M}_A \subset \mathcal{G}_A$.*

(e) *Every $M \in \mathcal{M}_A$ is absolutely convex.*

Proof. (a) *implies* (b). Assume (a), and let I be an ideal in A. Define $\mathfrak{Z} = \{Z(f^*)\colon f \in I\}$; to prove (b), it is clearly sufficient to show that $\mathfrak{Z}$ has the finite intersection property. Thus, let $f_1, f_2, \ldots, f_n \in I$; defining $g = f_1^2 + f_2^2 + \ldots + f_n^2 \in I$, we have $Z(g^*) = \bigcap_{i=1}^{n} Z(f_i^*)$. If $Z(g^*) = \emptyset$, then there exists $r \in \mathbf{R}$, $r > 0$, such that $g \geqslant r$; but then g is a unit of A, contradicting the fact that g belongs to an ideal in A. So $Z(g^*) \neq \emptyset$; hence $\mathfrak{Z}$ has the finite intersection property.

(b) *implies* (c). Let I be an ideal in A. By (b), choose some $p \in \beta X$ such that $g^*(p) = 0$ for all $g \in I$. But then, for $f \in I$, $fg \in I$ for all $g \in A$, whence $f \in M^p$.

(c) *implies* (d). Obvious.

(d) *implies* (e). Each M^p is absolutely convex.

(e) *implies* (a). Since no maximal ideal contains 1, every $f \in A$ with $f \geqslant 1$ is a unit of A.

We now classify the β-subalgebras of $C(X)$, as promised.

THEOREM 2.8. *The following are equivalent for a subalgebra A of $C(X)$.*

(a) *A is a β-subalgebra of $C(X)$.*

(b) *A is β-determining and closed under bounded inversion.*

Proof. (a) *implies* (b). Suppose that A is a β-subalgebra of $C(X)$. Then A is β-determining, by 2.6, and closed under bounded inversion, by 2.7.

(b) *implies* (a). Suppose that A is β-determining and closed under bounded inversion. By 2.6, τ is a homeomorphism of βX onto $\mathcal{G}$, and by 2.7, $\mathcal{M} \subset \mathcal{G}$. Since $\mathcal{G}$ is T_1, no two ideals of $\mathcal{G}$ are comparable. Clearly then $\mathcal{M} = \mathcal{G}$.

The topology of uniform convergence, or *u-topology*, is defined on $C(X)$ by taking as a neighborhood base for $g \in C$ the ε-neighborhoods $U_\varepsilon(g) = \{f \in C\colon |f-g| < \varepsilon\}$. A discussion of the u-topology may be found in [8]. We now give a simple characterization of u-closed β-subalgebras of $C(X)$; this characterization clearly provides a large class of examples of β-subalgebras.

THEOREM 2.9. *A subalgebra A of $C(X)$ is a u-closed β-subalgebra if and only if $C^*(X) \subset A$.*

Proof. Assume that A is a u-closed β-subalgebra, and let $A^* = A \cap C^*$; clearly A^* is a u-closed subalgebra of C^*. Next, A^* separates points in βX. For, let $p, q \in \beta X$ with $p \neq q$. Since A is β-determining, there exists $f \in A$ such that $f^*(p) = 0$, $f^*(q) \neq 0$. Since A is closed under bounded inversion, $g = (1+f^2)^{-1} \in A^*$; clearly $g^\beta(p) = 1$, $g^\beta(q) \neq 1$. By the Stone–Weierstrass Theorem, $A^* = C^*$, whence $C^* \subset A$.

Suppose, conversely, that $C^* \subset A$. Now, A is u-closed; for let $f \in C$ be in the u-closure of A. Then there exists $g \in A$ such that $|f-g| < 1$, which means that $f = (f-g)+g \in C^*+A \subset A$. Since C^* is β-determining, A is also. Clearly A is closed under bounded inversion.

As a corollary, $C^*(X)$ and $C(X)$ itself are u-closed β-subalgebras of $C(X)$. We remark that a u-closed subalgebra of $C(X)$ need not be β-determining or closed under bounded inversion. An example is the algebra of all real-valued polynomials on $\mathbf{R}$.

