Fixed points of arcwise connected spaces

by

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This paper is a continuation of the papers of Borsuk [1] and Young [4]. It follows from some results of Borsuk that his fixed point theorem of [1] is a special case of Young's earlier fixed-point theorem of [4]. The ideas used by Borsuk and Young are different. Our methods are similar to Borsuk's methods.

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§ 1. Borsuk-Young arcwise connected spaces (B-spaces).

(1.1) Definition. We say that an arcwise connected space X is a *B-space* if it has the following property:

For any one-to-one continuous mapping f of the ray $[0, \infty)$ into X, the closure of the set $P = f([0, \infty))$ is a simple arc.

This condition is taken from a lemma of Borsuk's paper [1] (see Section 3 of [1]). A similar condition equivalent in the class of Hausdorff spaces, but not equivalent for T_1-spaces (see (1.18) and (1.19)), was formulated by Young in the earlier paper [4].

First we shall give some fundamental properties of the class of B-spaces.

(1.2) Proposition. *A B-space does not contain a homeomorphic image of a circle. It does not even contain the image of a circle under a one-to-one continuous function.* (The latter assertion proves stronger if applied to non-Hausdorff spaces.)

As a corollary (see also (1.5)) we obtain the following

(1.3) Proposition. *For any two points x, y of a B-space X there exists a unique simple arc $\overset{\frown}{x,y}$ with end-points x, y (if $x = y$, then $\overset{\frown}{x,y} = \{x\}$).*

(1.4) Proposition. *A B-space does not contain, as a closed subset, a one-to-one continuous image of the ray* $[0, \infty)$.

(1.5) Proposition. *Any simple arc in a B-space is a closed set. Any one-to-one image of a segment in a B-space is a simple arc.*

(1.6) Proposition. *Any B-space is a T_1-space.*

(1.7) Example. The space $\{\alpha\colon \alpha < \Omega\} \times [0, 1)$, under the interval topology induced by lexicographic ordering, is a B-space. This space is called the *long half-line*. Point $(0, 0)$ is the end-point of the long half-line. The one-point union of two long half-lines, where the common point is the end-point of each of them, is called the *long line*. Also the long line is a B-space.

(1.8) Proposition. *If a Hausdorff space X is a one-to-one continuous image of the long half-line or of the long line, then X is a B-space.*

(1.9) Proposition. *If A is a simple arc in a B-space, or a subset which is homeomorphic to an open interval, then A is contained in a maximal simple arc or in a one-to-one continuous image of the long half-line.*

(1.10) Proposition. *Every closed arcwise connected subspace of a B-space is a B-space.*

A connected space X is said to be *unicoherent* if for every two connected closed subsets such that $X = A \cup B$ the set $A \cap B$ is connected. A connected space is said to be *hereditarily unicoherent* if all its closed and connected subsets are unicoherent.

(1.11) Proposition. *Every hereditarily unicoherent arcwise connected Hausdorff space X, not containing a closed subset, which is a one-to-one continuous image of the ray $[0, \infty)$ is a B-space.*

Proof. Let f be a one-to-one continuous mapping of $[0, \infty)$ into X. We put $P = f([0, \infty))$. There exists a point $q \in \overline{P} \setminus P$. Let Q be the (unique) simple arc with $f(0)$ and q as the end-points. Then $Q \subseteq \overline{P}$ (since $Q \cup \overline{P}$ is a connected closed subset of X and X is hereditarily unicoherent) and there exists a sequence of positive real numbers $a_1, a_2, \ldots$ such that $\lim_{n=\infty} a_n = \infty$ and $f(a_i) \in Q$ for any $i = 1, 2, \ldots$ (indeed, $q \in \overline{Q \setminus \{q\}} \subseteq \overline{P}$ and $q \notin \overline{f([0, a])}$), hence

$$q \in \overline{Q \setminus \{q\} \setminus f([0, a])} \subseteq \overline{Q \cap f((a, \infty))}$$

and there exists $a' > a$ such that $f(a') \in Q$ for any $a \geqslant 0$). Thus $Q \subseteq P$ and consequently $Q = \overline{P}$.

(1.12) Proposition. *Any arcwise connected hereditarily unicoherent compact Hausdorff space X is a B-space.*

Proof. Let f be a one-to-one continuous mapping of the ray $[0, \infty)$ into X. We shall show that the set $P = f([0, \infty))$ is not closed in X.

Otherwise P is a compact subspace of X. The set $P_n = f([0, n])$ is a closed subset of P for $n = 1, 2, \ldots$ Hence, it follows from Baire's theorem that there exists a positive integer N such that the interior I_N of P_N in P is a non-empty set. Similarly, for any $k = 0, 1, 2, \ldots$ there exists an integer $N > k$ such that the set

$$P_{k,N} = f([k, N])$$

has a non-empty interior $I_{k,N}$ in the subspace $\overline{f([k, \infty))}$, since

$$\overline{f([k, \infty))} = \left[f([0, k]) \cap \overline{f([k, \infty))}\right] \cup \bigcup_{n=k}^{\infty} P_{k,n}$$

and the interior of the set $f([0, k]) \cap \overline{f([k, \infty))}$ is empty in the subspace $\overline{f([k, \infty))}$. Then also

$$I_{k,N} \setminus f(\{0, 1, 2, \ldots\}) \neq \emptyset .$$

Evidently, $I_{k,n} \setminus \{f(k)\} \subseteq I_n$ (for $k < n$). Hence

$$\bigcup_{n=1}^{\infty} I_n \supseteq \bigcup_{k=0}^{\infty} \bigcup_{n=k+1}^{\infty} I_{k,n} \setminus f(\{0, 1, 2, \ldots\})$$

and

(1.13) $$\sup\{x \in R\colon f(x) \in \bigcup_{n=1}^{\infty} I_n\} = \infty .$$

Now we shall show that if $0 \leqslant a < N$ and $f(a) \in I_N$, then

$$f([0, a]) \subseteq I_N .$$

Indeed, $f([0, N]) \cap \overline{f([N, \infty))}$ is a connected subset of P, since X is hereditarily connected, and we have

$$f(a) \notin f([0, N]) \cap \overline{f([N, \infty))}$$

and

$$f(N) \in f([0, N]) \cap \overline{f([N, \infty))} .$$

Hence if $0 \leqslant b < a$, then

$$f(b) \notin f([0, N]) \cap \overline{f([N, \infty))}, \quad \text{i.e.} \quad f([0, a]) \subseteq I_N .$$

It follows from the above and from (1.13) that

$$\bigcup_{n=1}^{\infty} I_n = f([0, \infty)) = P ,$$

which is in contradiction to the compactness of P.

Thus, by Proposition (1.11), X is a B-space.

(1.14) Remark. Proposition (1.12), in the case of (metric) continua, is another formulation of the lemma from Section 3 of [1] (but our proof is not analogous to Borsuk's proof of that lemma).

(1.15) Example. *There exists a B-space which is not unicoherent.*

Indeed, let

$$p_n = \left(1, 1+\frac{1}{n}\right), \quad q_n = \left(2, \frac{1}{n}\right) \in \mathcal{R}^2, \quad n = 1, 2, \ldots$$

We put $\langle x, y\rangle$ for the closed segment with the end-points $x, y \in \mathcal{R}^2$. Then the subspace

$$X = \langle(0,0),(2,0)\rangle \cup \bigcup_{n=1}^{\infty} \langle(0,0), p_n\rangle \cup \bigcup_{n=1}^{\infty} \langle p_n, q_n\rangle$$

of the plane $\mathcal{R}^2$ is a B-space and X is not unicoherent, since it is the union of the connected closed subsets $\langle(0,0),(2,0)\rangle$ and

$$\bigcup_{n=1}^{\infty} \langle(0,0), p_n\rangle \cup \bigcup_{n=1}^{\infty} \langle p_n, q_n\rangle \cup \{(2,0)\},$$

and the intersection of these sets is not connected (since it is the two-point set $\{(0,0),(2,0)\}$).

