$(T_a \cap P - K_a)$. Let S_0 be an uncountable Lusin subset of T_0. Finally, let $E = \bigcup S_a$. Then, since $S_a \subset E \cap T_a$, $E \notin \mathcal{K}$. Moreover, recalling that a countable union of Lusin sets is an element of $\mathcal{C}$ and, a fortiori, of $\mathcal{N}$, it follows that for $a > 0$,

$$\mu_a(E) \leqslant \mu_a(\bigcup_{\beta<a} S_\beta) + \mu_a(I - K_a) = 0 .$$

Therefore, assuming the continuum hypothesis, if $\mathcal{K} = \mathcal{N}$, then $\mu_0^*(E) > 0$.

References

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Novak's result by Henkin's metod

by

H. C. Doets* (Bussum)

1. In [1], Novak proved among other things that, if Zermelo-Fraenkel set theory (ZF) is consistent, so is von Neumann–Bernays–Gödel set theory (NBG). Mostowski extended the result (see [2]) by noting that any theorem of NBG which speaks about sets only can already be derived in ZF. By making use of the method of Henkin's proof of the completeness theorem for first order theories [3] we show how a very simple proof of the above-stated fact may be obtained. Essentially, this is done as in [1] by showing that, assuming the consistency of a ZF-like theory, a model may be obtained for a related NBG-like theory.

However, professor Mostowski notified me of the fact that J. R. Shoenfield's proof of the theorem (JSL 19 (1954), pp. 21-28) remains the best result by showing that a primitive recursive function exists yielding proofs in ZF from proofs in NBG for ZF-sentences, while from our proof (as well as from Novak's and Rosser–Wang's, JSL 15 (1950), pp. 113-129) there results a *general* recursive function only (cf. Shoenfield's introduction to his paper).

2. Our symbolism will be one of the usual kinds and accordingly will not be explained. Free variables are indicated between brackets as usual; the same for substitution of terms in formulas; it is assumed that the necessary changes always are made to avoid clash of variables. Semantical notions like satisfaction (a finite sequence of a model M may satisfy a formula without indicating the relation between objects and variables too precisely) and (M-) truth are assumed to be known but use will be made of very elementary properties of these notions only (as in [3]). For NBG, refer to [4]. We make however the following slight change: let $S(x)$ be the formula $\bigvee y(x \in y)$; erase axioms A1 and A2; rewrite all axioms in one kind of variables relativizing former set variables to S. We denote axioms of the new system by the names of the corresponding axioms in the old system.

* Institute for Foundational Research, Amsterdam

3. THEOREM. *Let T be a (deductively) consistent set of sentences in our object language, including the* ZF *axiom of general extensionality (which permits no individuals). Then a model M exists for which the following holds:*

(i) A3 *of* [4] *is M-true and if T contains the* ZF *pairing axiom, then also* A4, B1-B8 *of* [4] *are M-true.*

(ii) *The relativization to S of any sentence of T is M-true (taking care of* Cl-3 [4]).

(iii) *If T contains every instance of the* ZF *replacement schema, then* C4 *of* [4] *(the* NBG *class-form of the replacement axiom) is M-true if* A4 *is.*

Remark. Actually, we shall considerably generalize part (iii) of the theorem to arbitrary sentence-schemas.

Proof of theorem. First we will define the Henkin-model. According to Henkin–Hasenjaeger, we may extend T in a language which has an additional denumerable set of individual constants C to a maximal consistent set of sentences T^* with the property that, if some existentially quantified formula $\bigvee x\varphi(x)$ is in T^*, there is an individual constant c in C such that $\varphi(c)$ also belongs to T^*. Let F be the set of formulas in the extended object language in one free variable. Define $M = \langle F, \varepsilon, \approx\rangle$, where:

1. $\varphi \approx \psi$ if and only if $\bigwedge x(\varphi(x) \leftrightarrow \psi(x))$ is in T^*,

2. $\varphi\varepsilon\psi$ if and only if c in C exists such that $\bigwedge x(\varphi(x) \leftrightarrow x \in c)$ and $\psi(c)$ are in T^*,

3. an individual constant c from C denotes the formula $(x \in c)$ in the model; $\in$, $\equiv$ are taken as denoting resp. ε and $\approx$.