3. The A-points of $\beta X \setminus X$. Let A be a β-subalgebra of $C(X)$. We shall now associate with A a set of points in $X^* = \beta X \setminus X$ called the *A-points* of X^*. Three examples of β-subalgebras A and their A-points will be examined separately in Sections 4, 5 and 7. First, we introduce some notation. By 2.6, the collection $\{S_A(f): f \in A\}$ is a base for the closed sets in βX. For $f \in A$, define $S_A^*(f) = S_A(f) \cap X^*$; then the collection $\{S_A^*(f): f \in A\}$ is clearly a base for the closed sets in X^*—a *natural* base associated with A. When no confusion can arise, we shall write $S^*(f)$ for $S_A^*(f)$. Since most of our topological considerations will take place in X^*, let us agree that the symbols "cl", "int", and "∂", without subscripts, refer to the topology of X^*.

DEFINITION 3.1. Let A be a β-subalgebra of $C(X)$. A point $p \in X^*$ is called an *A-point* of X^* if, for all $f \in A$, $p \notin \partial S_A^*(f)$.

Clearly a point $p \in X^*$ is an A-point if and only if $S^*(f)$ is a neighborhood of p whenever $f \in A$ and $p \in S^*(f)$. The set of A-points is precisely the set $\bigcap_{f \in A} (X^* \setminus \partial S^*(f))$, an intersection of a family of $|A|$ dense open subsets of X^*.

Let us now prove an existence theorem for A-points. A space X is said to have the *G_δ-property* if every nonvoid G_δ-subset of X has a nonvoid interior; equivalently, if every nonvoid zero-set in X has a nonvoid interior ([5], 3.11 (b)). The following analogue of the Baire category theorem is essentially proved in [11], 4.2.

PROPOSITION 3.2. *Let Y be a nonvoid locally compact Hausdorff space with the G_δ-property. If $\mathfrak{D}$ is a family of at most $\aleph_1$ dense open subsets of Y, then $\bigcap \mathfrak{D}$ is dense in Y. If, in addition, Y has no isolated points, then $|\bigcap \mathfrak{D}| \geqslant 2^{\aleph_1}$.*

Proof. We may write $\mathfrak{D} = \{U_\alpha: \alpha < \omega_1\}$. Suppose that G is an arbitrary nonvoid open set in Y; we shall show that $(\bigcap \mathfrak{D}) \cap G \neq \emptyset$. Let $\alpha < \omega_1$, and suppose that there is a collection $\{V_\beta: \beta < \alpha\}$ of nonvoid open sets in G satisfying the three conditions

(a) $\mathrm{cl}_Y V_\beta$ *is compact for* $\beta < \alpha$,

(b) $V_\beta \subset U_\beta$ *for* $\beta < \alpha$, *and*

(c) $\bigcap_{\beta<\alpha} V_\beta \neq \emptyset$.

Now $\bigcap_{\beta<\alpha} V_\beta$ is a G_δ-subset of Y, and therefore has a nonvoid interior which must meet the dense open set U_α. By local compactness, there is a nonvoid open set V_α in Y such that $\mathrm{cl}_Y V_\alpha$ is compact and $\mathrm{cl}_Y V_\alpha \subset U_\alpha \cap (\bigcap_{\beta<\alpha} V_\beta) \subset U_\alpha \cap G$; in fact, if Y has no isolated points, there are two such V_α's with disjoint closures. Thus, $\{V_\alpha: \alpha < \omega_1\}$ is defined inductively in such a way that $\{\mathrm{cl}_Y V_\alpha: \alpha < \omega_1\}$ is a collection of compact subsets with the finite intersection property satisfying $\mathrm{cl}_Y V_\alpha \subset U_\alpha \cap G$ for all $\alpha < \omega_1$. So $(\bigcap \mathfrak{D}) \cap G \supset \bigcap_{\alpha<\omega_1} \mathrm{cl}_Y V_\alpha \neq \emptyset$. If Y has no isolated points, at each stage of the construction, there are two choices of V_α with disjoint closures; hence $|\bigcap \mathfrak{D}| \geqslant 2^{\aleph_1}$.

Let us agree to use the symbol "[CH]" to indicate that we are assuming the continuum hypothesis ($c = \aleph_1$). A space X is said to be *realcompact* if, for every $p \in X^*$, there is a $Z \in Z(\beta X)$ such that $p \in Z \subset X^*$.

THEOREM 3.3. [CH]. *Let X be locally compact and realcompact but not compact. If A is a β-subalgebra of $C(X)$ with $|A| = c$, then X^* has a dense subset of 2^c A-points.*

Proof. Clearly X^* is a nonvoid compact set. In [2], 3.1, it is shown that, if X is locally compact and realcompact, then X^* has the G_δ-property. The realcompactness of X prevents isolated points in X^*. For suppose that p were isolated in X^*. Then there would be a zero-set neighborhood Z_1 of p in βX such that $Z_1 \cap X^* = \{p\}$, and by realcompactness, there would be a $Z_2 \in Z(\beta X)$ such that $p \in Z_2 \subset X^*$. But then we would have $\{p\} = Z_1 \cap Z_2 \in Z(\beta X)$, which by [5], 9.6, would be impossible.