(1.16) Proposition. *The cone CX over X is a B-space if and only if X is a T_1-space and any subset of X that is an image of a non-degenerate interval under a continuous mapping is a one-point set. In particular, if X is a Hausdorff space, then CX is a B-space if and only if X does not contain any non-degenerate arc. If the space X is also connected (and in addition not unicoherent), then CX is not hereditarily unicoherent. What is more, if in addition X is not unicoherent, then obviously even the base of CX is not unicoherent.*

The following example was presented to me by A. Lelek during a conversation.

(1.17) Example (A. Lelek). *There exists in $\mathcal{R}^3$ a compact B-space, which is not a hereditarily unicoherent continuum.*

Indeed, let K be Knaster's (plane) hereditarily indecomposable continuum. Then a homeomorphic image of CK is contained in $\mathcal{R}^3$ and it is a continuum which is not hereditarily unicoherent.

Instead of Knaster's continuum we can take any other plane continuum X, without any non-degenerate simple arc, which is not unicoherent. Then the cone CX has a base that is not unicoherent. But this cone is a compact B-space, topologically imbeddable in $\mathcal{R}^3$.

(1.18) Example. Let X be a space of power at least of the continuum, with the weakest T_1-topology. Then the cone over X is not a B-space and yet has the following property (Young [4]):

(1.19) *In the space X the union of every monotone increasing sequence of simple arcs is contained in a simple arc.*

Let us remark that an arcwise connected space is a B-space if and only if condition (1.19) and the following two conditions hold:

(1.20) *any simple arc in the space is a closed subset;*

(1.21) *any subset of the space which is a one-to-one continuous image of a simple arc is a simple arc in this space.*

§ 2. A game on B-space. Let X be a B-space and let $Q \subseteq X \times X$ be a set such that

(2.1) *$Q \cap (Z \times Z)$ is a closed subset of $Z \times Z$ for any arc Z in X,*

(2.2) $$Q \cap \Delta = \emptyset \quad (\text{i.e. } (x, x) \notin Q \text{ for any } x \in X).$$

If $(x, y) \notin Q$ for any $x, y \in \widetilde{a, b}$ (where $a, b \in X$), then we put $a d_Q b$. More generally, we put $A \in d_Q$, where A is a subset of X, if $(x, y) \notin Q$ for any $x, y \in A$. Evidently

(2.3) $$\textit{if } a d_Q b, \quad \textit{then} \quad (a, b) \notin Q.$$

We shall define the following two-person game with full information.

Rules: There are two players. They alternately pick points out of the space X. A sequence

(2.4) $$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots,$$

where a_i, b_i $(i = 1, 2, \ldots)$ denote points chosen by the first and second player respectively, will be called a *game*. The game (2.4) is considered to be won by the first player if for an integer i

(2.5) *there exists an $x \in \widetilde{b_i, b_{i+1}}$ such that $a_{i+1} \in \widetilde{a_i, x}$ and $a_i d_Q x$.*

We can say informally that the second player lost already in the $(i+1)$-move, so that the remaining moves are insignificant. The second player wins if there are no integers satisfying (2.5).

(2.6) Theorem. *The first player always has a winning strategy.*

More precisely, *there exist a point $a_1 \in X$ and a sequence of functions*

$$g_n \colon X^{2n} \to X, \quad n = 1, 2, \ldots$$

such that for any sequence (2.4) satisfying

(2.7) $$a_{n+1} = g_n(a_1, b_1, a_2, b_2, \ldots, a_n, b_n), \quad n = 1, 2, \ldots$$

there exists an integer i such that (2.5) holds true.

(2.8) Remark. We shall show a little more, namely that the first player has a winning strategy independly of the choice of a_1.

Proof. First we shall define g_1 and g_2. Let $x, y \in X$. If $x d_Q y$, then let

$$g_1(x, y) = y.$$

Next, if the relation xd_Qy does not hold, then there exists such a point y' that

(2.9) $$y' \in \overset{\frown}{x, y}\,,$$

(2.10) *the relation* xd_Qy' *does not hold*

(2.11) *if* $x' \in \overset{\frown}{x, y'}\backslash\{y'\}$ *then* xd_Qx'.

In this case let $g_1(x, y)$ be a point such that

(2.12) $$g_1(x, y) \in \overset{\frown}{x, y'}\backslash\{y'\} \quad \text{and} \quad g_1(x, y)\, d_Q y'\,.$$

The existence of such y' and $g_1(x, y)$ is a consequence of (2.1) and (2.2). We put also

(2.13) $$g_2(x, y, u, v) = \begin{cases} v & \text{if} \quad ud_Qv\,, \\ y' & \text{if} \quad u = g_1(x, y) \text{ and } \overset{\frown}{u, v} \notin d_Q\,, \\ y & \text{otherwise} \end{cases}$$

for any $u, v \in X$. Now g_{2n+1} and g_{2n+2}, $n = 1, 2, \ldots$, are defined by

(2.14) $$g_{2n+1}(x_1, y_1, \ldots, x_{2n+1}, y_{2n+1}) = g_1(x_{2n+1}, y_{2n+1})$$

and

$$g_{2n+2}(x_1, y_1, \ldots, x_{2n+2}, y_{2n+2}) = g_2(x_{2n+1}, y_{2n+1}, x_{2n+2}, y_{2n+2})\,.$$

Let a_1 be an arbitrarily chosen point of X and let (2.4) be a sequence such that (2.7) holds. We shall show that condition (2.5) holds for a certain positive integer i.

Otherwise the relation $a_id_Qb_i$ does not hold (since if $a_id_Qb_i$, then $a_{i+1} = b_i$) and, by (2.11)-(2.14),

(2.15) $$a_i d_Q a_{i+1}$$

and, by (2.10) and (2.13),

(2.16) *the relation* $a_{2i-1}d_Qa_{2i+1}$ *does not hold for any* $i = 1, 2, \ldots$

Thus it follows from (2.15) that

(2.17) $$\overset{\frown}{a_i, a_{i+1}} \cap \overset{\frown}{b_i, b_{i+1}} = \emptyset \quad \text{for any } i = 1, 2, \ldots$$

Let us remark that a_{2i} and a_{2i+1} belong to $\overset{\frown}{a_{2i-1}, b_{2i-1}}$, and $a_{2i} \in \overset{\frown}{a_{2i-1}, a_{2i+1}}$ (see (2.9), (2.12) and (2.13)). We have also $a_{2i+1} \in \overset{\frown}{a_{2i}, b_{2i}}$.

