Some properties of βX-X for complete spaces *

by

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1. Introduction. In [1] and [2], Fine and Gillman establish several properties for $\beta X - X$ when X is realcompact. In particular, they show that for X realcompact and locally compact every zero set of $\beta X - X$ is the closure of its interior. They prove also that if X is realcompact and H is a subset of $\beta X - X$, then $X \cup H$ is pseudocompact implies that H is dense in $\beta X - X$. If X is assumed to be locally compact as well as realcompact, the converse is also true. We shall establish the validity of these results under a weaker hypothesis than realcompactness; we replace realcompactness with the condition that X be complete in some compatible uniform structure. (Recall that a realcompact space is complete in the structure generated by $C(X)$, whereas the existence of measurable cardinals would imply that a complete space need not be realcompact.) We employ these results to prove that if X is locally compact and complete, $vX - X$ is a nowhere dense subset of $\beta X - X$. We also extend a well-known theorem of Rudin [5]: we show (assuming the continuum hypothesis) that if X is locally compact and complete, $\beta X - X$ contains a dense set of P-points. In the final section, we prove that if X is a locally compact metric space, $\beta X - X$ contains a dense set of remote points. This generalizes results in [2] and [4].

We use the terminology and conventions of [3]. All spaces are assumed to be completely regular and uniform structures are defined by families of pseudo-metrics. We recall that βX denotes the Stone-Čech compactification of X and that vX is the Hewitt-Nachbin realcompactification. We let $X^* = \beta X - X$ and $Z(X)$ denote the collection of zero sets of X. For $f \in C^*(X)$, the ring of bounded and continuous real valued functions on X, we let f^β denote its continuous extension to βX. We recall the familiar facts that if d is a member of a uniformity for X and if S is a d-discrete subset of X, then S is C-embedded in X (all continuous functions in $C(S)$ are extendable to functions in $C(X)$); and if X is a non-compact

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and complete space it admits unbounded continuous functions into the reals, and hence contains a d-discrete infinite set.

2. The main results.

THEOREM 1. *If Y is locally compact and admits a complete uniform structure, then each $Z \in Z(Y^*)$ is the closure of its interior.*

Proof. It suffices to show that each such Z has a non-void interior. For if $p \in Z - \mathrm{cl}_{Y^*}(\mathrm{int}_{Y^*} Z)$, there exists $F \in Z(Y^*)$ such that $p \in F \cap Z$, while $F \cap \mathrm{int}_{Y^*} T = \emptyset$. Thus, $F \cap Z$ is a non-empty zero set with a void interior.

Observe that local compactness of Y implies that Y^* is compact, and hence is C-embedded in βY. Therefore, $Z \in Z(Y^*)$ implies $Z = Z_{f^\beta} - Y$ for some $f \in C(Y)$. Consider Z_f. In case Z_f is not compact, completeness implies that it is not pseudocompact and therefore that it contains a d-discrete copy, N, of the natural numbers. Let $N = \{x_i\colon i \in N\}$, and let $g(x_i) = i$. Extend g continuously to all of Y, and observe that the function $\hat{g}$, defined by

$$\hat{g}(y) = \frac{1}{g^2(y)+1} \quad \text{for} \quad y \in Y\,,$$

is a positive unit of $C(Y)$ and that $g^\beta(p) = 0$ for $p \in \mathrm{cl}_{\beta Y} N - Y$. Thus if $h^\beta = |g^\beta| + |f^\beta|$, we observe that $\emptyset \neq Z_{h^\beta} \subset Z_{f^\beta} - Y = Z$. Finally select for each $i \in N$ a compact neighborhood V_i of x_i satisfying

(i) $V_i \cap V_j = \emptyset$ for $i \neq j$, and

(ii) $|h(y) - h(x_i)| < 1/i$ for $y \in V_i$.

For each i, let $u_i \in C(Y)$ be such that $u_i(x_i) = 1$ and $u_i[Y - V_i] = 0$. Since h is positive on Y, (ii) implies that $\bigcup\{V_i\colon i \in N\}$ is closed. Hence, $u = \sum_{n=1}^{\infty} u_n$ is continuous and vanishes on $Y - \bigcup\{V_i\colon i \in N\}$. If $q \in Y^*$ and if $u^\beta(q) \neq 0$, then $q \in \mathrm{cl}_{\beta Y}[\bigcup\{V_i\colon i \in N\}]$. This implies that $h^\beta(q) = 0$, so that we have $\emptyset \neq Y^* - Z_{u^\beta} \subset Z_{h^\beta} \subset Z$.