A series of lemmas will show that M satisfies the theorem. The first one assures that the identity-relation of the model is a good one. Its proof is trivial from the definitions and the fact that T^* is deductively closed:

LEMMA 1. $\approx$ *is a congruence relation in the relational system M.*

LEMMA 2. A3, *the* NBG *extensionality axiom, is valid in M.*

Proof. (1) If not $\varphi \approx \psi$, then, by definition and properties of T^*, it follows that for some c in C: $\neg(\varphi(c) \leftrightarrow \psi(c))$ is in T^*. As a result, we may easily derive that $\langle (x \in c), \varphi, \psi\rangle$ satisfies $S(y) \wedge \neg(y \in z \leftrightarrow y \in w)$ in M.

Before we proceed, we have the

DEFINITION. For any formula φ among whose free variables are $x_1, \ldots, x_n$, occurring at the right-hand side of the epsilon-sign only, and any n formulas $\psi_1, \ldots, \psi_n$ from F, $\varphi(\psi_1, \ldots, \psi_n)$ will be the formula obtained

(1) Added in proof. Here and in the sequel, the following two trivial properties may be of help: (i) φ satisfies S in M iff for some c in C, $\varphi \approx (x \in c)$; (ii) $(x \in c)\varepsilon\varphi$ iff $\varphi(c)$ is in T^*.

from φ by replacing every prime formula $(\sigma \in x_i)$ in which x_i occurs by the formula $\psi_i(\sigma)$ $(i = 1, \ldots, n)$.

The following lemma is a slight but potent generalization of Henkin's result from which all other lemmas needed merely follow as corollaries:

LEMMA 3. *Let $\varphi, \psi_1, \ldots, \psi_n$ be as in the definition with the addition that φ contains no more free variables than $x_1, \ldots, x_n$. For $\varphi(\psi_1, \ldots, \psi_n)$ to be in T^* it is necessary and sufficient that $\langle \psi_1, \ldots, \psi_n\rangle$ in M satisfies $\varphi^{(S)}$, the relativization of φ to S.*

Proof. As Henkin, we proceed by structural induction on φ. For obvious reasons we do not treat universal quantification. We distinguish the following cases:

(i) φ is a prime formula $(a \equiv b)$ for some a, b in C. The following assertions are equivalent: $(a \equiv b)$ is in T^*; $\bigwedge x(x \in a \leftrightarrow x \in b)$ is in T^*; $(x \in a) \approx (x \in b)$; $(a \equiv b)$ is M-true.

(ii) φ is a prime formula $(a \in b)$ for some a and b in C. Equivalent are: $(a \in b)$ is in T^*; $\bigwedge x(x \in a \leftrightarrow x \in a)$, $(a \in b)$ are in T^*; $(x \in a)\varepsilon(x \in b)$; $(a \in b)$ is M-true.

(iii) φ is a prime formula $(a \in x_i)$ for some a in C and $i \leqslant n$. Then $\varphi(\psi_1, \ldots, \psi_n)$ is $\psi_i(a)$. Equivalent are: $\psi_i(a)$ is in T^*; $\bigwedge y(y \in a \leftrightarrow y \in a)$, $\psi_i(a)$ are in T^*; $(y \in a)\ \varepsilon\psi_i$; ψ_i satisfies $(a \in x_i)$ in M.

(iv) φ is a negation or a conjunction (etc.); this is trivial.

(v) φ is $\bigvee y\psi(y)$ for some variable y and formula ψ.

Necessity. If $\varphi(\psi_1, \ldots, \psi_n)$ is in T^* then, for some c in C, $\psi(c)(\psi_1, \ldots, \psi_n)$ is in T^*. Thus, by the inductive hypothesis, $\langle \psi_1, \ldots, \psi_n\rangle$ satisfies $\psi^{(S)}(c)$ and by an elementary rule, $\langle (z \in c), \psi_1, \ldots, \psi_n\rangle$ satisfies $\psi^{(S)}(x_0)$. ([1].)

Moreover, since $\langle (z \in c), (z \equiv c)\rangle$ satisfies $(x \in y)$, we may conclude that $(z \in c)$ satisfies S. Because of this fact and [1], $\langle (z \in c), \psi_1, \ldots, \psi_n\rangle$ satisfies $S(x_0) \wedge \psi^{(S)}(x_0)$, that is, $\langle \psi_1, \ldots, \psi_n\rangle$ satisfies $\varphi^{(S)}$.