Let $\mathfrak{D} = \{X^* \setminus \partial S^*(f): f \in A\}$, a family of c $(= \aleph_1)$ dense open subsets of X^*. Letting X^* play the role of Y in 3.2, we conclude that $\bigcap \mathfrak{D}$ is a dense subset of X^* with cardinality at least 2^c. But, since A is a β-subalgebra of $C(X)$, $|X^*| \leqslant 2^{|A|} = 2^c$, so that $|\bigcap \mathfrak{D}| = 2^c$. As we have pointed out, $\bigcap \mathfrak{D}$ is the set of A-points of X^*.

Suppose that $\{A_\alpha: \alpha \in \Lambda\}$ is a family of β-subalgebras of $C(X)$. The set of points in X^* that are simultaneously A_α-points for all $\alpha \in \Lambda$ is given by

$$\bigcap_{\alpha \in \Lambda} \bigcap_{f \in A_\alpha} (X^* \setminus \partial S_{A_\alpha}^*(f))$$

An obvious modification of the proof of 3.3 gives the following generalization.

THEOREM 3.4. [CH]. *Let X be locally compact and realcompact but not compact. If $\{A_\alpha: \alpha \in \Lambda\}$ is a family of β-subalgebras of $C(X)$ with $|A_\alpha| = c$ for each $\alpha \in \Lambda$ and with $|\Lambda| \leqslant c$, then X^* has a dense subset of 2^c points which are simultaneously A_α-points for all $\alpha \in \Lambda$.*

If X is separable and A is a β-subalgebra of $C(X)$, then obviously

$|A| = c$. Thus, if X is separable, then the cardinality restrictions on the β-subalgebras in 3.3 and 3.4 are redundant. However, a locally compact, realcompact, and noncompact space X may be nonseparable and still satisfy $|C(X)| = c$. For example, let X be a nonclosed cozero-set in $\mathbf{N}^*$ (such exists by [5], 4K.1).

Since the maximal ideal space of a β-subalgebra is Hausdorff, we can apply many of the results of [4] to β-subalgebras. For example, every prime ideal in a β-subalgebra A is contained in a unique maximal ideal M^p of A ([4], 3.4). Following [4], we may define for a β-subalgebra A of $C(X)$,

$$O_A^p = \{f \in A\colon\ p \in \operatorname{int}_{\beta X} S_A(f)\}\,,$$

where $p \in \beta X$. Clearly O_A^p is an ideal in A contained in M_A^p. We shall often write O^p for O_A^p. By [4], 2.6, each O^p is an intersection of prime ideals in A, and by [4], 3.4, a prime ideal in A is contained in M^p if and only if it contains O^p. Clearly then M^p properly contains some prime ideal in A if and only if $O^p \neq M^p$.

Proposition 3.5. *If A is a β-subalgebra of $C(X)$ and $p \in X^*$, then $M_A^p = O_A^p$ implies that p is an A-point of X^*.*

Proof. Suppose that $M^p = O^p$. If, for $f \in A$, we have $p \in S^*(f)$, then $p \in \operatorname{int}_{\beta X} S(f)$, whence $p \in \operatorname{int} S^*(f)$. Thus, p is an A-point of X^*.

The converse of 3.5 is false. For we know, by 3.3, that [CH] $\mathbf{N}^*$ has a dense subset of 2^c $C^*(\mathbf{N})$-points; however, $M_{C^*}^p = O_{C^*}^p$ is never true for $p \in \mathbf{N}^*$.

4. C^*-points.

We now discuss a simple example of A-points, namely, the C^*-points. A point $p \in X$ is a *P-point* of X if any G_δ-subset (equivalently, any zero-set) of X containing p is a neighborhood of p.

Theorem 4.1. *A point in X^* is a $C^*(X)$-point if and only if it is a P-point of X^*.*

Proof. Evidently, a point in X^* is a P-point of X^* if and only if it is not an element of the X^*-boundary of any zero-set of X^*, and is a $C^*(X)$-point if and only if it is not an element of the X^*-boundary of the trace on X^* of any zero-set of βX. Certainly then, every P-point of X^* is a $C^*(X)$-point.