In case Z_f is compact, let $p \in Z$ and select $g^\beta \in C(\beta Y)$ such that $g^\beta(p) = 0$ and $g^\beta[Z_f] = 1$. Again we consider the function $h^\beta = |f^\beta| + |g^\beta|$; we select a countable set $S = \{x_n\colon n \in N\} \subset Y - Z_f$ such that $0 < h(x_{n+1}) < h(x_n) < 1/n$, for each n. As before, we select a compact neighborhood V_n for each x_n such that $V_n \cap V_m = \emptyset$ for $n \neq m$, and $|h(y) - h(x_n)| < \frac{1}{2} h(x_n)$ for $y \in V_n$. We now repeat the arguments above to construct a function u which is 0 on $Y - \bigcup\{V_n\colon n \in N\}$ and 1 on S; we note that $\emptyset \neq Y^* - Z_{u^\beta} \subset Z$, and this concludes the proof.

Completeness is not necessary. For a non-trivial example, we observe that the space $Y = \beta R - (\beta N - N)$ is a pseudocompact space in which

$Y^* = N^*$ and so satisfies the result above. We are not able to find any condition weaker than local compactness in the above theorem.

In [4], Plank introduces the notion of A-points and β-subalgebras of $C(X)$. In the presence of the continuum hypothesis, he is able to prove the following theorem:

If X is locally compact and realcompact, and if $\{A_m\colon m \in M\}$ is a family of β-subalgebras, where $\mathrm{card}(A_m) \leqslant c$ and $\mathrm{card}(M) \leqslant c$, then X^* contains a dense set of 2^c points which are simultaneously A_m-points for all $m \in M$.

If one notes that if X is complete, then X^* has no isolated points and employs the preceeding theorem in place of the Fine-Gillman result, a repetition of the remaining arguments of Plank will yield the same result for spaces that are complete and locally compact.

THEOREM 2. *Let S admit a complete uniform structure, let $H \subset S^*$, and let $X = S \cup H$. If X is pseudocompact, then H is dense in S^*. Conversely, if H is dense in S^* and S is locally compact as well as complete, then X is pseudocompact.*

Proof. We establish the second result first. If X is not pseudocompact, there exists $f^\beta \in C(\beta X)$ such that $\emptyset \neq Z_{f^\beta} \subset X^* \subset S^*$. According to theorem 1, $Z_{f^\beta} = \mathrm{cl}_{S^*}[\mathrm{int}_{S^*} Z_{f^\beta}]$, and this implies that $\mathrm{int}_{\overline{S^*}} Z_{f^\beta}$ is a non-empty open subset of S^* that is disjoint from H.

To prove the first statement, assume that H is not dense in S^*. Let $F_\beta \in Z(\beta X)$ such that $S^* - F_\beta \neq \emptyset$ and let $H \cap (S^* - F_\beta) = \emptyset$. Let $p \in S^* - F_\beta$ and let $Z_{f^\beta} \in Z(\beta X)$ be a zero set neighborhood of p in X that is disjoint from F_β.

Let $Z = Z_{f^\beta} \cap S$. Since $p \in \mathrm{cl}_{\beta S} Z$, Z is not compact; hence completeness implies it contains a d-discrete copy of N, N, that is C-embedded in S. Furthermore, since N and H are contained in disjoint zero sets of X, they are completely separated in that space. But this information implies that N is C-embedded in X, i.e., X is not pseudocompact.

COROLLARY. *If X admits a complete uniform structure, $\nu X - X$ has a void interior in X^*.*

Proof. Since $X \cup (\beta X - \nu X)$ is pseudocompact, the conclusion follows from the theorem above.

THEOREM 3. *If X is locally compact and complete, $\nu X - X$ is nowhere dense in X^*.*

Proof. Let $0 = \bigcup\{\mathrm{int}_{X^*} Z_{f^\beta}\colon Z_{f^\beta} \in Z(\beta X) \text{ and } Z_{f^\beta} \subset X^*\}$. By Theorem 1, we have that

$$\mathrm{cl}_{X^*}\dot{0} \supset \bigcup\{Z_{f^\beta}\colon Z_{f^\beta} \in Z(\beta X) \text{ and } Z_{f^\beta} \subset X^*\} \supset \beta X - \nu X\,.$$

By the preceeding corollary, we have that $\mathrm{cl}_{X^*} 0 = X^*$. Since $0 \cap \nu X = \emptyset$, we conclude that $\nu X - X$ is nowhere dense.

