since for any bounded set $G \subseteq \mathcal{R}^2$ we have

$$\mathrm{Fr}\big(p(G)\big) \subseteq p(\mathrm{Fr}\, G) \cdot$$

Thus

$$p(G_n) \cap p \circ f_a(A)$$

is a closed-open set in the subspace $p \circ f_a(A)$ for $n = 1, 2, \ldots$, and, since $p \circ f_a(A)$ is connected,

$$p \circ f_a(A) \subseteq \bigcap_{n=1}^{\infty} p(G_n) .$$

Hence $A \subseteq u, v$. This proves Lemma (7.4).

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Rank theory of modules

by

Vlastimil Dlab (Canberra)

1. Preliminaries. The application [2] of the general algebraic dependence scheme of [1] to modules resulted in obtaining some basic information on dependence over modules. The aim of the present paper is to extend these investigations and build up a rank theory of modules parallel to that of abelian groups (cf. e.g. L. Fuchs [6]). In particular, the theory offers a generalization of some results of A. W. Goldie [8] and, when applied to injective modules, it enables us to generalize some results of E. Matlis [9]. In the latter, invariants $r_{\mathfrak{J}}(M)$ are derived which coincide with the invariants of P. Gabriel and U. Oberst in [7]. The value of our approach rests on the fact that, in contrast to [7], we define $r_{\mathfrak{J}}(M)$ for an R-module M without any reference to its injective hull $H(M)$ and can then use these cardinals $r_{\mathfrak{J}}(M)$ to characterize $H(M)$.

Throughout the paper, R denotes a fixed (associative) ring with unity, $\mathfrak{L}$ — the family of all its proper (i.e. $\neq R$) left ideals and $\mathfrak{J} \subseteq \mathfrak{L}$ — the subfamily of all (meet —) irreducible ideals. For $L \in \mathfrak{L}$ and $\varrho \in R$, the symbol $L\colon \varrho$ stands for the (left) ideal consisting of all $\chi \in R$, such that $\chi\varrho \in L$. Following [3], a subfamily $\mathfrak{K}$ of $\mathfrak{L}$ is said to be a Q-*family* if

$$\text{(Q)} \qquad \forall L, \varrho\,(L \in \mathfrak{K} \wedge \varrho \in R \setminus L \rightarrow L\colon \varrho \in \mathfrak{K}) .$$

Denote the least Q-family containing a given ideal $L \in \mathfrak{L}$ by $\mathfrak{Q}_L$; thus, $\mathfrak{Q}_L = \{L\colon \varrho | \varrho \in R \setminus L\}$. Define in the set $\boldsymbol{Q}$ of all Q-families $\mathfrak{K}$ the "*duality*" map ∂ by

$$(\partial) \qquad L \in \partial\mathfrak{K} \leftrightarrow L \in \mathfrak{L} \wedge \mathfrak{Q}_L \cap \mathfrak{K} = \emptyset .$$

Thus, ∂ defines in $\boldsymbol{Q}$ a Galois connection (cf. O. Ore [10]). In particular,

$$\mathfrak{K}_1 \subseteq \mathfrak{K}_2 \rightarrow \partial\mathfrak{K}_1 \supseteq \partial\mathfrak{K}_2 ,$$

and ∂^2 is an (idempotent) closure operator; in fact, $\partial^{2n+i}\mathfrak{K} = \partial^i\mathfrak{K}$ for any two positive integers n and i. Making use of ∂, we can introduce the symmetric relation $\nabla \subseteq \boldsymbol{Q} \times \boldsymbol{Q}$ by

$$(\nabla) \qquad [\mathfrak{K}^1, \mathfrak{K}^2] \in \nabla \leftrightarrow \partial^2\mathfrak{K}^1 = \partial\mathfrak{K}^2 (\leftrightarrow \partial\mathfrak{K}^1 = \partial^2\mathfrak{K}^2) .$$

The square ∇^2 is easily seen to be an equivalence $\overset{Q}{\sim}$ in Q. As a matter of fact,

$$(\overset{Q}{\sim}) \qquad \mathcal{K}_1 \overset{Q}{\sim} \mathcal{K}_2 \leftrightarrow \partial\mathcal{K}_1 = \partial\mathcal{K}_2 \leftrightarrow \forall \mathfrak{Q}_L(\mathfrak{Q}_L \subseteq \mathcal{K}_1 \cup \mathcal{K}_2 \to \mathfrak{Q}_L \cap \mathcal{K}_1 \cap \mathcal{K}_2 \neq \emptyset)$$

and $\partial^2\mathcal{K} = \bigcup_{\mathcal{X} \overset{Q}{\sim} \mathcal{K}} \mathcal{X}$ is the greatest element in its $\overset{Q}{\sim}$-equivalence class. If $\mathcal{K}_1 \overset{Q}{\sim} \mathcal{K}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $\mathcal{K}_1$ is said to be *essential* in $\mathcal{K}_2$. Notice that

$$[\mathcal{K}^1, \mathcal{K}^2] \in \nabla \to \mathcal{K}^1 \cap \mathcal{K}^2 = \emptyset$$

and that ∇ is stable under the equivalence $\overset{Q}{\sim}$, i.e.