But the converse holds. For let $p \in \partial Z_1$ where $Z_1 \in Z(X^*)$. There is a G_δ-subset S of βX such that $S \cap X^* = Z_1$. By complete regularity, there exists $Z_2 \in Z(\beta X)$ such that $p \in Z_2 \subset S$. Surely then $p \in \partial(Z_2 \cap X^*)$.

Combining 4.1 and 3.3 gives us the following special case of a well-known result. For an even stronger result, see [5], 9M.3.

Corollary 4.2 (Rudin). [CH]. *Let X be locally compact and realcompact but not compact. If $|C(X)| = c$, then X^* has a dense subset of 2^c P-points.*

5. C-points.

In this section, we shall turn our attention to the C-points of X^*; thus, we shall consider $C(X)$ as a β-subalgebra of itself. We shall relate the concept of C-point with that of remote point, defined by Fine and Gillman.

Proposition 5.1. *If X is completely uniformizable, in particular if X is realcompact or metrizable, then $\operatorname{int} S^*(f) = (\operatorname{int}_{\beta X} S(f)) \cap X^*$ for all $f \in C(X)$.*

Proof. Obviously, $(\operatorname{int}_{\beta X} S(f)) \cap X^* \subset \operatorname{int} S^*(f)$. Let $p \in \operatorname{int} S^*(f)$; then there exists $g \in C$ such that $p \in X^* \setminus S^*(g) \subset S^*(f)$. But then, $g \notin M^p$ and $fg \in C_0 = \bigcap_{q \in X^*} M^q$. In [10] it is shown that, if X is completely uniformizable, then C_0 consists of all $h \in C$ with compact support. Thus, $p \notin \operatorname{cl}_{\beta X} Z(g)$ (see 2.3), and $K = \operatorname{cl}_X \operatorname{Coz}(fg)$ is compact. Hence, $p \in \beta X \setminus (K \cup \operatorname{cl}_{\beta X} Z(g)) \subset \operatorname{cl}_{\beta X} Z(f)$, so that $p \in \operatorname{int}_{\beta X} S(f)$.

Definition 5.2. A point $p \in \beta X$ is called a *remote point* in βX if p is not in the βX-closure of any discrete subset of X.

A remote point in βX necessarily lies in X^*. Following [5], we associate with each maximal ideal M_C^p in $C(X)$ the *z-ultrafilter*

$$A^p = \{Z(f)\colon f \in M_C^p\} = \{Z \in Z(X)\colon\ p \in \operatorname{cl}_{\beta X} Z\} \quad \text{(see 2.3)}\,.$$

Theorem 5.3. *Let $p \in X^*$ where X is a metric space, and consider the following four conditions.*

(a) *p is a C-point of X^*.*

(b) *A^p has no member which is nowhere dense.*

(c) *$M_C^p = O_C^p$.*

(d) *p is a remote point in βX.*

Conditions (a), (b) *and* (c) *are mutually equivalent and are implied by* (d). *All four conditions are equivalent if X has no isolated points.*

Proof. (a) *implies* (b). Suppose that p is a C-point, and let $Z \in A^p$. Then $p \in \operatorname{int}(\operatorname{cl}_{\beta X} Z \setminus X)$, and by Proposition 5.1, $p \in V = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z$. Thus, $\emptyset \neq V \cap X \subset Z$, and Z is not nowhere dense.

(b) *implies* (c). Assume (b), and let $f \in M^p$. Since X is a metric space, we may find $g \in C(X)$ such that $Z(g) = \operatorname{cl}_X \operatorname{Coz}(f)$; hence $X = Z(f) \cup Z(g)$. Now, if $p \in \operatorname{cl}_{\beta X} Z(g)$, then $p \in \operatorname{cl}_{\beta X}(Z(f) \cap Z(g)) = \operatorname{cl}_{\beta X} \partial_X Z(f)$, contradicting our hypothesis, since $\partial_X Z(f)$ is nowhere dense. Thus, $p \in \beta X \setminus \operatorname{cl}_{\beta X} Z(g) \subset \operatorname{cl}_{\beta X} Z(f)$, so that $f \in O^p$.

(c) *implies* (a). This follows from 3.5.

(d) *implies* (b). Suppose that A^p has a nowhere dense member Z. It is shown in [7], p.138 (VIII), that, if Z is a closed nowhere dense set in the metric space X, then there is a discrete subset D of X such that $D \cup Z = \operatorname{cl}_X D$ and $D \cap Z = \emptyset$. Thus $p \in \operatorname{cl}_{\beta X} Z \subset \operatorname{cl}_{\beta X} D$, so that p is not a remote point.