Local compactness, or something like it, is necessary here. For if X_n is a discrete set of measurable cardinal for each $n \in N$, we let $X = \Pi\{X_n: n \in N\}$. Every closed neighborhood V in βX contains a C-embedded copy of some X_n; hence $\beta X_n \subset V$. In particular, $\nu X_n - X_n \subset V \cap X^*$, and since $\nu X_n - X_n \subset \nu X - X$, we have that $\mathrm{cl}_{X^*}(\nu X - X) = X^*$.

LEMMA. *If X admits a complete uniform structure and $p \in X^*$, then every neighborhood of p in X^* contains a copy of $\beta N - N$.*

Proof. Let 0 be a neighborhood of p in X^* and let $H = X^* - 0$. Since H is not dense, the space $Y = X \cup H$ is not pseudocompact. Therefore, Y contains a C-embedded copy, $\boldsymbol{N}$, of the natural numbers, where $\boldsymbol{N}$ is contained in X, a dense subset of Y. Thus, $\beta\boldsymbol{N} - \boldsymbol{N} \subset Y^* \subset 0$.

THEOREM 4. *In the presence of the continuum hypothesis, if X is locally compact and complete, then X^* contains a dense set of P-points.*

Proof. It is well known ([3], p. 138) that if T is locally compact and contains a C-embedded copy, $\boldsymbol{N}$, of the natural numbers, then every P-point of N^* is a P-point of T^*. Rudin [5] has shown (using the continuum hypothesis) that N^* contains a dense set of 2^c P-points. Thus, we may combine these facts with the preceeding lemma to achieve the desired conclusion.

3. Applications to metric spaces. Let M be a metric space without isolated points. Let G denote the set of remote points of βM, i.e., those points of M^* not in the closure of any discrete subset of M; and let $D = M^* - G$. In case $M = R$, the real line, Fine and Gillman [2] show that both D and G are dense subsets of R^*. Their proof of the non-emptiness of G requires the continuum hypothesis. In [4] Plank gives several interesting characterizations of remote points in βM, for arbitrary M and (assuming the continuum hypothesis) proves:

If M is a separable metric space, G is a dense subset of M^* and $\mathrm{card}(G) = 2^c$.

Since M contains no copy of the natural numbers C-embedded in $M \cup D$, it follows that $M \cup D$ is pseudocompact, i.e., D is dense in M^*. (Recall that every metric space is complete in some compatible uniform structure.) We shall now consider the situation with regard to G; it is more complicated and we must employ the continuum hypothesis and assume that M is locally compact as well.

LEMMA. *Let M be a metric space and 0 a non-empty open subset of M. If $S = \mathrm{cl}_M O$, then every remote point of βS $(= \mathrm{cl}_{\beta M} S)$ is a remote point of βM.*

Proof. It is known ([4], theorem 5.3) that if M is a metric space without isolated points, p is a remote point of βM if and only if $p \notin \mathrm{cl}_{\beta M} H$, for any closed nowhere dense subset $H \subset M$. We shall assume p is a remote point of βS, while $p \in \mathrm{cl}_{\beta M} H$ for some closed nowhere dense $H \subset M$, and proceed to reach a contradiction. Since S is the closure of an open set, $H \cap S$ is a nowhere dense closed subset of S. Therefore, our assumption implies that $p \notin \mathrm{cl}_{\beta M}(H \cap S)$. Since p is in the closure of H, we must have $p \in \mathrm{cl}_{\beta M}(H - \mathrm{int}_M S)$. Since S and $(H - \mathrm{int}_M S)$ are zero sets of M and since p is in the closure of each, it follows that $p \in \mathrm{cl}_{\beta M}[S \cap (H - \mathrm{int}_M S)]$. But $S \cap (H - \mathrm{int}_M S) \subset S - \mathrm{int}_M S$, a nowhere dense closed subset of S. This contradicts our assumption that p is a remote point of βS.