$$[\mathcal{K}_1^1, \mathcal{K}_1^2] \in \nabla \wedge \mathcal{K}_1^i \overset{Q}{\sim} \mathcal{K}_2^i \ (i = 1, 2) \to [\mathcal{K}_2^1, \mathcal{K}_2^2] \in \nabla .$$

Also, if $[\mathcal{K}^1, \mathcal{K}^2] \in \Delta$, then the Q-family $\mathcal{K} = \mathcal{K}^1 \cup K^2$ is $\overset{Q}{\sim}$-equivalent to $\mathfrak{L}$; thus $\mathcal{K}$ is essential (in $\mathfrak{L}$). A set $\{\mathcal{K}_\omega |\ \omega \in \Omega\}$ of essential Q-families $\mathcal{K}_\omega$ is said to be *centred* if $\bigcap_{\omega \in \Omega} \mathcal{K}_\omega$ is essential; evidently, every finite set of essential Q-families is centred.

Let M (always) denote a (unital left) R-module; put $M^{\#} = M \setminus \{0\}$. The order (annihilator) of $m \in M$ is denoted by $O(m)$; hence, $O(m) \in \mathfrak{L}$ if and only if $m \in M^{\#}$. Evidently, $O(\varrho m) = O(m): \varrho$ for any $\varrho \in R$ and $m \in M^{\#}$. Also, for a cyclic R-submodule $\langle m \rangle$ generated by m we have $\langle m \rangle \cong R \bmod O(m)$.

A subset $X \subseteq M^{\#}$ of M is said to be *independent* if

$$\langle X \rangle = \bigoplus_{x \in X} \langle x \rangle = \bigoplus_{x \in X} Rx;$$

otherwise, X is said to be *dependent*. X is a *maximal independent subset* of a set $S \subseteq M$ if it is the only independent subset of S containing X. Two independent subsets X_1 and X_2 are defined to be *ε-related* if both X_1 and X_2 are maximal independent subsets of $X_1 \cup X_2$. Thus, any two maximal independent subsets of a set $S \subseteq M$ are *ε-related*. The following extension of the definition of an essential R-submodule of an R-module will be also needed: A subset $S_1 \subseteq M$ is called *essential* in $S_2 \subseteq M$ if every maximal independent subset of S_1 is a maximal independent subset of S_2.

We refer to [1] and [2] for the following basic result:

(A) *Let X_1 and X_2 be two independent ε-related subsets of M. If $O(x)$ is irreducible for every $x \in X_1$, then*

$$\operatorname{card}(X_1) \geqslant \operatorname{card}(X_2) .$$

Also, the following two simple results (cf. Lemma 2.1 and 3.2 of [2]) will be used repeatedly:

(B) *An ideal $L \in \mathfrak{L}$ is irreducible if and only if, for every α and β of $R \setminus L$, there are μ and ν in R such that*

$$\mu\alpha - \nu\beta \in L \quad \text{and} \quad \mu\alpha \notin L .$$

(C) *A subset $X \subseteq M^{\#}$ is dependent if and only if there exist $x_i \in X$ and $\varrho_i \in R (1 \leqslant i \leqslant k)$ such that*

$$\sum_{i=1}^{k} \varrho_i x_i = 0 \quad \textit{with} \quad R \neq O(\varrho_1 x_1) = O(\varrho_i x_i) \ \textit{for}\ 1 \leqslant i \leqslant k .$$

Our investigations will be closely connected with the subsets $M_{\mathcal{K}}$ of M; for $\mathcal{K} \subseteq \mathfrak{L}$, $M_{\mathcal{K}}$ is defined by

$$m \in M_{\mathcal{K}} \leftrightarrow m \in M \wedge O(m) \in \mathcal{K} .$$

First, we present two simple preliminary results.

PROPOSITION 1. *$\mathcal{K} \subseteq \mathfrak{L}$ is a Q-family if and only if, for every R-module M, $M_{\mathcal{K}} \cup \{0\} \subseteq M$ is the union of cyclic R-submodules.*

PROPOSITION 2. *Two Q-families $\mathcal{K}_1$ and $\mathcal{K}_2$ are $\overset{Q}{\sim}$-equivalent if and only if, for every R-module M, any two maximal independent subsets X_1 and X_2 of $M_{\mathcal{K}_1}$ and $M_{\mathcal{K}_2}$, respectively, are ε-related. In particular, if $X_1 \subseteq M_{\mathcal{K}_2}$ then X_1 is also a maximal independent subset of $M_{\mathcal{K}_2}$. Thus, $\mathcal{K}_1$ is essential in $\mathcal{K}_2$ if and only if, for every R-module M, $M_{\mathcal{K}_1}$ is essential in $M_{\mathcal{K}_2}$ (or, equivalently, if $M_{\mathcal{K}_1} \cap \langle m \rangle \neq \{0\}$ for every $m \in M_{\mathcal{K}_2}$).*

Proof. Let $\mathcal{K}_1 \overset{Q}{\sim} \mathcal{K}_2$ and $m \in M_{\mathcal{K}_2} \setminus X_1$, so $O(m) \in \mathcal{K}_2$. By $(\overset{Q}{\sim})$, there is $\varrho \in R$ such that $\varrho m \in M_{\mathcal{K}_1}$. Therefore, in view of (C), $X_1 \cup (m)$ is dependent, as required.