Assume that X has no isolated points; we shall prove that (b) implies (d). Suppose then that p is not a remote point; then there is a discrete subset D of X such that $p \in \mathrm{cl}_{\beta X} D$. Since any point common to D and $\mathrm{int}_X \mathrm{cl}_X D$ would be isolated, one easily sees that $Z = \mathrm{cl}_X D$ is nowhere dense; clearly $Z \in A^p$.

The equivalence of (b) and (d) appears in [3] for $X = \mathbf{R}$; we wish to thank Mark Mandelker for communicating (b) implies (c).

THEOREM 5.4. [CH]. *If X is a separable, locally compact, noncompact metric space without isolated points, then βX has a collection of 2^c remote points which forms a dense subset of X^*.*

Proof. Since X is a separable metric space, it is clear that X is realcompact and $|C(X)| = c$. (In fact, [CH] for a metric space X, the separability of X is equivalent to the condition $|C(X)| = c$.) By 3.3, X^* has a dense subset of 2^c C-points, and by 5.3, the C-points are precisely the remote points in βX.

An obvious corollary to 5.4 is that [CH] $\beta\mathbf{R}$ has a collection of remote points which is dense in $\mathbf{R}^*$. This result was proved by Fine and Gillman in [3] by another method. Our proof appears to be simpler than the Fine–Gillman proof, but their method has wider application; they show that [CH] $\beta\mathbf{Q}$ has remote points, whereas our method fails in this case ($\mathbf{Q}^*$ does not have the G_δ-property). Using the methods of [3], we now extend 5.4 to include the case $X = \mathbf{Q}$ by removing the local compactness from the hypotheses.

THEOREM 5.5. [CH]. *If X is a separable, noncompact metric space without isolated points, then βX has a collection of 2^c remote points which forms a dense subset of X^*.*

Proof. Let V be a closed neighborhood in βX of any point in X^*. Since X is a separable metric space, X is realcompact and has no more than $\aleph_1$ $(= c)$ dense open subsets. By [3], 2.3, there exists a family $\mathcal{F}$ of zero-sets of X such that $\mathcal{F}$ has the finite-intersection property, $\bigcap \mathcal{F} = \emptyset$, and every dense open subset of X contains a member of $\mathcal{F}$. Since X is realcompact, we may construct $\mathcal{F}$ such that each of its members is contained in V (see [3], 2.5). Now let $\Delta = \{p \in \beta X\colon \mathcal{F} \subset A^p\} = \bigcap_{Z \in \mathcal{F}} \mathrm{cl}_{\beta X} Z$, a nonvoid compact subset of $V \cap X^*$. A simple modification of the proof of [3], 2.3, guarantees that Δ is infinite; hence, by [5], 9.11, we have $|\Delta| \geqslant 2^c$. As in the proof of 3.3, $|X^*| \leqslant 2^c$, whence $|\Delta| = 2^c$. Now, for $p \in \Delta$, A^p contains no member which is nowhere dense; each such p is remote by 5.3.

Thus, [CH] $\mathbf{Q}^*$ has C-points but no C^*-points (see [5], 6 O.5). We remark that 5.3 and 5.5 remain true if we assume only that the set of solated points in X has compact closure.

6. Remote points in $\beta\mathbf{R}$ vs. P-points in $\beta\mathbf{R}\backslash\mathbf{R}$.

We now concentrate on the case $X = \mathbf{R}$. Let P denote the set of P-points of $\mathbf{R}^*$, R denote the set of remote points in $\beta\mathbf{R}$, $\widetilde{P} = \mathbf{R}^*\backslash P$, and $\widetilde{R} = \mathbf{R}^*\backslash R$. We shall now show that no inclusions hold between the sets P, R, $\widetilde{P}$ and $\widetilde{R}$. First we prove a preliminary result. We call X an *F-space* if every cozero-set in X is C^*-embedded in X. Every C^*-embedded subset of an F-space is an F-space ([5], 14.26), $\mathbf{N}^*$ and $\mathbf{R}^*$ are compact F-spaces ([5], 14.27), and every countable subset of an F-space is C^*-embedded ([5], 14N.5).