THEOREM 5. *Assuming the continuum hypothesis, every locally compact metric space without isolated points contains a set of remote points dense in M^*.*

Proof. Let V be a closed subset of βM such that $\mathrm{int}_{M^*}(V \cap M^*) \neq \emptyset$ Since $\nu M \cap M^*$ is nowhere dense in M^*, there exists $q \in \mathrm{int}_{\beta M} V$ with $q \notin \nu M$. Let $g^\beta \in C(\beta M)$ such that $g^\beta(q) = 1$ and $g^\beta[(\beta M - V)] = 0$. Let $f \in C(M)$ such that f is unbounded on every neighborhood of q and let $h = |f \cdot g|$. Thus, h is unbounded on V, so we may select a sequence $\{x_n: n \in N\} \subset V$ such that $h(x_{n+1}) > h(x_n) > n$ for each n. Now let $I_n = [h(x_n) - r_n, h(x_n) + r_n]$, where r_n is a positive number chosen so that $I_n \cap I_m = \emptyset$ for $n \neq m$. For each n, let O_n be an open set such that $x_n \in O_n \subset h^{-1}[I_n]$ and $\mathrm{cl}_M O_n$ is compact and a subset of V. The set $S = \bigcup\{\mathrm{cl}_M O_n: n \in N\}$ is σ-compact and is equal to $\mathrm{cl}_M \bigcup \{O_n: n \in N\}$. (For if $p \in \mathrm{cl}_{\beta M} \bigcup \{O_n: n \in N\}$, and if $p \notin S$, then every neighborhood of p must meet infinitely many of the sets O_n. But this implies that h is unbounded on every neighborhood of p, and hence that $p \in M^*$.)

Since S is a σ-compact metric space, it is separable. Therefore, the above quoted theorem of Plank implies the existence of remote points of $\beta S = \mathrm{cl}_{\beta M} S$, and since V is closed, these remote points are contained in V. Our previous lemma yields the conclusion that these remote points of βS are also remote points of βM.

In the above theorem, we did not use the full force of local compactness: something like local separability would have done the trick. Similarly, we could replace metrizability with the requirement that M be complete and perfectly normal. We have not been able to determine whether perfect normality can be replaced by normality.

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Fredholm σ-proper maps of Banach spaces

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Introduction. The theory of framed cobordism was introduced by L. Pontrjagin in order to study homotopy groups of spheres. Pontrjagin has shown in [9] that the problem of homotopy classification of continuous maps of S^m into S^n is equivalent to the problem of cobordism classification of $(m-n)$-dimensional framed submanifolds of S^m. Afterwards it has turned out to be easier to solve this homotopy classification problem by quite different methods. But it also turned out that Pontrjagin's methods allows to translate some problems in differential topology to homotopy theory.

Using the idea of Pontrjagin, S. Smale has suggested the following notion of degree for certain maps of differential Banach manifolds. Let X and Y be connected C^p Banach manifolds and $f\colon X \to Y$ a proper Fredholm C^p map of index n, with $p > n+1$. It follows from Smale's version of the Sard Theorem [10] that except for a set of the first category all points of Y are regular values of f. If y is a regular value of f, then $f^{-1}(y)$ is a C^p compact n-dimensional submanifold of X or is empty. Moreover, it is shown in [10] that if y_0 and y_1 are regular values of f, then $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are cobordant as unoriented n-dimensional manifolds. Thus there is defined an element $\gamma(f)$ (generalized degree mod 2 of f) of the unoriented bordism group $\mathfrak{R}(X)$.

The purpose of this paper is to find a link between the invariant $\gamma(f)$ and the homotopy theory of so-called *compact fields* ([4], [5]). Instead of X and Y we consider two infinite dimensional Banach spaces E and F. We assume that there is given a subset Γ of the set $\Phi(E, F)$ of all Fredholm operators from E to F, satisfying certain conditions (see Section E). As an example of such a Γ we can take $V =$ a convex subset of $\Phi(E, F)$ and let $\Gamma = \{A \in \Phi(E, F);\ A = B + C,\ B \in V,\ C \text{ is compact}\}$.

Let U be an open subset of E. An n-dimensional C^p *Γ-framed submanifold* of U is a pair (M, φ) where M is an n-dimensional C^p submanifold of U and $\varphi\colon M \to \Gamma$ is a continuous map such that $\operatorname{Ker}\varphi(x) =$ the subspace tangent to M at x, for all $x \in M$. In the set of all Γ-framed compact submanifolds of U, there is a natural cobordism relation which we call *ω-cobordism*.