On the other hand, take $L \in \mathcal{K}_2 \setminus \mathcal{K}_1$ and consider $M = R \bmod L = \langle m \rangle$ with $X_2 = \{m\}$. According to (C),

$$O \neq \varrho m = \sum_{i=1}^{k} \varrho_i x_i \quad \text{with} \quad O(\varrho m) = O(\varrho_i x_i) \in \mathcal{K}_1$$

for suitable $\varrho, \varrho_i \in R$ and $x_i \in M_{\mathcal{K}_1}$ $(1 \leqslant i \leqslant k)$. Hence, $L: \varrho \in \mathcal{K}_1$, i.e. $\mathfrak{Q}_L \cap \mathcal{K}_1 \neq \emptyset$ and thus $\mathcal{K}_1 \overset{Q}{\sim} \mathcal{K}_2$.

The rest of the proposition follows easily.

2. Concept of rank. Let R and M be a fixed ring and R-module, respectively.

Let Ω be an index set. For every $\omega \in \Omega$, consider a pair of Q-families $\mathcal{K}_\omega^1, \mathcal{K}_\omega^2$ such that

$$[\mathcal{K}_\omega^1, \mathcal{K}_\omega^2] \in \nabla ;$$

put

$$\mathcal{K}_\omega = \mathcal{K}_\omega^1 \cup \mathcal{K}_\omega^2 \quad \text{and} \quad \mathcal{K} = \bigcap_{\omega \in \Omega} \mathcal{K}_\omega .$$

Consider the set 2^Ω of all mappings of Ω into $\{1, 2\}$ and, for each $f \in 2^\Omega$, define the subset M_f of an R-module M by

$$m \in M_f \leftrightarrow m \in M \wedge O(m) \in \bigcap_{\omega \in \Omega} \mathcal{K}_\omega^{f(\omega)} .$$

Since $\bigcap_{\omega \in \Omega} \mathcal{K}_\omega^{f(\omega)}$, as well as all $\mathcal{K}_\omega$ and $\mathcal{K}$, are Q-families, $M_f \subseteq M^{\#}$ and $M_f \cup \{0\}$ is a union of cyclic submodules of M. Moreover, it is obvious that, for $f_i \in 2^\Omega (i = 1, 2)$,

(*) $$M_{f_1} \cap M_{f_2} \neq \emptyset \leftrightarrow f_1 = f_2 \quad (\leftrightarrow M_{f_1} = M_{f_2})$$

and that

$$\bigcup_{f \in 2^\Omega} M_f = \{m \mid m \in M \wedge O(m) \in \bigcap_{\omega \in \Omega} \mathcal{K}_\omega\} = M_{\mathcal{K}} .$$

The following three lemmas form the background of our investigations.

LEMMA 1. *Let X be a maximal independent subset of M such that $X \subseteq M_{\mathcal{K}}$. Then, for every $f \in 2^\Omega$,*

$$X_f = X \cap M_f$$

is a maximal independent subset of M_f.

LEMMA 2. *Let, for every $f \in 2^\Omega$, X_f be a maximal independent subset of M_f. Then*

$$X = \bigcup_{f \in 2^\Omega} X_f$$

is a maximal independent subset of $M_{\mathcal{K}}$. Thus, if $\{\mathcal{K}_\omega \mid \omega \in \Omega\}$ is centred then X is a maximal independent subset of M. (1)

LEMMA 3. *Let $\mathfrak{K}$ be a Q-family. Let N be an R-submodule of M. Then, there exists a maximal independent subset X of $M_{\mathfrak{K}}$ such that*

(i) *$Y = X \cap N$ is a maximal independent subset of $N_{\mathfrak{K}}$,*

(ii) *$x_1 \neq x_2$ with $x_i \in X \setminus Y$ $(i = 1, 2)$ implies $x_1 \bmod N \neq x_2 \bmod N$ and*

(iii) *$\overline{X} = \{x \bmod N \mid x \in X \setminus Y\}$ is an independent subset of $(M/N)_{\mathfrak{K}}$. If, moreover, $\mathfrak{K}$ contains no essential ideals of R (i.e. if, for every $L \in \mathfrak{K}$, there is a non-zero $\varrho \in R$ such that $L \cap R\varrho = \{0\}$), then X can be chosen so that, in addition, $\overline{X}$ is a maximal independent subset of $(M/N)_{\mathcal{K}}$.*

(1) The conclusion does not hold, in general, if $\{\mathcal{K}_\omega \mid \omega \in \Omega\}$ is not centred (consider the R-module $R \bmod L$ with $\mathfrak{Q}_L \cap \bigcap_{\omega \in \Omega} \mathcal{K}_\omega = \emptyset$).

Proof of Lemma 1. Only maximality requires to be proved. For $x \in M_f$, we have, according to (C), a relation

$$0 \neq \varrho x = \sum_{i=1}^{k} \varrho_i x_i , \quad \text{where} \quad O(\varrho x) = O(\varrho_i x_i) \text{ for } 1 \leqslant i \leqslant k$$

with suitable $\varrho, \varrho_i \in R$ and $x_i \in X$. Hence, for each i, $1 \leqslant i \leqslant k$,

$$O(\varrho_i x_i) = O(x) : \varrho \in \mathcal{K}_\omega^{f(\omega)} \quad \text{for every } \omega \in \Omega .$$

Consequently, because of (*),

$$O(x_i) \in \bigcap_{\omega \in \Omega} \mathcal{K}_\omega^{f(\omega)} , \quad \text{i.e.} \quad x_i \in X \cap M_f = X_f ,$$

as required.