PROPOSITION 6.1. *If X is an infinite compact F-space, then X contains at least 2^c non-P-points.*

Proof. Let X be an infinite compact F-space. Then, by [5], 0.13, X contains a countable discrete set $D = \{p_n\colon n \in \mathbf{N}\}$. As a countable set, D is C^*-embedded in X, whence $\mathrm{cl}_X D = \beta D$ ([5], 6.9(a)). Define $f \in C^*(X)$ by letting $f(p_n) = n^{-1}$ for $n \in \mathbf{N}$ and extending over X. Then, for every $p \in D^* = \mathrm{cl}_X D\backslash D$, $p \in Z(f)$, but $Z(f)$ is not a neighborhood of p. Thus, every one of the 2^c points in D^* is a non-P-point of X.

As a corollary, $\mathbf{N}^*$ and $\mathbf{R}^*$ each have 2^c non-P-points.

THEOREM 6.2. [CH]. *The sets $P \cap R$, $P \cap \widetilde{R}$, $\widetilde{P} \cap R$ and $\widetilde{P} \cap \widetilde{R}$ are each dense subsets of $\mathbf{R}^*$ of cardinal 2^c.*

Proof. ($P \cap R$). Apply 3.4 to the family $\{C(\mathbf{R}), C^*(\mathbf{R})\}$ of β-subalgebras of $C(\mathbf{R})$.

($P \cap \widetilde{R}$ and $\widetilde{P} \cap \widetilde{R}$). Let V be a closed neighborhood in $\beta\mathbf{R}$ of any point in $\mathbf{R}^*$. Then $V \cap \mathbf{R}$ is nonpseudocompact and is C-embedded in $\mathbf{R}$ ([5], 1F.4); hence $V \cap \mathbf{R}$ contains a copy D of $\mathbf{N}$ which is C-embedded in $\mathbf{R}$ ([5], 1.20). Then $D^* = \mathrm{cl}_{\beta\mathbf{R}} D\backslash D \subset V \cap \mathbf{R}^*$, since D is closed and C^*-embedded in $\mathbf{R}$. A point in D^* is a P-point of D^* if and only if it is a P-point of $\mathbf{R}^*$ ([5], 4L.2, 9M.2). But D^* is homeomorphic with $\mathbf{N}^*$, so that D^* has 2^c non-P-points by 6.1 and [CH] 2^c P-points by 4.2. Clearly, no point of D^* is a remote point in $\beta\mathbf{R}$.

($\widetilde{P} \cap R$). Let V be a closed neighborhood in $\beta\mathbf{R}$ of any point in $\mathbf{R}^*$. As in the proof of 5.5, construct an infinite compact set Δ of remote points in $\beta\mathbf{R}$. Since $\mathbf{R}^*$ is an F-space, the C^*-embedded subset Δ is also an F-space. Then, by 6.1, Δ has 2^c non-P-points, and each of these is a non-P-point of $\mathbf{R}^*$. Thus, $V \cap \mathbf{R}^*$ has 2^c points which are non-P-points of $\mathbf{R}^*$ and remote points in $\beta\mathbf{R}$.

7. The algebra H.

In this section, we shall let $C(X)$ denote the algebra (over the complex numbers $\mathbf{C}$) of *complex-valued* continuous functions on X and $C^*(X)$ the subalgebra of bounded functions. A *subalgebra of $C(X)$* will mean a subalgebra in the usual sense which contains the constant functions and which is *self-adjoint* (closed under the formation of complex conjugates). By an *ideal* we shall mean a proper self-adjoint ideal. With these conventions, it is not difficult to see that all the results that we have obtained for subalgebras of $C(X)$ in the real case are true in the complex case as well.

Following R. M. Brooks [1], let us define

$$H = \{f \in C(\mathbf{N}): \limsup_{n\to\infty} \bar{f}(n) \leqslant 1\}$$

where $\bar{f}(n) = |f(n)|^{1/n}$ for $n \in \mathbf{N}$. It is shown in [1] that H is a subalgebra of $C(\mathbf{N})$ containing $C^*(\mathbf{N})$, so by 2.9, H is a u-closed β-subalgebra of $C(\mathbf{N})$. Thus, $\mathcal{M}_H$ is homeomorphic with $\beta\mathbf{N}$ ([1], 2.4).

PROPOSITION 7.1. $H = \{f \in C(\mathbf{N}): \bar{f}^\beta \leqslant 1$ on $\mathbf{N}^*\}$. *A function* $f \in H$ *is a unit of* H *if and only if* $Z(f) = \emptyset$ *and* $\bar{f}^\beta = 1$ *on* $\mathbf{N}^*$.