Indeed, by condition (2.5) of winning and (2.11)

$$\overset{\frown}{b_{2i-1}, b_{2i}} \cap (\overset{\frown}{a_{2i}, a_{2i+1}}\backslash\{a_{2i+1}\}) = \emptyset\,.$$

Hence, for a point $p \in \overset{\frown}{a_{2i}, a_{2i+1}}\backslash\{a_{2i}, a_{2i+1}\}$ the points b_{2i-1}, b_{2i} and a_{2i+1} belong to the same component and the point a_{2i} to different component of arcwise connectivity in the subspace $X\backslash\{p\}$, whence $p \in \overset{\frown}{a_{2i}, b_{2i}}$. Thus $a_{2i+1} \in \overset{\frown}{a_{2i}, b_{2i}}$. This show that $a_{i+1} \in \overset{\frown}{a_i, b_i}$ for any $i = 1, 2, \ldots$

Thus, by (2.17), a_i does not belong to the component of arcwise connectivity of b_{i+1} in the subspace $X\backslash\{a_{i+1}\}$ ($a_{i+1} \neq b_i$ since $a_id_Qa_{i+1}$ but not $a_id_Qb_i$). Hence

$$(\overset{\frown}{a_i, a_{i+1}}\backslash\{a_{i+1}\}) \cap (\overset{\frown}{a_{i+1}, b_{i+1}}\backslash\{a_{i+1}\}) = \emptyset$$

and, since $\overset{\frown}{a_{i+1}, a_{i+2}} \subseteq \overset{\frown}{a_{i+1}, b_{i+1}}$, we obtain

(2.18) $$\overset{\frown}{a_i, a_{i+1}} \cap \overset{\frown}{a_{i+1}, a_{i+2}} = \{a_{i+1}\}\,.$$

Furthermore, we have

(2.19) $$\overset{\frown}{a_1, a_i} \cap \overset{\frown}{a_i, a_{i+1}} = \{a_i\} \quad \text{for any } i = 1, 2, \ldots$$

Indeed, if $i = 1$, then (2.19) is trivial. Thus let us assume that (2.19) is true for $i = k$. Then we shall show that

$$\overset{\frown}{a_1, a_{k+1}} \cap \overset{\frown}{a_{k+1}, a_{k+2}} = \{a_{k+1}\}\,.$$

Indeed, by (2.18)

$$(\overset{\frown}{a_1, a_k} \cup \overset{\frown}{a_{k+1}, a_{k+2}}) \cap \overset{\frown}{a_k, a_{k+1}} = \{a_k, a_{k+1}\}\,.$$

Hence, by Proposition (1.2),

$$\overset{\frown}{a_1, a_k} \cap \overset{\frown}{a_{k+1}, a_{k+2}} = \emptyset\,.$$

Thus

$$\overset{\frown}{a_1, a_{k+1}} \cap \overset{\frown}{a_{k+1}, a_{k+2}} \subseteq (\overset{\frown}{a_1, a_k} \cup \overset{\frown}{a_k, a_{k+1}}) \cap \overset{\frown}{a_{k+1}, a_{k+2}} = \{a_{k+1}\}\,.$$

Equality (2.19) is proved.

Thus the set $P = \bigcup\limits_{n=1}^{\infty} \overset{\frown}{a_n, a_{n+1}}$ is a one-to-one continuous image of the ray $[0, \infty)$ and $\overline{P}$ is a non-degenerate simple arc.

Let $q \in \overline{P}\backslash P$. Then pd_Qq for some $p \in P$ and there exists a positive integer i such that a_{2i-1} and a_{2i+1} belong to $\overset{\frown}{p, q}$, which contradicts (2.16). Theorem (2.6) is proved.

(2.20) *The first player has a winning strategy also if the game* (2.4) *is considered to be won by him if for an integer* i *either*

$$\overset{\frown}{a_i, b_i}\backslash\{a_i, b_i\} \in d_Q$$

or

$$\overset{\frown}{a_i, a_{i+1}}\backslash\{a_i, a_{i+1}\} \in d_Q \quad \text{and} \quad \overset{\frown}{a_i, a_{i+1}} \cap \overset{\frown}{b_i, b_{i+1}} \neq \emptyset\,.$$

The proof of (2.20) is analogous to the proof of Theorem (2.6).

(2.21) **Remark.** It is easy to see that Theorem (2.6) holds also if for the space X conditions (1.19) and (1.20) are satisfied (cf. Remark (1.18)).

We shall now proceed to a description of a discrete version of our game. (This part of the paper is not essential for the understanding of the whole.) We shall call *a graph* the ordered pair (V, G) where V is a non-empty set and G is a family of pairs (not ordered) of different elements of V. The graph (V, G) is called a (generalized) *tree* if it is connected (i.e. for any two distinct points $a, b \in V$ there is a finite sequence $x_0 = a, x_1, x_2, \ldots, x_n = b$ such that $\{x_i, x_{i+1}\} \in G$ for $i = 0, 1, 2, \ldots, n-1$) and V does not contain an infinite sequence $a_1, a_2, \ldots \in V$ such that $\{a_i, a_{i+2}\} \in G$ and $a_i \neq a_{i+2}$ for any $i = 1, 2, \ldots$

The players choose vertices of the graph alternately, one at a time so that again we obtain a sequence of the form (2.4) which we call a game. The first player wins if for an integer i we have

(i) $a_i = b_i$ or

(ii) $\{a_i, a_{i+1}\} \in G$ and the vertices b_i, b_{i+1} are contained in the same component and for any sequence

$$x_0 = b_i, x_1, x_2, \ldots, x_n = b_{i+1}$$

such that $\{x_k, x_{k+1}\} \in G$ for $k = 0, 1, 2, \ldots, n-1$ we have

$$\{a_i, a_{i+1}\} = \{x_l, x_{l+1}\}$$

for some $l \in \{0, 1, 2, \ldots, n-1\}$.

In the opposite case the second player wins.

(2.21) THEOREM. *The first player has a winning strategy if and only if the graph (V, G) is a tree.*

§ 3. Fixed point properties of B-space.

The following theorem is a generalization of Borsuk's Theorem 2 from [1] and Young's Theorem from [4] (see Corollary (3.3)).

(3.1) THEOREM. *Every continuous mapping $f\colon X \to X$ of a B-space into itself such that*

$$f(\overset{\frown}{x), f(}y) \subseteq f(\overset{\frown}{x, y}) \tag{3.2}$$

for any $x, y \in X$ has a fixed point.

If X is a Hausdorff B-space, then condition (3.2) is satisfied automatically. Hence we obtain

(3.3) COROLLARY (Young [4]). *Every continuous mapping of a Hausdorff B-space into itself has a fixed point.*

Now, let $f\colon X \to X$ be a continuous mapping, where X is a B-space, and let

$$Q = \{(x, y) \in X \times X\colon\ y = f(x)\}.$$

Then the set

$$Q \cap (A \times A) = \{(x, y) \in A \times A\colon\ y = f(x)\}$$
$$= \{(x, y) \in (f^{-1}(A) \cap A) \times A\colon\ y = f(x)\}$$

is, by Proposition (1.5), a closed subset of $A \times A$ for any simple arc A in X. Hence (2.1) holds.

Theorem (3.1) is a simple consequence of the following more general one

(3.4) THEOREM. *Let X be a B-space and let $Q \subseteq X \times X$ be a non-empty relation such that conditions* (2.1) *and*

(3.5) *there exists a function $h\colon Q \times X \to X$ such that $(a', h(a, b, a')) \in Q$ and $\overset{\frown}{b, h}(a, b, a') \subseteq \bigcup_{p \in \overset{\frown}{a, a'}} \{q\colon (p, q) \in Q\}$ for any $(a, b) \in Q$ and $a' \in X$,*

are satisfied. Then there exists an $x \in X$ such that $(x, x) \in Q$ (i.e. $Q \cap \Delta \neq \emptyset$).

(3.6) Remark. It follows from (3.5) that for any $x \in X$ there exists an $y \in X$ such that $(x, y) \in Q$.