Proof of Lemma 2. The independence of X is again a simple consequence of (C). Also, X is obviously a maximal independent subset of $\bigcup_{f \in 2^\Omega} M_f = M_{\mathcal{K}}$.

Since $\{\mathcal{K}_\omega \mid \omega \in \Omega\}$ is centred, i.e. since $\mathcal{K}$ is essential, $M_{\mathcal{K}}$ is essential in M according to Proposition 2, and Lemma 2 follows.

Proof of Lemma 3. Denote, for $m \in M$, by $\overline{m}$ the corresponding coset $m \bmod N$ of M/N. Further, denote by S the subset of all elements $m \in M_{\mathfrak{K}}$ such that

$$O(m) = O(\overline{m}) ,$$

and put

$$M_{\mathfrak{K}}^* = N_{\mathfrak{K}} \cup S \subseteq M_{\mathfrak{K}} .$$

Take a maximal independent subset Y of $N_{\mathfrak{K}}$ and extend it to a maximal independent subset $X \supseteq Y$ of $M_{\mathfrak{K}}^*$.

First of all, X is a maximal independent subset of $M_{\mathfrak{K}}$. For, if $m \in M_{\mathfrak{K}} \setminus M_{\mathfrak{K}}^*$, i.e. if $O(m) \subsetneqq O(\overline{m})$, then

$$0 \neq \varrho m \in N .$$

Hence, by (C), $Y \cup \{m\}$ is dependent.

Secondly, (ii) is evident and $\overline{X} = \{\overline{x} \mid x \in X \setminus Y\}$ is independent in M/N. For, assume in accordance with (C), that $x_i \in X \setminus Y$ and $\varrho_i \in R$ $(1 \leqslant i \leqslant k)$ exist such that

$$\sum_{i=1}^{k} \varrho_i \overline{x}_i = \overline{0} \quad \text{with} \quad O(\varrho_i \overline{x}_i) = L \in \mathfrak{K} ,$$

i.e.

$$\sum_{i=1}^{k} \varrho_i x_i = n \in N \quad \text{with} \quad O(\varrho_i x_i) = L .$$

Then, since $O(\sum_{i=1}^{k} \varrho_i x_i) = L$, $n \in N_{\mathcal{K}}$. Hence,

$$Y \cup \{x_1, x_2, \ldots, x_k\} \subseteq X$$

is dependent — a contradiction of independence of X.

Finally, assuming that $\mathcal{K}$ contains no essential ideals of R, $\bar{X}$ is, in fact, a maximal independent subset of $(M/N)_{\mathcal{K}}$. For, if $\bar{m} \in (M/N)_{\mathcal{K}}$ and so $O(\bar{m}) \in \mathcal{K}$, then there is a non-zero $\varrho \in R$ such that

$$O(\bar{m}) \cap R\varrho = \{0\},$$

i.e.

$$\bar{0} \neq \varrho\bar{m} \quad \text{and} \quad O(\varrho\bar{m}) = \{0\}:\ \varrho = O(\varrho m) \quad \text{with} \quad m \in \bar{m}.$$

Therefore, $\varrho m \in M^*_{\mathcal{K}}$. Using (C) again, the proof can be easily completed.

In order to get basic invariants of an R-module, let us first apply our results in the case of $\Omega = \{1\}$, $\mathcal{H}_1^1 = \mathcal{J}$, $\mathcal{H}_1^2 = \partial\mathcal{J}$. Note that the family $\mathcal{J} \subseteq \mathcal{L}$ of irreducible ideals is a Q-family (cf. Lemma 2.2 of [2]); $\partial\mathcal{J}$ is the family of what we shall call *strongly reducible* ideals. The subsets of all elements of an R-module M whose orders belong to $\mathcal{J}$ and $\partial\mathcal{J}$ denote by M_1 and M_2, respectively. In view of Lemmas 1 and 2 and (A), we can formulate

THEOREM 1. *Any R-module M possesses maximal independent subsets X consisting of elements of irreducible and strongly reducible orders; denote the family of all such subsets X by $\mathfrak{X}_M$. In fact, X belongs to $\mathfrak{X}_M$ if and only if X is the union of maximal independent subsets X_1 and X_2 of M_1 and M_2, respectively. The cardinality* $\mathrm{card}(X_1)$ *is an invariant of M in the sense that, for any $X' \in \mathfrak{X}_M$,* $\mathrm{card}(X' \cap M_1) = \mathrm{card}(X_1)$. *If $M_2 \neq \emptyset$, then* $\sup_{X \in \mathfrak{X}_M} \mathrm{card}(X \cap M_2) \geqslant \aleph_0$.