Proof. The first part follows by observing that $\limsup_{n\to\infty} f(n) = \sup\{f^\beta(p): p \in \mathbf{N}^*\}$ for any real-valued $f \in C^*(\mathbf{N})$. The second part is clear since $\overline{fg}^\beta = \bar{f}^\beta \bar{g}^\beta$ for $f, g \in H$.

Following Brooks, let us define, for $p \in \mathbf{N}^*$, the collection $J^p = \{f \in H: \bar{f}^\beta(p) < 1\}$ of non-units of H.

PROPOSITION 7.2. *For* $p \in \mathbf{N}^*$, J^p *is a prime ideal in* H *contained in* M^p, *whence* $O^p \subset J^p \subset M^p$.

Proof. We first note that $f \in J^p$ implies $f^*(p) = 0$. For suppose that $\bar{f}^\beta(p) < 1$. Then there exists $\delta < 1$ and a neighborhood V of p in $\beta\mathbf{N}$ such that $|f(n)|^{1/n} \leqslant \delta$ whenever $n \in V \cap \mathbf{N}$; that is, $|f(n)| \leqslant \delta^n$ whenever $n \in V \cap \mathbf{N}$. If U is a neighborhood of p in $\beta\mathbf{N}$, then $U \cap V$ contains arbitrarily large $n \in \mathbf{N}$ yielding arbitrarily small positive values of $|f(n)|$; hence $f^*(p) = 0$.

J^p is easily seen to be an ideal (see [1], 2.3.4, 2.3.5) and is clearly prime, since $\overline{fg}^\beta = \bar{f}^\beta \bar{g}^\beta$. Suppose $f \in J^p$, whence $fg \in J^p$ for all $g \in H$; then $(fg)^*(p) = 0$ for all $g \in H$, whereby $f \in M^p$. Since $J^p \subset M^p$, it follows from [4], 3.4, that $O^p \subset J^p$.

By considering H as a topological ring, it is shown in [1], 4.9, that H has at least one nonmaximal prime ideal. We can now improve on this result.

PROPOSITION 7.3. *H has 2^c nonmaximal prime ideals.*

Proof. Since $|H| = c$, H has no more than 2^c nonmaximal prime ideals. By [4], 2.6, 3.4, it suffices to prove that $M^p \neq O^p$ for $p \in \mathbf{N}^*$. Thus, define $f(n) = n^{-n}$ for $n \in \mathbf{N}$, and let $p \in \mathbf{N}^*$ be arbitrary. Since $\bar{f}(n) = n^{-1}$, clearly $f \in J^p \subset M^p$. It is easy to see that $O^p = O^p_C \cap H$. Therefore $f \notin O^p$, since $Z(f) = \emptyset$.

Let us now give a simple characterization of the basic closed set $S^*(f)$ for $f \in H$ (cf. 2.3). First we state a lemma.

LEMMA 7.4. *Let* $p \in \mathbf{N}^*$ *and* $f \in H$. *If* $\bar{f}^\beta = 1$ *on some* $\mathbf{N}^*$*-neighborhood of* p, *then* $f \notin M^p$.

Proof. Suppose that $\bar{f}^\beta = 1$ on some $\mathbf{N}^*$-neighborhood V of p. We may assume that $V = \mathrm{cl}_{\beta\mathbf{N}} E \setminus E$ for some subset E of $\mathbf{N}$ and that $\bar{f}(n) \geqslant \frac{1}{2}$ for $n \in E$. Define $g \in C(\mathbf{N})$ by letting $g(n) = f(n)^{-1}$ for $n \in E$ and $g(n) = 1$ for $n \notin E$. Then $\lim_{n\to\infty} \bar{g}(n) = 1$, so that $g \in H$. Furthermore, $(fg)^*(p) = 1$, so that $f \notin M^p$.

PROPOSITION 7.5. *For* $f \in H$, $S^*(f)$ *is a regular closed subset of* $\mathbf{N}^*$; *moreover*, $S^*(f) = \mathrm{cl}\{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\}$ *and* $\mathrm{int}\, S^*(f) = \{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\}$.

Proof. By 7.2, it is clear that $\mathrm{cl}\{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\} \subset S^*(f)$. Suppose that $p \in S^*(f)$. By 7.4, in every $\mathbf{N}^*$-neighborhood of p, there is a point q such that $\bar{f}^\beta(q) < 1$; that is, $p \in \mathrm{cl}\{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\}$.