Sufficiency. Let $\langle \psi_1, \ldots, \psi_n\rangle$ satisfy $\varphi^{(S)}$, that is, for some η in F, $\langle \eta, \psi_1, \ldots, \psi_n\rangle$ satisfies $S(x_0) \wedge \psi^{(S)}(x_0)$. Because η satisfies the left-hand side of the conjunction, it is easy to show that we may take η to be of the form $(z \in c)$ for some variable z and c in C. By the inverse of the same elementary rule as before, we conclude that $\langle \psi_1, \ldots, \psi_n\rangle$ satisfies $\psi^{(S)}(c)$, thus, by the inductive hypothesis, $\psi(c)(\psi_1, \ldots, \psi_n)$ is in T^*. It follows that $\varphi(\psi_1, \ldots, \psi_n)$ is in T^*.

As said, rest of the proof is easy-going; it is immediate from Lemma 3 that:

LEMMA 4. *For any sentence in T, the relativization of the sentence to S is M-true. In particular* A4 *(the* NBG *pairing axiom) is M-true if the* ZF *pairing axiom belongs to T.*

Lemma 5. *Let φ be an arbitrary formula in the free variables $y, x_1, \ldots, x_n$. Then the sentence*

$$(\divideontimes) \qquad \wedge x_1 \ldots \wedge x_n \vee x \wedge y\big(S(y) \to (y \in x \leftrightarrow \varphi^{(S)}(y, x_1, \ldots, x_n))\big)$$

is M-true.

Remarks

1. It is well known that, as a consequence, B1-B8 of [4] are M-true if A4 is.

2. In [1], it is stated (p. 90) that B1-B8 follow from ($\divideontimes$) with the restriction that φ contains the free variable y *only*. This is a mistake.

Proof. First we note (cf. [4], bottom p. 9, top p. 10) that by the logically true formula

$$x_i \in z \leftrightarrow \vee y\big(S(y) \wedge x_i \equiv y \wedge y \in z\big)$$

and the M-true formula

$$x_i \equiv z \leftrightarrow \wedge z\big(S(z) \to (z \in x_i \leftrightarrow z \in y)\big)$$

we may assume that φ satisfies the restrictions of the definition stated just before Lemma 3. In this way, the truth of ($\divideontimes$) is easily established: Select $\psi_1, \ldots, \psi_n$ from F. We want to show that $\varphi(y)(\psi_1, \ldots, \psi_n)$ does the trick. For let η be any "set" from the model. As before, we may assume that η is $(z \in c)$ for some c in C. Then the following assertions are equivalent:

$$(z \in c)\, \varepsilon \varphi(y)(\psi_1, \ldots, \psi_n); \quad \varphi(c)(\psi_1, \ldots, \psi_n) \text{ is in } T^*;$$

$$\langle\psi_1, \ldots, \psi_n\rangle \text{ satisfies } \varphi^{(S)}(c) \text{ in } M; \quad \langle\eta, \psi_1, \ldots, \psi_n\rangle \text{ satisfies } \varphi^{(S)} \text{ in } M.$$

Lemma 6. (i) *If T contains every instance of the Zermelo subset schema, then*

$$\wedge z\Big(S(z) \to \vee x\big(S(x) \wedge \wedge y\,(S(y) \to (y \in x \leftrightarrow y \in z \wedge y \in w))\big)\Big)$$

is M-true.

(ii) *If T contains the* ZF *pairing axiom and every instance of the Fraenkel replacement schema, then* C4 *of* [4] *is M-true.*

(iii) *Analogously for other sentence-schemas.*

Proof. We shall work out the procedure for C4 only; it will be seen that, assuming the pairing axiom to be in T, the lemma is valid for arbitrary possible sentence schemas. In the replacement schema, the formula-parameter 'φ' functions as a "binary relation": $\varphi(x, y)$. By means of the pairing axiom (from which we can develop the theory of ordered pairs, cf. [4], pp. 3-4) we may equivalently work with an "unary relation", that is, we may replace every instance of '$\varphi(x, y)$' in the replacement schema by '$\vee z(\varphi(z) \wedge z = \langle x, y\rangle)$'. In this new replacement schema, replace every occurrence of '$\varphi(z)$' by '$z \in w$' where w is supposed to be a new variable. Let the resulting formula be $\psi(w)$.

Now, C4 is trivially equivalent in M to the sentence $\wedge w \psi^{(S)}(w)$ which we shall prove to be valid in M.

For w, select some η from F. We have to show that η satisfies $\psi^{(S)}(w)$. Now consider $\psi(\eta)$. This is exactly an instance of the revised replacement schema and so it is in T^*. By Lemma 3, η *does* satisfy $\psi^{(S)}(w)$ in M. This completes the proof of the theorem.

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I did not refer to but hereby acknowledge my indebtness to:

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