DEFINITION 1. Define the *rank* $r(M)$ of an R-module M by

$$r(M) = \mathrm{card}_{X \in \mathfrak{X}_M}(X \cap M_1).$$

For the sake of completeness, we can also define, in addition to the (*irreducible*) rank $r(M)$, the *reducible rank* $r^\circ(M)$ and the *total rank* $\bar{r}(M)$ of M by

$$r^\circ(M) = \sup_{X \in \mathfrak{X}_M} \mathrm{card}(X \cap M_2)$$

and

$$\bar{r}(M) = r(M) + r^\circ(M).$$

Notice that $r^\circ(M) = 0$ or $r^\circ(M) \geqslant \aleph_0$. If $r^0(M) = 0$, i.e. if $M_2 = \emptyset$, M will be called *tidy*. In [2], the property (J) (defined there) of a ring R has been shown to be equivalent to the fact that every R-module is tidy. Since every (left) noetherian ring has (J) (see [2]), the above definition extends the definition of rank of A. W. Goldie [8] to arbitrary R-modules.

Following the foregoing pattern, we can get an invariant of an R-module M corresponding to any Q-family:

DEFINITION 2. Let $\mathcal{K}$ be a Q-family. Define the $\mathcal{K}$-*rank* $r_{\mathcal{K}}(M)$ of an R-module M as the cardinality of a maximal independent subset of the set of all elements of M whose orders belong to $\mathcal{K} \cap \mathcal{J}$.

Thus, $r(M) = r_{\mathcal{J}}(M) = r_{\mathcal{J} \cup \partial\mathcal{J}}(M)$. Also, in an obvious way, $r^\circ_{\mathcal{K}}(M)$ and $\bar{r}_{\mathcal{K}}(M)$ can be defined. Notice that $\bar{r}_{\mathcal{K}}(M) = 0$ if and only if M has no elements of orders belonging to $\mathcal{K}$ and that $\bar{r}(M) = \bar{r}_{\mathcal{K}}(M) + \bar{r}_{\partial\mathcal{K}}(M) = 0$ if and only if $M = \{0\}$.

THEOREM 2. *Let $\mathcal{K}$ be a Q-family. Then,*

(i) *$r_{\mathcal{K}}(N) \leqslant r_{\mathcal{K}}(M)$ for any R-submodule N of M;*

(ii) *$M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ implies $r_{\mathcal{K}}(M) = \sum_{\gamma \in \Gamma} r_{\mathcal{K}}(M_\gamma)$;*

(iii) *$r_{\mathcal{K}}(M) \leqslant r_{\mathcal{K}}(N) + r_{\mathcal{K}}(M/N)$ for any R-submodule N; if $\mathcal{K} \cap \mathcal{J}$ contains no essential ideals of R, then*

$$r_{\mathcal{K}}(M) = r_{\mathcal{K}}(N) + r_{\mathcal{K}}(M/N).$$

Proof. (i) is trivial. Also, since $\bigcup_{\gamma \in \Gamma} (M_\gamma)_{\mathcal{K} \cap \mathcal{J}}$ is obviously essential in $M_{\mathcal{K} \cap \mathcal{J}}$, (ii) holds. Finally, (iii) is an immediate consequence of Lemma 3.

THEOREM 3. (i) *For any Q-family $\mathcal{K}$, $r_{\mathcal{K}}(M) + r_{\partial\mathcal{K}}(M) = r(M)$ and $r_{\partial^2\mathcal{K}}(M) = r_{\mathcal{K}}(M)$.*

(ii) *If Q-families $\mathcal{K}_1$ and $\mathcal{K}_2$ are $\overset{Q}{\sim}$-equivalent, then $r_{\mathcal{K}_1}(M) = r_{\mathcal{K}_2}(M)$* (2).

Proof. Applying Lemmas 1 and 2 together with (A) in the case $\Omega = \{1, 2\}$. $\mathcal{H}_1^1 = \mathcal{J}$, $\mathcal{H}_1^2 = \partial\mathcal{J}$, $\mathcal{H}_2^1 = \mathcal{K}$, $\mathcal{H}_2^2 = \partial\mathcal{K}$, we get the first part of (i). The second part is a consequence of (ii) which in turn, follows from Proposition 2 and (A); for, $\mathcal{K}_1 \overset{Q}{\sim} \mathcal{K}_2$ implies readily $\mathcal{K}_1 \cap \mathcal{J} \overset{Q}{\sim} \mathcal{K}_2 \cap \mathcal{J}$.

In order to get the most refined invariants $r_{\mathcal{K}}(M)$ let us consider the smallest, in the sense of Theorem 3 significant, Q-families contained in $\mathcal{J}$, viz. the families $\partial^2\mathfrak{Q}_L \cap \mathcal{J}$ for $L \in \mathcal{J}$.

LEMMA 4. *Let $\mathcal{K} \in \boldsymbol{Q}$ and $L \in \mathcal{J}$. Then*

(i) $\mathfrak{Q}_L \cap \mathcal{K} \neq \emptyset \leftrightarrow \mathfrak{Q}_L \cap \partial\mathcal{K} = \emptyset\ (\leftrightarrow \mathfrak{Q}_L \subseteq \partial^2\mathcal{K})$;

therefore

(ii) $\mathcal{J} \subseteq \partial K \cup \partial^2\mathcal{K}$, *and thus* $\mathcal{J} \cup \partial\mathcal{J} \subseteq \partial\mathfrak{Q}_L \cup \partial^2\mathfrak{Q}_L$;

(iii) $\partial^2\mathfrak{Q}_L \cap \mathcal{K} = \emptyset$ *implies* $\partial^2\mathfrak{Q}_L \subseteq \partial^2\mathcal{K}$.