By Proposition 7.2, we have $\{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\} \subset \mathrm{int}\, S^*(f)$. Suppose that $p \in \mathrm{int}\, S^*(f)$ and $\bar{f}^\beta(p) = 1$; we shall deduce a contradiction. Let $(n_k)_{k\in\mathbf{N}}$ be an increasing sequence in $\mathbf{N}$ such that $\lim_{k\to\infty} \bar{f}(n_k) = 1$. Letting $E = \{n_k: k \in \mathbf{N}\}$, we may assume that $\mathrm{cl}_{\beta\mathbf{N}} E \setminus E \subset S^*(f)$. Then $\bar{f}^\beta = 1$ on the nonvoid open subset $\mathrm{cl}_{\beta\mathbf{N}} E \setminus E$ of $S^*(f)$, and this contradicts 7.4.

In [1], it is stated that $M^p = J^p$, for all $p \in \mathbf{N}^*$. We now show that this is false; in fact, the equality holds precisely when p is a P-point of $\mathbf{N}^*$.

THEOREM 7.6. *The following are equivalent for a point* $p \in \mathbf{N}^*$.

(a) $J^p = M^p$.

(b) *p is an H-point of* $\mathbf{N}^*$.

(c) *p is a P-point of* $\mathbf{N}^*$.

Proof. (a) *implies* (b). Suppose that $J^p = M^p$. If $p \in S^*(f)$, then $p \in \{q \in \mathbf{N}^*: \bar{f}^\beta(q) < 1\} = \mathrm{int}\, S^*(f)$. Hence, p is an H-point of $\mathbf{N}^*$.

(b) *implies* (c). Let p a non-P-point of $\mathbf{N}^*$, and let $g \in C(\beta\mathbf{N})$ be a real-valued function which is nonconstant on every $\mathbf{N}^*$-neighborhood of p; we may assume that $0 \leqslant g \leqslant 1$ and $g(p) = 1$. Let $f(n) = g(n)^n$ for $n \in \mathbf{N}$; then $\bar{f} = g|\mathbf{N}$, so that $\bar{f}^\beta = g$. Thus $f \in H$, and by 7.5, $p \notin \mathrm{int}\, S^*(f)$. Now, in every $\mathbf{N}^*$-neighborhood of p, there is a point q such that $\bar{f}^\beta(q) < 1$, of f. So $p \in S^*(f)$, by 7.5. Hence, p is not an H-point of $\mathbf{N}^*$.

(c) *implies* (a). Suppose that $f \in M^p$ and $f \notin J^p$. Then $\bar{f}^\beta(p) = 1$, but by 7.4, $\bar{f}^\beta$ is not identically 1 on any $\mathbf{N}^*$-neighborhood of p. Clearly then, p is not a P-point of $\mathbf{N}^*$.

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Fundamental retracts and extensions of fundamental sequences

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In order to extend some standard notions of the homotopy theory onto arbitrary compacta X, Y lying in the Hilbert space H, I introduced in [2] the notion of the *fundamental sequence from X to Y*, defined as an ordered triple $\underline{f} = \{f_k, X, Y\}$ consisting of X, Y and of a sequence $\{f_k\}$ of (continuous) maps of H into itself satisfying the following condition:

For every neighborhood V of Y (neighborhoods are understood here always in the space H) there exists a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \ \text{in } V \quad \text{for almost all } k.$$

The set X will be said to be the *domain*, and the set Y—the *range* of the fundamental sequence $\underline{f}$.

Setting $i_k(x) = x$ for every point $x \in H$, we immediately see that for every compactum $X \subset H$ the triple $\{i_k, X, X\}$ is a fundamental sequence $\underline{i}_X$, called the *fundamental identity sequence for X*.

If c is a point of a compactum $X \subset H$, then setting $c(x) = c$ for every point $x \in H$, we get a fundamental sequence $\underline{c}_X = \{c, X, X\}$ called a *constant fundamental sequence for X*.

Let us observe that if $\hat{X}$ is a closed subset of a compactum $X \subset H$, and Y is a closed subset of a compactum $\hat{Y} \subset H$, and if $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence, then $\underline{\hat{f}} = \{f, \hat{X}, \hat{Y}\}$ is also a fundamental sequence.

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are said to be *homotopic* (in symbols: $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there exists a neighborhood U of X such that

$$f_k/U \simeq g_k/U \ \text{in } V \quad \text{for almost all } k.$$

The fundamental sequences from X to Y may be considered as a generalization of the maps of X into Y, and the classes of all homotopic fundamental sequences from X to Y (called *fundamental classes from X to Y*) may be considered as a generalization of the homotopy classes of maps of X into Y.