Proof. The following condition is a consequence of (3.5):

(3.7) *if $x \in \overset{\frown}{b, h}(a, b, a')$ and $a' \in \overset{\frown}{a, x}$, then*

$$(\overset{\frown}{a, x} \times \overset{\frown}{a, x}) \cap Q \neq \emptyset, \quad \textit{where} \quad (a, b) \in Q \textit{ and } a' \in X.$$

Indeed, if (3.5) holds and $x \in \overset{\frown}{b, h}(a, b, a')$ and $a' \in \overset{\frown}{a, x}$, then $(p, x) \in Q$ for some $p \in \overset{\frown}{a, a'}$, whence $(p, x) \in (\overset{\frown}{a, x} \times \overset{\frown}{a, x}) \cap Q$.

Now if (2.2) holds, then let $(a_1, b_1) \in Q$, $a_{n+1} = g_n(a_1, b_1, \ldots, a_n, b_n)$, as in Theorem (2.6), and $b_{n+1} = h(a_n, b_n, a_{n+1})$, $n = 1, 2, \ldots$ Then (2.5) holds for a positive integer i, in contradiction to (3.7). This proves Theorem (3.4).

(3.8) Remark. It is easy to see that in Theorem (3.4) condition (3.5) can be replaced by the following weaker condition:

There exists a function $h\colon Q \times X \to X$ such that $(a', h(a, b, a')) \in Q$ for any $(a, b) \in Q$, $a' \in X$, and condition (3.7) *holds.*

Consequently also in Theorem (3.1) condition (3.2) can be replaced by the following weaker one:

If $x \in f(\overset{\frown}{a), f(}a')$ and $a' \in \overset{\frown}{a, x}$, then $(\overset{\frown}{a, x} \times \overset{\frown}{a, x}) \cap Q \neq \emptyset$, for any $a, a' \in X$.

But it is not true that any B-space has the fixed-point property; in § 5 we shall give an example of a B-space which does not possess this property.

We say that a metric space (X, d) is the *space of a graph* (V, G) (see Section 2) if

(a) $V \subseteq X$,

(b) for any set $\{a, b\} \in G$ there exists in X exactly one simple arc $\overset{\frown}{a, b}$ with end-points a, b,

(c) any arc $\widetilde{a, b}$, where $\{a, b\} \in G$, is isometric to the closed interval $I = [0, 1]$,

(d) $X = \bigcup_{\{a,b\} \in G} \widetilde{a, b}$ and $\widetilde{a, b} \cap \widetilde{c, d} \subseteq \{a, b\}$ for any $\{a, b\}, \{c, d\} \in G$,

(e) the metric d is, under the conditions (c) and (d), the greatest metric such that $d(a, b) = 1$ for any different $a, b \in V$.

Evidently the metric space of a graph is a 1-dimensional space. If a graph is not connected, then the space of that graph does not possess the fixed-point property (since it is not connected). Next, if a graph is connected but is not a tree, then it contains, as a closed subset, a homeomorphic image of a circle or of the ray $[0, \infty)$ and this set is a retract of the space of the graph. Hence this space does not possess the fixed-point property. Thus from Theorem (2.6), since the space of a tree is a B-space, we obtain

(3.9) THEOREM. *The space of a (generalized) graph has the fixed-point property if and only if that graph is a tree.*

(3.10) Remark. In [4] it has been proved that the spike of a denumerable family of closed intervals (in a Hilbert space) has the fixed-point property (this is a very special case of Theorem (3.9)). In respect of generality, the simple result contained in Proposition 7 of [2] cannot be compared with Theorem (3.1) or even (3.4).

(3.11) Remark. One can verify that Theorem (3.4) and the other results of this paragraph hold if we replace the condition that the arcwise connected space X is a B-space by two conditions, (1.19) and (1.20) (see Remark (2.21)).

§ 4. Continuous mappings of compact Hausdorff spaces onto T_1-spaces. Let us remark that

(4.1) THEOREM. *If $f\colon X \to X$ is a continuous mapping of a B-space into itself such that for any simple arc A in X:*

(4.2) *there exist a Hausdorff space Y_A and continuous mapping $g_A\colon A \to Y_A$, of A onto Y_A, and a one-to-one continuous mapping $h_A\colon Y_A \to f(A)$ of Y_A onto $f(A)$ such that $f|A = h_A \circ g_A$,*

then f has a fixed point.

Proof. If $x, y \in f(A)$, then there exists a simple arc Z in Y_A with the end-points $h_A^{-1}(x)$, $h_A^{-1}(y)$. Hence, by (1.5), $h_A(Z) \subseteq f(A)$ is a simple arc in X which contains x and y. Thus condition (3.2) holds and, by (3.1), Theorem (4.1) is true.

Thus it is interesting to find out when the following proposition is true for a continuous mapping $f\colon X \to Y$ of a compact Hausdorff space onto a T_1-space Y:

(4.3) *there are a continuous mapping $g\colon X \to Z$ onto a Hausdorff space Z and a one-to-one continuous mapping $h\colon Z \to Y$ such that $f = h \circ g$.*

The answer is contained in the following theorem:

(4.4) THEOREM. *Let $f\colon X \to Y$ be a continuous mapping of a Hausdorff compact space X onto a T_1-space Y. Then (4.3) holds if and only if*

(4.5) $\quad f^{-1}(f(F))$ *is a closed subset of X for any closed $F \subseteq X$.*

Proof. Evidently, (4.3) implies (4.5). Hence, let us assume that condition (4.5) is satisfied. Then let $h\colon Z \to Y$ be a one-to-one function of a set Z onto Y, $Z \neq Y$. We shall introduce a topology in Z by a basis of open sets. A set G belongs to this basis if and only if $G = Z \setminus h^{-1} \circ f(F)$ for a closed subset F of X; if also $G' = Z \setminus h^{-1} \circ f(F')$, where F' is closed n X, then

$$G \cap G' = Z \setminus [h^{-1} \circ f(F) \cup h^{-1} \circ f(F')] = Z \setminus h^{-1} \circ f(F \cup F')$$

s a set of the base. Next if H is an open subset of Y, then

$$h^{-1}(H) = Z \setminus h^{-1} \circ f(F),$$

where $F = f^{-1}(Y \setminus H)$ is a closed subset of X. Thus h is continuous and consequently Z is a T_1-space.

Obviously, we must have $g = h^{-1} \circ f\colon X \to Z$, and such a g is a mapping onto. Let F be a closed subset of X. Then

$$g^{-1}\big(Z \setminus h^{-1} \circ f(F)\big) = X \setminus f^{-1}\big(f(F)\big)$$

is an open subset of X. Thus the mapping g is continuous.

Finally let $x, y \in Z$ be a pair of different points. Then $g^{-1}(x), g^{-1}(y)$ are disjoint closed subset of X. Let $G \supseteq g^{-1}(x)$ and $H \supseteq g^{-1}(y)$ are disjoint open subset of X. Then

$$U = Z \setminus h^{-1} \circ f(X \setminus G) = Z \setminus g(X \setminus G) \text{ and } V = Z \setminus h^{-1} \circ f(X \setminus H) = Z \setminus g(X \setminus H)$$

are neighbourhoods of x and y respectively, and

$$U \cap V = Z \setminus \big(g(X \setminus G) \cup g(X \setminus H)\big) = Z \setminus g\big((X \setminus G) \cup (X \setminus H)\big) = \emptyset .$$

The theorem is proved.