Proof. (i) follows immediately from (B). Furthermore, (ii) and (iii) (because $\partial^2\mathfrak{Q}_L \cap \mathcal{K} \neq \emptyset$ is equivalent to $\mathfrak{Q}_L \cap \mathcal{K} \neq \emptyset$) is a simple consequence of (i).

(2) On the other hand, if $r_{\mathcal{K}_1}(M) = r_{\mathcal{K}_2}(M)$ for every R-module M, then $\mathcal{K}_1 \cap \mathcal{J} \overset{Q}{\sim} \mathcal{K}_2 \cap \mathcal{J}$.

THEOREM 4. *The set of all $\partial^2\mathfrak{Q}_L$ for $L \in \mathfrak{J}$ is a partition of $\bigcup_{L\in\mathfrak{J}} \partial^2\mathfrak{Q}_L \supseteq \mathfrak{J}$. Let $L_\omega \in \mathfrak{J}$, $\omega \in \Omega$, be a set of representatives of the "equivalence classes" $\partial^2\mathfrak{Q}_L$ and put*

$$\mathfrak{T}_\omega = \partial^2\mathfrak{Q}_L \cap \mathfrak{J}\,.$$

Then $\{\mathfrak{T}_\omega \cup \partial\mathfrak{T}_\omega |\ \omega \in \Omega\}$ is centred and thus, for any R-module M,

$$r(M) = \sum_{\omega\in\Omega} r_{\mathfrak{T}_\omega}(M)\,.$$

Moreover, if $\mathfrak{K}$ is a Q-family, then

$$\partial^2\mathfrak{K} \cap \mathfrak{J} = \bigcup_{\omega\in\Omega''} \mathfrak{T}_\omega \quad \textit{for a certain } \Omega' \subseteq \Omega$$

and

$$\partial\mathfrak{K} \cap \mathfrak{J} = \bigcup_{\omega\in\Omega'} \mathfrak{T}_\omega \quad \textit{with} \quad \Omega'' = \Omega \setminus \Omega';$$

hence, for any R-module M,

$$r_{\mathfrak{K}}(M) = \sum_{\omega\in\Omega'} r_{\mathfrak{T}_\omega}(M) \quad \textit{and} \quad r_{\partial\mathfrak{K}}(M) = \sum_{\omega\in\Omega''} r_{\mathfrak{T}_\omega}(M)\,.$$

Proof. Clearly, $\partial^2\mathfrak{Q}_{L_1} \cap \partial^2\mathfrak{Q}_{L_2} = \emptyset$ or $\partial^2\mathfrak{Q}_{L_1} = \partial^2\mathfrak{Q}_{L_2}$ by (iii) of the preceding lemma. In view of (ii) of the same lemma,

$$\bigcap_{\omega\in\Omega} (\mathfrak{T}_\omega \cup \partial\mathfrak{T}_\omega) \supseteq \mathfrak{J} \cup \partial\mathfrak{J}\,,$$

and thus, $\{\mathfrak{T}_\omega \cup \partial\mathfrak{T}_\omega |\ \omega \in \Omega\}$ is centred. The remaining statements of the theorem follow then in the previously established pattern from Lemmas 1, 2 and (A), in combination with (ii) and (iii) of Lemma 4.

The second part of Theorem 4 enables us to introduce the concept of torsion rank $r_t(M)$ and torsion-free rank $r_t(M)$ of an R-module M. The question of "torsion" in the theory of modules has been dealt with in terms of so-called *R-families* in [4]; two particular definitions of torsion have been suggested in [5]. Here, we show that only one of them is acceptable provided that we want to retain the relation

$$(+) \qquad r_f(N) + r_f(M/N) = r_f(M) \quad \text{for every } R\text{-module } N \subseteq M\,.$$

Denote by $\mathcal{E}$ the family of all (proper) *essential* ideals of R; then, $\mathcal{C}_* = \partial^2\mathcal{E}$ is the family of all (proper) *maxi* ideals of R of [5]. Referring to [5], we remark here briefly that, in any R-module M, all elements of orders belonging to $\mathcal{C}_*$ form, together with 0, an R-submodule $T(M)$ of M and that $T\big(M/T(M)\big) = \{0\}$. Accordingly, an R-module M is said to be *torsion* (*$*$-torsion*) or *torsion-free* (*$*$-torsion-free*) if $T(M) = M$ or $T(M) = \{0\}$, respectively.

DEFINITION 3. Define the *torsion rank* $r_t(M)$ and the *torsion-free rank* $r_f(M)$ of an R-module M by

$$r_t(M) = r_{\mathcal{C}_*}(M) \qquad \big(= r_{\mathcal{E}}(M)\big)$$

and

$$r_f(M) = r_{\partial\mathcal{C}_*}(M) \qquad \big(= r_{\partial\mathcal{E}}(M)\big)\,,$$

respectively.

Now, in view of Theorem 4,

$$\mathcal{C}_* \cap \mathfrak{J} = \bigcup_{\omega\in\Omega_t} \mathfrak{T}_\omega \quad \text{for} \quad \Omega_t \subseteq \Omega$$

and

$$\partial\mathcal{C}_* \cap \mathfrak{J} = \bigcup_{\omega\in\Omega_f} \mathfrak{T}_\omega \quad \text{with} \quad \Omega_f = \Omega \setminus \Omega_t\,.$$

In fact, $\omega \in \Omega_t$ if and only if $\mathfrak{T}_\omega \cap \mathcal{E} \neq \emptyset$. Making use of Lemma 3, we get from Theorem 4 immediately

COROLLARY 1.

