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Some remarks on Hausdorff measure

by

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Let us begin with some notation and terminology. Denote by $\mathcal{F}$ the class of non-decreasing functions h on $(0, \infty)$ with $\lim\limits_{\varepsilon\to 0} h(\varepsilon) = 0$. If $h \in \mathcal{F}$ and $E \subset I = [0,1]$, then the *h-Hausdorff outer measure $m_h(E)$ of E is the extended real number*

$$\liminf_{\varepsilon\to 0}\Big\{\sum h(b_i - a_i);\ E \subset \bigcup (a_i, b_i),\ \sup(b_i - a_i) < \varepsilon\Big\}.$$

Denote by $\mathcal{K}$ the collection of subsets E of I such that $m_h(E) = 0$ for all $h \in \mathcal{F}$. Denote by $\mathcal{T}$ the collection of regular non-atomic probability measures μ on the Borel subsets $\mathcal{B}$ of I, and denote by $\mathcal{N}$ the collection of subsets E of I satisfying $\sup\{\mu^*(E);\ \mu \in \mathcal{T}\} = 0$. Denote by $\mathcal{C}$ the set of concentrated subsets of I (i.e., $E \in \mathcal{C} \Longleftrightarrow$ there is a sequence $\{x_i\}$ of elements of I such that if $\{\varepsilon_i\}$ is a sequence of positive numbers, then $E - \bigcup N(x_i, \varepsilon_i)$ is, at most, a countable set, where $N(x. \varepsilon) = (x - \varepsilon/2, x + \varepsilon/2)$). Finally, denote by $\mathcal{D}$ the collection of enumerations $\{x_i\}$ of countable, dense subsets of I and by $\mathcal{E}$ the collection of sequences of positive numbers.

It is easy to show that $\mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{N}$, and the author showed [1] that if the continuum hypothesis is satisfied, then $\mathcal{C} \neq \mathcal{N}$. The purpose of this note is to show, assuming the continuum hypothesis, that $\mathcal{C} \neq \mathcal{K}$. To this end, let us begin by giving the following characterizations of the elements of $\mathcal{K}$.

LEMMA 1. *Each of the following conditions is necessary and sufficient in order that a subset E of I be an element of $\mathcal{K}$.*

(i) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is a sequence $\{x_i\}$ of points of I such that $E - \bigcup N(x_i, \varepsilon_i)$ is countable.*

(ii) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is $\{x_i\} \in \mathcal{D}$ such that $E - \bigcup N(x_i, \varepsilon_i)$ is countable.*

(iii) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is a sequence $\{x_i\}$ of points of I such that $E \subset \bigcup N(x_i, \varepsilon_i)$.*

(iv) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is $\{x_i\} \in \mathcal{D}$ such that $E \subset \bigcup N(x_i, \varepsilon_i)$.*

Proof. One checks without difficulty that conditions (i)-(iv) are equivalent and that they imply $E \in \mathcal{K}$. Suppose that (iii) is not satisfied. Under this supposition there exists a monotonic sequence $\{\varepsilon_i\} \in \mathcal{E}$ such that (1) $\lim \varepsilon_i = 0$ and (2) if $\{x_i\} \subset I$, then $E \not\subset \bigcup N(x_i, \varepsilon_i)$. Let $h \in \mathcal{F}$ be defined as follows: $h(\varepsilon_{2^n-1}) = 2^{-(n-1)}$, $h(x) = 1$ if $x > \varepsilon_1$, and h is linear on $[\varepsilon_{2^{n+1}-1}, \varepsilon_{2^n-1}]$. Suppose that $E \subset \bigcup (a_i, b_i)$ where the sequence $\{b_i - a_i\}$ is non-increasing. Then there are a poistive integer j and a positive integer n satisfying $\varepsilon_i < b_i - a_i$ and $2^{n+1}-1 > i \geqslant 2^n - 1$. Thus

$$b_j - a_j \geqslant \varepsilon_{2^{n+1}-1} \quad \text{for} \quad j \leqslant 2^n - 1$$

and, hence,

$$\sum h(b_j - a_j) \geqslant (2^n-1) h(\varepsilon_{2^{n+1}-1}) = (2^n - 1) 2^{-n} > 2^{-1}$$

which implies that $m_h(E) > 0$.