(4.6) COROLLARY. *If a mapping $g'\colon X \to Z'$ of a compact Hausdorff space X onto a Hausdorff space Z' is not a continuous mapping, but $g'|X_0$ is continuous for a certain closed subset X_0 of X such that $g'(X_0) = Z'$ (especially, if $Z' = X_0 = \overline{X}_0 \subseteq X$ and $g'\colon X \to X_0$ is a non-continuous retraction), and $h'\colon Z' \to Y$ is a one-to-one continuous mapping of Z' onto a T_1-space Y, then (4.3) does not hold for $f = h' \circ g'$.*

Proof. Let F' be a closed subset of Z' such that $g'^{-1}(F')$ is not a closed subset of X. But $F = g'^{-1}(F') \cap X_0$ is a closed subset of X and

$$f^{-1}\big(f(F)\big) = g'^{-1}\big(h'^{-1}\big(h' \circ g'(F)\big)\big) = g'^{-1}\big(g'(F)\big) = g'^{-1}(F')$$

is not a closed subset of X. Thus, by Theorem (4.4), condition (4.3) is not satisfied.

(4.7) Example. Let $r\colon [0,2]\to[0,1]$ be a retraction, not continuous and such that $r^{-1}(x)$ is a finite set for any $x \in [0,1]$. Next let $h'\colon [0,1]\to Y$ be a one-to-one mapping of $[0,1]$ onto a T_1-space Y with the minimal T_1-topology (closed subsets of Y are finite). Then $f = h' \circ r$ is a continuous mapping of $[0,2]$ onto the T_1-space Y such that condition (4.3) is not satisfied.

Now we introduce a generalization of the notion of arc-wise connectivity.

(4.8) Definition. A space X is said to be *weakly arcwise connected* if and only if for any different points $x, y \in X$ there exists a one-to-one continuous mapping $f\colon [0,1]\to X$ such that $f(0) = x$ and $f(1) = y$.

For example, if X is a T_1-space with the weakest T_1-topology and the power of X is at least the continuum, then X is a weakly arcwise connected space which is not an arcwise connected one. Obviously any arcwise connected space is weakly arcwise connected.

(4.9) Let $Y = [0,1) \cup \{a,b\}$, where $a \neq b$ and $\{a,b\} \cap [0,1) = \emptyset$, and let a mapping $g\colon Y\to[0,1]$ be given by

$$g(x) = \begin{cases} x & \text{if} \quad x \in [0,1)\,, \\ 1 & \text{if} \quad x = a \text{ or } x = b\,. \end{cases}$$

Let, by definition, a subset G be open in Y if and only if $f(G)$ is an open subset of $[0,1]$, for the usual topology of the segment $[0,1]$. Hence Y is a T_1-space (and g is a continuous mapping). The space Y is an image of the closed segment $[-1,1]$ under the continuous mapping $f\colon [-1,1]\to Y$

$$f(x) = \begin{cases} a & \text{if} \quad x = -1\,, \\ |x| & \text{if} \quad -1 < x < 1 \\ b & \text{if} \quad x = 1\,, \end{cases}$$

but there exists no one-to-one continuous mapping $h\colon A\to Y$ of a simple arc A into Y such that $a, b \in h(A)$. Thus T_1-space Y is a continuous image of a simple arc, which is not a weakly arcwise connected space. This seems to be the simplest example of this kind.

Now we give an extremal example of this type.

(4.10) Example. We shall give an example of a T_1-space X, containing at least two different points, such that X will be a continuous image of a simple arc but X will not contain any one-to-one image of the segment $[0,1]$.

Let $f'\colon I\to X'$ be a light continuous mapping of the unit interval $I = [0,1]$ onto a dendrite X' such that the following two conditions hold:

(4.11) *If x', y' are different points of X', then there exists a point $z \in X'$ of order 3* [1] *such that for every $p \in f'^{-1}(z)$ there exists an $\varepsilon_0 > 0$ such that the set $f'\big((p-\varepsilon, p+\varepsilon)\big)$ is disjoint with exactly one component of connectivity of x' or y' in the subspace $X'\setminus\{f'(p)\}$ of X', for any positive $\varepsilon < \varepsilon_0$.*

(4.12) *The set X_3' of all the points of order 3 in X is countable.*

Let $X = (X'\setminus X_3') \cup f'^{-1}(X_3')$ (we assume that $X' \cap I = \emptyset$). Thus we can define the mappings $f\colon I\to X$ and $g\colon X\to X'$ as follows:

$$f|f'^{-1}(X'\setminus X_3') = f'|f'(X'\setminus X_3') \quad \text{and} \quad f|f'^{-1}(X_3') \text{ is an identity,}$$

$$g|f'^{-1}(X_3') = f'|f'^{-1}(X_3') \quad \text{and} \quad g|X'\setminus X_3' \text{ is an identity}\,.$$

These mappings are onto and $f' = g \circ f$. Obviously, $f^{-1}(x)$ is a closed subset of I for any $x \in X$.

Let us consider X with the strongest topology such that $f\colon I\to X$ will be continuous. Under such a topology X is a T_1-space and $g\colon X\to X'$ is a continuous mapping.

We shall show that

(4.13) *if a connected subset S of X contains at least two different points, then there are $p, q \in I \cap S$ such that $f'(p) = f'(q) \in X_3'$ and for any positive ε the set*

$$f'\big((p-\varepsilon, p+\varepsilon) \cup (q-\varepsilon, q+\varepsilon)\big)$$

has common points with any of the components of $X'\setminus\{f'(p)\}$, but

$$f'\big((p-\varepsilon_0, p+\varepsilon_0)\big)\,, \quad \textit{as well as} \quad f'\big((q-\varepsilon_0, q+\varepsilon_0)\big)\,,$$

is disjoint with a component of $X'\setminus\{f'(p)\}$.

Indeed, because f' is light, $g(S)$ has at least two different points $x' = g(x)$, $y' = g(y)$, where $x, y \in S$. Let z be a point of X' as in (4.11) and let C_x', C_y', C' be the components of x', y' and the remaining component of $X'\setminus\{z\}$. The sets C_x', C_y' and C' are open in X'. Thus, by the definition of topology in X, also the sets

$$C_x = g^{-1}(C_x') \cup \big\{p \in I \cap X\colon f'\big((p-\varepsilon, p+\varepsilon)\big) \subseteq C_x' \cup \{z\} \text{ for some } \varepsilon > 0\big\}\,,$$

$$C_y = g^{-1}(C_y') \cup \big\{p \in I \cap X\colon f'\big((p-\varepsilon, p+\varepsilon)\big) \subseteq C_z' \cup \{z\} \text{ for some } \varepsilon > 0\big\}\,,$$

[1] A point p of a dendrite D *has order k* if the subspace $D\setminus\{p\}$ has k different components.

$$D_x = g^{-1}(C'_x \cup C') \cup \{p \in I \cap X \colon f'((p-\varepsilon, p+\varepsilon)) \subseteq C'_x \cup C' \cup \{z\} \text{ for some } \varepsilon > 0\},$$

$$D_y = g^{-1}(C'_y \cup C') \cup \{p \in I \cap X \colon f'((p-\varepsilon, p+\varepsilon)) \subseteq C'_y \cup C' \cup \{z\} \text{ for some } \varepsilon > 0\}$$

are open. We have

$$C_x \cap D_y = C_y \cap D_x = \emptyset$$

and

$$D_x \cap S \supseteq C_x \cap S \neq \emptyset \neq C_y \cap S \subseteq D_y \cap S,$$

whence

(4.14) $$S \setminus (C_x \cup D_y) \neq \emptyset \neq S \setminus (C_y \cup D_x).$$

The last inequalities prove (4.13).