(i) $r_t(M) = \sum_{\omega\in\Omega_t} r_{\mathfrak{T}_\omega}(M)$ *and* $r_f(M) = \sum_{\omega\in\Omega_f} r_{\mathfrak{T}_\omega}(M)$;

(ii) $r(M) = r_t(M) + r_f(M)$;

(iii) $r_t(M) = r_t\big(T(M)\big)$ *and* $r_t\big(M/T(M)\big) = 0$;

(iv) $(+)$ *holds; in particular,* $r_f(M) = r_f\big(M/T(M)\big)$ *and* $r_f\big(T(M)\big) = 0$.

Let us remark that the $\circ$-torsion-free rank $r_{f_0}(M)$ of M corresponding to the family $\mathcal{C}_0 \subseteq \mathfrak{L}$ of [5] does not satisfy $(+)$. (There, $\mathcal{C}_0 \subseteq \mathcal{E}$ is the family of all proper *strong* ideals of R: $L \in \mathfrak{L}$ is strong if, for any $\varrho \in R \setminus L$ and any $0 \neq \sigma \in R$, always $(L : \varrho)\sigma \neq \{0\}$.) Let R_0 be the ring of all triples (x, y, z) of integers modulo 2 with component-wise addition and multiplication defined by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1x_2,\, x_1y_2 + y_1x_2,\, x_1z_2 + z_1x_2);$$

clearly, R_0 has no strong ideals. Consider R_0 as an R_0-module and the ideal $L = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\} \subseteq R_0$ as its R_0-submodule. Then,

$$r_{f_0}(R_0) = 2\,, \quad r_{f_0}(L) = 2\,, \quad r_{f_0}(R_0/L) = 1\,,$$

and thus $(+)$ does not hold.

To conclude this section, let us formulate another simple consequence of Theorem 4; note that if R is a commutative noetherian ring, then there is a one-to-one correspondence between the families $\partial^2\mathfrak{Q}_L$ of Theorem 4 and the (proper) prime ideals of R.

Corollary 2. *Let R be a commutative noetherian ring, $\{P_\omega|\ \omega \in \Omega_t\}$ and $\{P_\omega|\ \omega \in \Omega_f\}$ the set of all (proper) prime essential and all prime non-essential ideals of R, respectively; put $\Omega = \Omega_t \cup \Omega_f$. Let M be an R-module. Then, for each $\omega \in \Omega$, the cardinality of a maximal independent subset of elements of order P_ω in M is an invariant: P_ω-rank $r_{P_\omega}(M)$ of M. Moreover,*

$$r_t(M) = \sum_{\omega \in \Omega_t} r_{P_\omega}(M), \qquad r_f(M) = \sum_{\omega \in \Omega_f} r_{P_\omega}(M)$$

and

$$r(M) = \bar{r}(M) = \sum_{\omega \in \Omega} r_{P_\omega}(M).$$

3. Injective R-modules. Here, we generalize some results of Eben Matlis [9] on injective R-modules. Let $H(M)$ be an injective hull of an R-module M; $H(M)$ can be characterised as a maximal essential extension of M. Thus, for any Q-family $\mathfrak{K}$, $M_{\mathfrak{K}}$ is essential in $[H(M)]_{\mathfrak{K}}$ and therefore we get

Theorem 5. *For any Q-family $\mathfrak{K}$,*

$$r_{\mathfrak{K}}(M) = r_{\mathfrak{K}}[H(M)].$$

In particular, rank, reducible rank, total rank, torsion rank or torsion-free rank of M equals to the respective rank of $H(M)$; also

$$r_{\mathfrak{I}_\omega}(M) = r_{\mathfrak{I}_\omega}[H(M)] \quad \text{for every } \mathfrak{I}_\omega \text{ of Theorem 4}.$$

Furthermore, using (B) we can easily prove

Lemma 5. *The following properties of an injective R-module H are equivalent:*

(i) *H is indecomposable.*

(ii) $\bar{r}(H) = 1$.

(iii) *For any $x \in H$, $O(x) \in \mathfrak{J}$ and $H = H(\langle x\rangle)$.*

(iv) *$H = H(R \bmod L)$ with a certain $L \in \mathfrak{J}$.*

Lemma 6. *Let $L_i \in \mathfrak{J}$ $(i = 1, 2)$. Then*

$$\partial^2\mathfrak{Q}_{L_1} = \partial^2\mathfrak{Q}_{L_2} \leftrightarrow H(R \bmod L_1) \cong H(R \bmod L_2).$$

Proof. If $\partial^2\mathfrak{Q}_{L_1} = \partial^2\mathfrak{Q}_{L_2}$, then, for suitable ϱ_1 and ϱ_2 of R,

$$L_1 : \varrho_1 = L_2 : \varrho_2 \neq R.$$

Thus, in view of Lemma 5,

$$H(R \bmod L_1) \cong H\big(R \bmod (L_1 : \varrho_1)\big) \cong H\big(R \bmod (L_2 : \varrho_2)\big) \cong H(R \bmod L_2).$$

On the other hand, let $R \bmod L_1 = \langle m_1\rangle$ and $R \bmod L_2 = \langle m_2\rangle$. Furthermore, let φ be an isomorphism of $H(\langle m_1\rangle)$ onto $H(\langle m_2\rangle)$. Then, there are suitable ϱ_1 and ϱ_2 of R such that, in $H(\langle m_2\rangle)$,

$$\varrho_1\varphi(m_1) = \varrho_2 m_2 \neq 0.$$

Since $\varrho_1\varphi(m_1) = \varphi(\varrho_1 m_1)$ and $R \neq O(\varrho_i m_i) = O(m_i) : \varrho_i = L_i : \varrho_i$ for $i = 1, 2$, the reverse implication follows, too.