Example 1. Assuming the continuum hypothesis, let $\{x^a\}$ and $\{\varepsilon^a\}$ be well orderings of $\mathfrak{D}$ and $\mathcal{E}$ such that each a has countably many predecessors. Let $y = \{y_i\} \in \mathfrak{D}$, let $U_a = \bigcup N(y_i, \varepsilon_i^a)$, and let $W_a = \bigcap_{\beta \leqslant a} U_\beta$. Then W_a is the complement in I of a first category F_σ subset of I. In what follows we shall use the fact that an uncountable Borel set B contains a perfect nowhere dense set P which in turn supports an element μ of $\mathfrak{T}$ (i.e., $\mu(P) = 1$). We shall also use the fact (cf. [1]), assuming the continuum hypothesis, that any uncountable closed subset Q of I contains an uncountable Lusin subset S (i.e., if $\{x_i\}$ is a sequence of points of Q which is dense in Q and $\{\varepsilon_i\} \in \mathcal{E}$, then $S - \bigcup N(x_i, \varepsilon_i)$ is countable). Let $\mu_0 \in \mathfrak{T}$ with support P_0, a nowhere dense subset of U_0. Let $\delta^0 \in \mathcal{E}$ satisfy $\sum \mu_0(N(x_i^0, \delta_i^0)) < 1$. Let S_0 be an uncountable Lusin subset of a perfect nowhere dense subset Q_0 of $P_0 - \bigcup_i N(x_i^0, \delta_i^0)$. Suppose that $\mu_\beta, P_\beta, \delta^\beta, Q_\beta$, and S_β have been defined for $\beta < a$. Then $X_a = \bigcup_{\beta < a} (P_\beta \cup Q_\beta)$ is a first category F_σ subset of I and, hence, $W_a - X_a$ contains a nowhere dense perfect set P_a which supports $\mu_a \in \mathfrak{T}$. Let δ^a satisfy $\mu_a(\bigcup_i N(x_i^a, \delta_i^a)) < 1$. Then let S_a be an uncountable Lusin subset of a perfect nowhere dense subset Q_a of $P_a - \bigcup_i N(x_i^a, \delta_i^a)$. Let $E = \bigcup_a S^a$, let $\{\varepsilon_i\} \in \mathcal{E}$, and let $\{x_i\} \in \mathfrak{D}$. Then there exist indices a and β such that $\{\varepsilon_{2i}\} = \varepsilon^a$ and $\{x_i\} = x^\beta$. Since S_β is an uncountable subset of $E - \bigcup_i N(x_i^\beta, \delta_i^\beta)$, $E \notin \mathcal{C}$. Moreover, $\bigcup_{\gamma \leqslant a} S_\gamma \subset \bigcup N(y_i, \varepsilon_{2i})$. Because a countable union $T = \bigcup_{\gamma < a} S_\gamma$ of Lusin subsets is an element of $\mathcal{C}$, there exists a sequence $\{z_i\}$ of points of I such that $T_a \subset \bigcup_i N(z_i, \varepsilon_{2i-1})$. Let $w_{2i} = y_i$ and $w_{2i-1} = z_i$ to obtain $E \subset \bigcup N(w_i, \varepsilon_i)$: $E \in \mathcal{K}$.

Although a determination of whether $\mathcal{K} = \mathcal{N}$ seems to be elusive, an example is given below to show that if $\mathcal{K} = \mathcal{N}$ and the continuum hypothesis is satisfied, then there is a subset E of I such that E is not measurable with respect to Lebesgue measure $m(0 = m_*(E) < m^*(E))$ but satisfies $\mu(E) = 0$ for every element μ of $\mathfrak{T}$ that is singular with respect to Lebesgue measure (i.e., $\mu \wedge m = 0$). In the course of constructing the example, it will be convenient to have the following lemma.

Lemma 2. *Suppose that $\nu \in \mathfrak{T}$ and that $\{X_i\}$ is a sequence of elements of $\mathfrak{B}$. Then there is a first category subset F of I satisfying*

(1) *$\nu(F) = 1$, and*

(2) *$F \cap X_i$ is a first category subset of X_i, $i \geqslant 1$.*

Proof. Let p be a positive integer, and let A_i and B_i be closed and nowhere dense subsets of X_i and $I - X_i$ satisfying $\nu(A_i \cup B_i) > 1 - (p2^i)^{-1}$. Let $F_p = \bigcap (A_i \cup B_i)$. Then F_p is nowhere dense in $I, X_1, X_2, \ldots$ and

$$\nu(I - F_p) = \nu\Big(U\big(I - (A_i \cup B_i)\big)\Big) \leqslant \Sigma \nu\big(I - (A_i \cup B_i)\big) < p^{-1}.$$

It suffices to let $F = \bigcup F_p$.