Now let $S = h([a, b])$, where $h\colon [a, b] \to X$ is a non-constant continuous mapping of an interval $[a, b]$ into X; we can assume, without loss of generality, that $x = h(a)$ and $y = h(b)$ are the different points. Then there exist points $p, q \in I \cap S$, as in (4.13). By (4.14) we can assume that

$$p \in S \setminus (C_x \cup D_y) \subseteq D_x \quad \text{and} \quad q \in S \setminus (C_y \cup D_x).$$

But $D_x \cup D_y = X$, whence $S \cap D_x \cap D_y \neq \emptyset$ is an open proper of subspace S of X. Thus

$$g^{-1}(z) \cap S \subseteq S \cap D_x \cap D_y \not\subseteq g^{-1}(z) \cap S$$

since $g^{-1}(z) \cap S$ is a closed subset of S. Hence there exists a point $u \in S \cap D_x \cap D_y \setminus g^{-1}(z)$, i.e. a point $u \in S$ such that $g(u) \in C'$. We have $u = h(c)$ for some c, where $a < c < b$. The power of the set

$$g \circ h([a, c]) \cap g \circ h([c, b]) \supseteq \widetilde{g(u), z}$$

is the continuum. By (4.12) the set X'_3 is countable. Hence, by definition of g, the power of the set $h([a, c]) \cap h([c, b])$ is also equal to the continuum and consequently h is not a one-to-one mapping. This proves that the space X possesses the required properties in (4.10). It remains to give a concrete example of a light continuous mapping $f'\colon I \to X'$, where $X' = f'(I)$ is a dendrite such that conditions (4.11) and (4.12) are satisfied for this mapping.

Let l^2 be, as usual, the Hilbert space and let

$$E_n = \{(x_1, x_2, \ldots) \in l^2 \colon x_i = 0 \text{ for } i > n\}$$

and

$$e_n = (\delta_1^n, \delta_2^n, \ldots)$$

where

$$\delta_i^n = \begin{cases} 0 & \text{if } i \neq n, \\ 1 & \text{if } i = 1. \end{cases} \quad \text{where} \quad n = 1, 2, \ldots$$

We shall define a sequence of polygonal trees $X_n \subseteq E_n$ and a sequence of simplicial mappings onto $f_n\colon I \to X_n$, $n = 1, 2, \ldots$, as follows:

$$X_1 = \{(x_1, x_2, \ldots) \in E_1 \colon 0 \leqslant x_1 \leqslant 1\}, \qquad f_1(t) = (t, 0, 0, \ldots)$$

for $t \in I$.

Next, let

$$0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = 1$$

be a sequence such that $f_n(t)$ is an end-point or point of ramification of a tree $X_n \subseteq E_n$ if and only if $t = t_i^n$ for some $i = 0, 1, \ldots, k(n)$. Then we put

$$f_{n+1}(t) = \begin{cases} f_n\left(t_i^n + \frac{3}{2}(t - t_i^n)\right) & \text{if } t_i^n \leqslant t \leqslant \frac{2}{3} t_i^n + \frac{1}{3} t_{i+1}^n \\ f_n\left(\frac{t_i^n + t_{i+1}^n}{2}\right) + \frac{3}{2}\left(t - \frac{2}{3} t_i^n - \frac{1}{3} t_{i+1}^n\right) e_{n+1} & \\ & \text{if } \frac{2}{3} t_i^n + \frac{1}{3} t_{i+1}^n \leqslant t \leqslant \frac{t_i^n + t_{i+1}^n}{2} \\ f_n\left(\frac{t_i^n + t_{i+1}^n}{2}\right) + \frac{3}{2}\left(\frac{t_{i+1}^n - t_i^n}{6} - t\right) e_{n+1} & \\ & \text{if } \frac{t_i^n + t_{i+1}^n}{2} \leqslant t \leqslant \frac{1}{3} t_i^n + \frac{2}{3} t_{i+1}^n \\ f_n\left(t_{i+1}^n + \frac{3}{2}(t - t_{i+1}^n)\right) & \text{if } \frac{1}{3} t_i^n + \frac{2}{3} t_{i+1}^n \leqslant t \leqslant t_{i+1}^n \end{cases}$$

for $n = 1, 2, \ldots$, $i = 0, 1, \ldots, k(n) - 1$, and

$$X_{n+1} = f_{n+1}(I).$$

It is easy to see that

(i) $$t_i^n = t_{4i}^{n+1},$$

(ii) $$f_n(t_i^n) = f_{n+1}(t_i^n) = f_{n+2}(t_i^n) = \ldots,$$

(iii) $$k(n) = 4^{n-1},$$

(iv) $f_n(t_i^n)$ *is an end-point of* X_n *if* i *is even*,

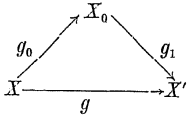
(v) $f_n(t_i^n)$ *is a ramification point of order* 3 *if* i *is odd and* $n > 1$.

The sequence $X_1, X_2, \ldots$ is increasing and the sequence of mappings $f_1, f_2, \ldots$ is uniformly converging to a continuous mapping f' defined on I.

We put $X' = f'(I)$. Then $X' = \overline{\bigcup_{n=1}^{\infty} X_n}$. Evidently $f'(t_i^n) = f_n(t_i^n)$ for

$n = 1, 2, \ldots, i = 0, 1, \ldots, 4^{n-1}$ and $f'(t)$ is a point of order 3 in X' if and only if $t = t_i^n$ for an integer $n > 1$ and an odd i such that $0 < i < 4^{n-1}$. (If i is even, then $f'(t_i^n)$ is an end-point in X'. Also if $x \in X' \setminus \bigcup_{n=1}^{\infty} X_n$, then x is an end-point. The other points of X' are the points of order 3). Thus X_3' is a countable set. One can easily verify that also condition (4.11) is satisfied.

(4.15) Remark. If X' and f': $I \to X'$ are constructed as above, then $g^{-1}(x)$ is a finite set for any $x \in X_3'$. The mapping g: $X \to X'$ can be factored,



$g = g_1 \circ g_0$, $g_0(X) = X_0$, in such a way that $g_1^{-1}(x)$ is a two-point set for any $x \in X_3'$, and X_0 has the required properties of X from (4.10).

Next, one can verify that, in this case, f is an open mapping of I onto X, whence X has a countable base of open sets.