Now, we are ready to formulate the basic

Theorem 6. *Lemma 6 yields a one-to-one correspondence Φ between the Q-families $\mathfrak{I}_\omega$ of Theorem 4 and the non-isomorphic indecomposable injective R-modules: write $\Phi(\mathfrak{I}_\omega) = H(\mathfrak{I}_\omega)$; every $H(\mathfrak{I}_\omega)$ is either torsion or torsion-free.*

Let

$$N = \bigoplus_{\substack{\omega \in \Omega \\ \gamma \in \Gamma_\omega}} H_{\omega,\gamma}, \qquad H_{\omega,\gamma} \cong H(\mathfrak{I}_\omega) \quad \text{for every } \gamma \in \Gamma_\omega,$$

be a direct sum contained in an injective hull $H(M)$ of an R-module M, and let N be maximal in the sense that there is no indecomposable injective submodule H of $H(M)$ such that $H \cap N = \{0\}$. Then

$$\operatorname{card}(\Gamma_\omega) = r'_{\mathfrak{I}_\omega}(M) \quad \text{for every } \omega \in \Omega.$$

In particular, any two direct decompositions of an R-module into direct sums of indecomposable injective R-modules are isomorphic and can be described by a cardinal-valued function defined on Ω.

Also, if M is tidy (in particular, if R has the property (J) *of* [2]*), then N is essential in $H(M)$ and $H(M)$ is fully characterized by the function f defined by M on Ω:*

$$f(\omega) = r_{\mathfrak{I}_\omega}(M).$$

Theorem 6 follows from the results of § 2 and from Lemmas 5 and 6 quite simply. We refrain also from formulating the consequence of Theorem 6 in the case of a (commutative) noetherian ring R which is easy to deduce (cf. [9]).

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AUSTRALIAN NATIONAL UNIVERSITY

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Some remarks on Hausdorff measure

by

R. B. Darst (Lafayette, Ind.)

Let us begin with some notation and terminology. Denote by $\mathcal{F}$ the class of non-decreasing functions h on $(0, \infty)$ with $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$. If $h \in \mathcal{F}$ and $E \subset I = [0,1]$, then the *h-Hausdorff outer measure $m_h(E)$ of E is the extended real number*

$$\liminf_{\varepsilon\to 0}\left\{\sum h(b_i - a_i);\ E \subset \bigcup (a_i, b_i),\ \sup(b_i - a_i) < \varepsilon\right\}.$$

Denote by $\mathcal{H}$ the collection of subsets E of I such that $m_h(E) = 0$ for all $h \in \mathcal{F}$. Denote by $\mathcal{P}$ the collection of regular non-atomic probability measures μ on the Borel subsets $\mathcal{B}$ of I, and denote by $\mathcal{N}$ the collection of subsets E of I satisfying $\sup\{\mu^*(E);\ \mu \in \mathcal{P}\} = 0$. Denote by $\mathcal{C}$ the set of concentrated subsets of I (i.e., $E \in \mathcal{C} \Longleftrightarrow$ there is a sequence $\{x_i\}$ of elements of I such that if $\{\varepsilon_i\}$ is a sequence of positive numbers, then $E - \bigcup N(x_i, \varepsilon_i)$ is, at most, a countable set, where $N(x.\ \varepsilon) = (x - \varepsilon/2, x + \varepsilon/2)$). Finally, denote by $\mathcal{D}$ the collection of enumerations $\{x_i\}$ of countable, dense subsets of I and by $\mathcal{E}$ the collection of sequences of positive numbers.

It is easy to show that $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{N}$, and the author showed [1] that if the continuum hypothesis is satisfied, then $\mathcal{C} \neq \mathcal{N}$. The purpose of this note is to show, assuming the continuum hypothesis, that $\mathcal{C} \neq \mathcal{H}$. To this end, let us begin by giving the following characterizations of the elements of $\mathcal{H}$.

LEMMA 1. *Each of the following conditions is necessary and sufficient in order that a subset E of I be an element of $\mathcal{H}$.*

(i) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is a sequence $\{x_i\}$ of points of I such that $E - \bigcup N(x_i, \varepsilon_i)$ is countable.*

(ii) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is $\{x_i\} \in \mathcal{D}$ such that $E - \bigcup N(x_i, \varepsilon_i)$ is countable.*

(iii) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is a sequence $\{x_i\}$ of points of I such that $E \subset \bigcup N(x_i, \varepsilon_i)$.*

(iv) *If $\{\varepsilon_i\} \in \mathcal{E}$, then there is $\{x_i\} \in \mathcal{D}$ such that $E \subset \bigcup N(x_i, \varepsilon_i)$.*