Example 2. Let $\{\varepsilon_i\} \in E$ such that $\sum \varepsilon_i < 2^{-1}$. Let P be a nowhere dense perfect subset of I satisfying $m(P) > 2^{-1}$. Let $\mathfrak{T}_1$ be a maximal collection of mutually singular elements of $\mathfrak{T}$ with $m \in \mathfrak{T}_1$. Suppose that $\{\mu_a\}$ is a well ordering of $\mathfrak{T}_1$ such that $\mu_0 = m$ and each a has countable many predecessors. If $\mu \in \mathfrak{T}$, then $\mu = \sum_{\nu \in \mathfrak{T}_1} \mu \wedge \nu$: if $\mu_a(E) = 0$ for $a > 0$, then $\mu(E) = 0$ for $\mu \wedge m = 0$.

If $a > 0$ and $\{\nu_i^a\}^{n_a}$ is an enumeration of $\{\mu_\beta\}_{\beta < a}$ where n_a is a positive integer or "infinity", then μ_a is singular with respect to

$$\nu_a = (1 - 2^{-n_a})^{-1} \sum_{i=1}^{n_a} 2^{-1} \nu_i^a.$$

Hence, employing Lemma 2, there is a first category subset H_a of I such that

(1) $H_a \cap P$ and $H_a - P$ are first category F_σ subsets of P and $I - P$,

(2) $\mu_\beta(H_a) = 0$, $\beta < a$, and

(3) $\mu_a(H_a) = 1$.

For $a > 0$, let $K_a = \bigcup_{0 < \beta \leqslant a} H_a$. Then $\mu_0(K_a) = 0$. Suppose that $\{x^a\}$ is a corresponding well ordering of $\mathfrak{D}$. Let $U_a = \bigcup_i N(x_i^a, \varepsilon_i)$, and let $T_a = I - U_a$. Then $\mu_0(T) > 2^{-1}$ and, since $\mu_0(P) > 2^{-1}$, $\mu_0(T_a \cap P) = \mu_0(T_a \cap P - K_a) > 0$. Let S_a be an uncountable Lusin subset of

$(T_a \cap P - K_a)$. Let S_0 be an uncountable Lusin subset of T_0. Finally, let $E = \bigcup S_a$. Then, since $S_a \subset E \cap T_a$, $E \notin \mathcal{K}$. Moreover, recalling that a countable union of Lusin sets is an element of $\mathcal{C}$ and, a fortiori, of $\mathcal{N}$, it follows that for $a > 0$,

$$\mu_a(E) \leqslant \mu_a(\bigcup_{\beta<a} S_\beta) + \mu_a(I - K_a) = 0 .$$

Therefore, assuming the continuum hypothesis, if $\mathcal{K} = \mathcal{N}$, then $\mu_0^*(E) > 0$.

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Novak's result by Henkin's metod

by

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1. In [1], Novak proved among other things that, if Zermelo-Fraenkel set theory (ZF) is consistent, so is von Neumann–Bernays–Gödel set theory (NBG). Mostowski extended the result (see [2]) by noting that any theorem of NBG which speaks about sets only can already be derived in ZF. By making use of the method of Henkin's proof of the completeness theorem for first order theories [3] we show how a very simple proof of the above-stated fact may be obtained. Essentially, this is done as in [1] by showing that, assuming the consistency of a ZF-like theory, a model may be obtained for a related NBG-like theory.

However, professor Mostowski notified me of the fact that J. R. Shoenfield's proof of the theorem (JSL 19 (1954), pp. 21-28) remains the best result by showing that a primitive recursive function exists yielding proofs in ZF from proofs in NBG for ZF-sentences, while from our proof (as well as from Novak's and Rosser–Wang's, JSL 15 (1950), pp. 113-129) there results a *general* recursive function only (cf. Shoenfield's introduction to his paper).

2. Our symbolism will be one of the usual kinds and accordingly will not be explained. Free variables are indicated between brackets as usual; the same for substitution of terms in formulas; it is assumed that the necessary changes always are made to avoid clash of variables. Semantical notions like satisfaction (a finite sequence of a model M may satisfy a formula without indicating the relation between objects and variables too precisely) and (M-) truth are assumed to be known but use will be made of very elementary properties of these notions only (as in [3]). For NBG, refer to [4]. We make however the following slight change: let $S(x)$ be the formula $\bigvee y\,(x \in y)$; erase axioms A1 and A2; rewrite all axioms in one kind of variables relativizing former set variables to S. We denote axioms of the new system by the names of the corresponding axioms in the old system.

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