§ 5. B-spaces without the fixed-point property. Let f: $I \to X$ be a continuous mapping of I onto a T_1-space X such that any one-to-one image in the X of a simple arc is a one-point set, and let $a = f(0)$ and $b = f(1)$ be different points (see Example (4.10)). Then the T_1-space Y obtained from the space $X \times \{0, 1\}$ by the identification of the point (a, i) with $(b, 1-i)$, for $i = 0, 1$, is a continuous image of I under the following mapping g: $I \to Y$:

$$g(t) = \begin{cases} (f(2t), 0) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ (f(2t-1), 1) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Any one-to-one image in the Y of a simple arc is also a one-point set. Next, the mapping h: $Y \to Y$ given by

$$h((x, i)) = (x, 1-i) \quad \text{for} \quad x \in Y, \; i = 0 \text{ or } 1$$

is a homeomorphism of Y onto itself without the fixed-point property. We can assume that

$$Y \subseteq \{(x_1, x_2) \in \mathcal{R}^2\colon x_2 = 1\}$$

(obviously, Y is not a topological subspace of $\mathcal{R}^2$). Let

$$Z = \{t \cdot x\}_{x \in Y,\, t \in I}$$

be a space with a topology such that a set $F \subseteq Z$ is closed in Z if and only if $F \cap Y$ and $F \cap \{t \cdot x\}_{t \in I}$ are closed subsets of Y (with the T_1-topology defined above) and of the segment $\{t \cdot x\}$, $t \in I$, (with the usual

Euclidean topology) respectively, for any $x \in Y$. Then Z is a B-space since any one-to-one continuous image of the ray $[i, \infty)$ in Z is contained in a set of the form

$$\{t \cdot x\}_{t \in I} \cup \{t \cdot y\}_{t \in I}, \quad x, y \in Y.$$

Space Y is a retract of Z. The retraction r: $Z \to Y$ can be defined as follows

$$r(t \cdot x) = g(t \cdot g'(x)), \quad \text{for} \quad t \in I, \; x \in Y,$$

where g': $Y \to I$ is an arbitrarily chosen function such that $g \circ g'(x) = x$. Evidently, r is continuous and $r(x) = x$ for $x \in Y$. Thus Z is a B-space without the fixed-point property. Loosely speaking, the space Z, as well as the space X from Example (4.10), is a 1-dimensional space. The space Z is not contractible.

§ 6. Topological squares without the fixed-point property. The following theorem, combined with Young's Theorem (3.1), is a generalization of the concrete result of the paper [4].

(6.1) THEOREM. *If an arcwise connected Hausdorff space X contains an infinite discrete closed subspace and X^2 is a normal space, then X^2 does not possess the fixed-point property.*

Proof. If X contains an infinite discrete closed subspace, then it contains an infinite countable discrete closed subspace $A = \{a_1, a_2, \ldots\}$. Let $\widetilde{a_n, a_{n+1}}$ be a simple arc with a_n and a_{n+1} as end-points, $n = 1, 2, \ldots$ We put

$$P = \bigcup_{n=1}^{\infty} \widetilde{a_{2n-1}, a_{2n}} \times \{a_{2n-1}\} \cup \bigcup_{n=1}^{\infty} \{a_{2n}\} \times \widetilde{a_{2n-1}, a_{2n}}.$$

Obviously P is a closed subset of the space $X \times X$ and P is homeomorphic to the ray $[0, \infty)$. Hence, P is a retract of $X \times X$ and consequently $X \times X$ does not possess the fixed point property.

Thus, we have obtained a large class of spaces with the fixed-point property whose squares do not have this property. This is the class of paracompact B-spaces, which contain infinite discrete subspace.

§ 7. Some examples. Now we shall give some examples of "good spaces" without the fixed-point property.

(7.1) EXAMPLE. There exists a 1-dimensional unicoherent plane continuum X without the fixed-point property for homeomorphisms into itself.

Indeed, we can put

$$X = \{(x, y) \in \mathcal{R}^2\colon x^2 + y^2 = 1\} \cup \cup \left\{\left((1+t)\cos\frac{\pi}{t}, (1+t)\sin\frac{\pi}{t}\right)\colon 0 < t \leqslant 1\right\}.$$

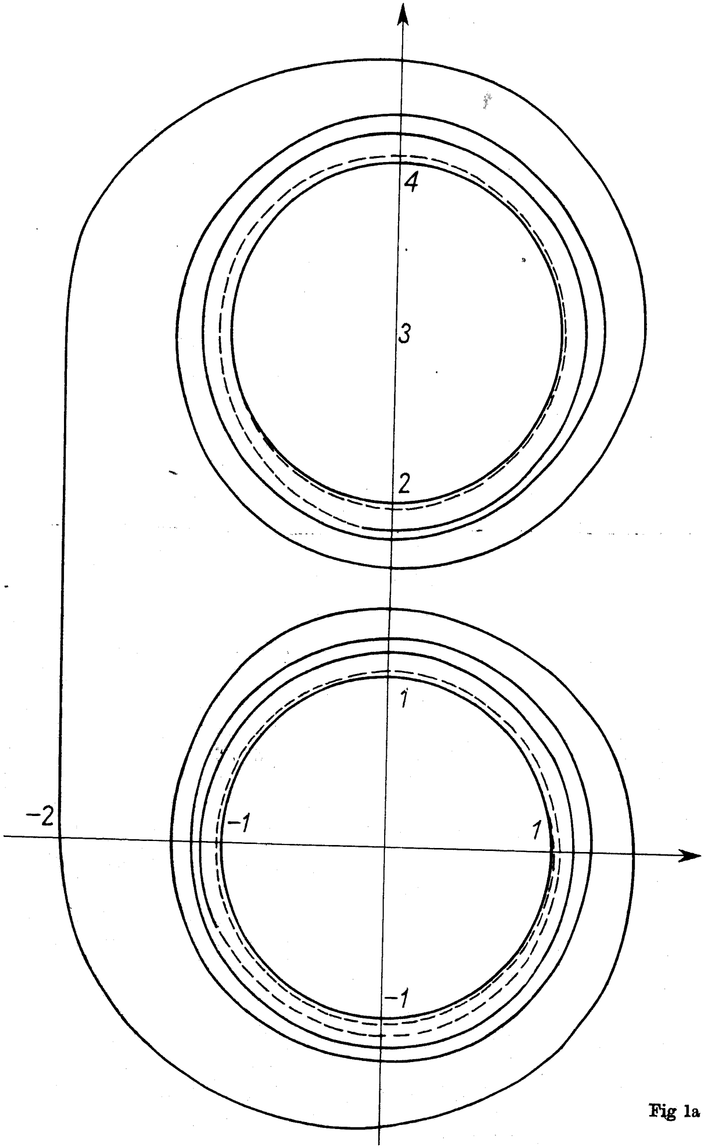


Fig 1a

xt, let

$$= X \cup \{(x, y) \in \mathcal{R}^2:\ x = -2 \text{ and } 0 \leqslant y \leqslant 3\} \cup \cup \{(x, y) \in \mathcal{R}^2:\ (x, 3-y) \in X\}.$$

Y is a 1-dimensional unicoherent plane continuum (see Fig. 1a) there exists a homeomorphism of Y onto itself without any fixed t

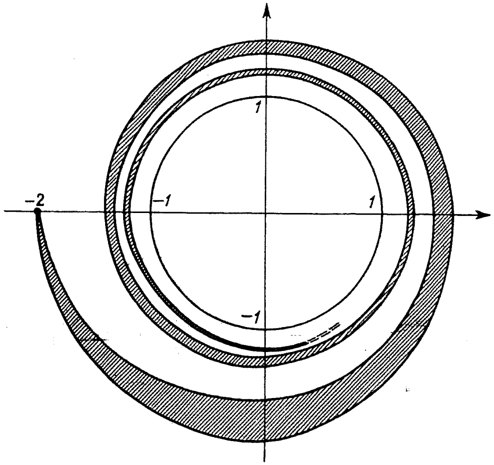


Fig. 1b

For the unicoherent plane continuum Z from Fig. 1b there also exists a homeomorphism of Z onto itself. This continuum is defined as follows:

$$Z = \{(x, y) \in \mathcal{R}^2:\ (x\cos t - y\sin t,\ x\sin t + y\cos t) \in X \text{ for some } t \text{ such that } 0 \leqslant t \leqslant \pi/2\}.$$

Evidently the spirals

$$S = \left\{\left((1+t)\cos\frac{\pi}{t},\ (1+t)\sin\frac{\pi}{t}\right):\ 0 < t \leqslant 1\right\}$$

and

$$S' = \{(x, y) \in \mathcal{R}^2:\ (-y, x) \in S\}$$

are disjoint.

(7.2) EXAMPLE. [2] There exists an arcwise connected 1-dimensional continuum X without the fixed-point property for continuous mappings

[2] This example is due to Young [5]. We give the formal description of this example (only one inessential detail has been changed).

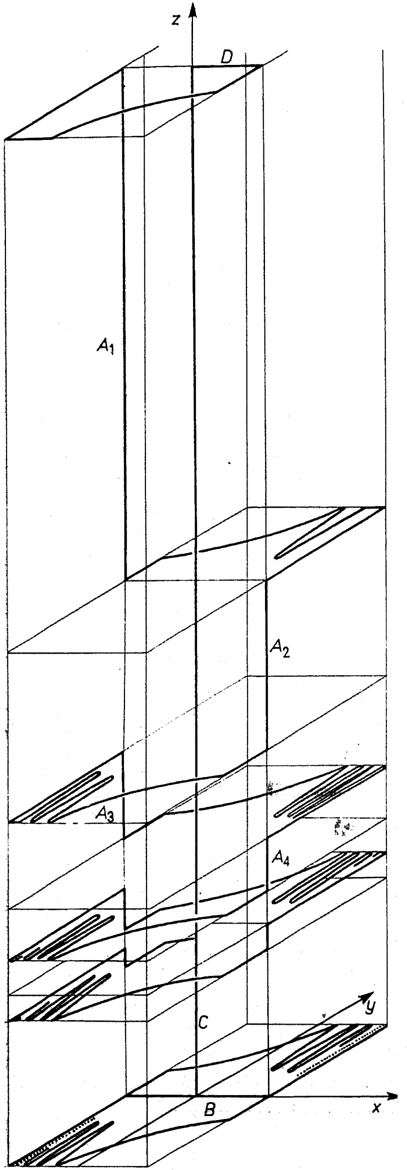


Fig. 2

onto itself, such that X does not contain any homeomorphic image of the circle (see Fig. 2).

Indeed, let

$$S(x) = 2+\sin\frac{\pi}{x-1} \quad \text{for} \quad -1 \leqslant x < 1$$

and

$$S_n(x) = \begin{cases} (-1)^n\cdot S\big((-1)^n x\big) & \text{for} \quad -1 \leqslant (-1)^n x \leqslant 1-\dfrac{2}{2n-1}, \\ (-1)^n\cdot 3 & \text{for} \quad 1-\dfrac{2}{2n-1} \leqslant (-1)^n x \leqslant 1 . \end{cases}$$

Then S_n, $n = 1, 2, \ldots$, are continuous mappings and

$$\lim_{n=\infty} S_{2n}(x) = S(x) \quad \text{for} \quad -1 \leqslant x < 1$$

and

$$\lim_{n=\infty} S_{2n-1}(x) = -S(-x) \quad \text{for} \quad -1 < x \leqslant 1 .$$

Now we put

$$X_0 = \{(x, y) \in \mathfrak{R}^2\colon (x = 1 \wedge -1 \leqslant y \leqslant 3) \vee (x = -1 \wedge -3 \leqslant y \leqslant 1) \vee \vee \big[|x| < 1 \wedge \big(y = S(x) \vee y = -S(-x)\big)\big]\} \times \{0\} \subseteq \mathfrak{R}^3 ,$$

$$X_n = \{(x, y) \in \mathfrak{R}^2\colon \big(|x| = 1 \wedge 0 \leqslant (-1)^n y \leqslant (-1)^n S_n(x)\big) \vee \vee \big(|x| \leqslant 1 \wedge y = S_n(x)\big)\} \times \left\{\frac{1}{n}\right\} \subseteq \mathfrak{R}^3 ,$$

$$A_n = \left\{(x, y, z) \in \mathfrak{R}^3\colon x = (-1)^n \wedge y = 0 \wedge \frac{1}{n+1} \leqslant z \leqslant \frac{1}{n}\right\},$$

$$B = \{(x, y, z) \in \mathfrak{R}^2\colon |x| \leqslant 1 \wedge y = z = 0\} ,$$

$$C = \{(x, y, z) \in Z^3\colon x = y = 0 \wedge 0 \leqslant z \leqslant 1\} ,$$

$$D = \{(x, y, z) \in \mathfrak{R}^3\colon 0 \leqslant x \leqslant 1 \wedge y = 0 \vee z = 1\} ,$$

and

$$X = \bigcup_{n=0}^{\infty} X_n \cup \bigcup_{n=1}^{\infty} A_n \cup B \cup C \cup D .$$

Obviously X is an arcwise connected 1-dimensional continuum and X does not contain any homeomorphic image of the circle.

Let $f\colon X \to X$ be a function defined as follows:

(a) $\quad f(x, y, 0) = (-x, -y, 0) \quad$ for any $(x, y, 0) \in X_0$.

Thus, we have defined $f|X_0$ so that we have

$$f(X_0) = X_0 \subseteq X .$$

(b) $$f(x, 0, 0) = \begin{cases} (-2x+\operatorname{sgn} x, 0, 0) & \text{if} \quad \tfrac{1}{2} \leqslant |x| \leqslant 1 , \\ (0, 0, \tfrac{1}{2}-|x|) & \text{if} \quad |x| \leqslant \tfrac{1}{2} . \end{cases}$$

Again, we have

$$f(B) \subseteq B \cup C \subseteq X .$$

(c)
$$f(0,0,z) = \begin{cases} (0,0,\tfrac{1}{2}+z) & \text{if} \quad 0 \leqslant z \leqslant \tfrac{1}{2}, \\ (2z-1,0,1) & \text{if} \quad \tfrac{1}{2} \leqslant z \leqslant 1 . \end{cases}$$

Again, $f(C) \subseteq C \cup D \subseteq X$.

(d)
$$f(x,0,1) = \begin{cases} (1,-12x,1) & \text{if} \quad 0 \leqslant x \leqslant \tfrac{1}{4}, \\ (3-8x, S_1(3-8x), 1) & \text{if} \quad \tfrac{1}{4} \leqslant x \leqslant \tfrac{1}{2}, \\ (-1,-9+12x,1) & \text{if} \quad \tfrac{1}{2} \leqslant x \leqslant \tfrac{3}{4}, \\ (-1,0,\tfrac{5}{2}-2x) & \text{if} \quad \tfrac{3}{4} \leqslant x \leqslant 1 . \end{cases}$$

Thus, we have defined $f|D$ and we have

$$f(D) = X_1 \cup A_1 \subseteq X .$$

(e)
$$f(x,y,z) = \begin{cases} \left(-x, S_{n+1}(-x, \dfrac{1}{n+1}\right) & \text{if} \quad (x,y,z) \in X_n \text{ and } |x| < 1, \\ \left(-x, -y, \dfrac{1}{n+1}\right) & \text{if} \quad (x,y,z) \in X_n \text{ and } |x| = 1 . \end{cases}$$

Thus we have defined $f|X_n$ and we have $f(X_n) = X_{n+1}$.

(f)
$$f(x,0,z) = \left(-x, 0, \frac{n}{n+2}\left(z-\frac{1}{n}\right)+\frac{1}{n+1}\right) \quad \text{if} \quad (x,0,z) \in A_n .$$

Thus we have defined $f|A_n$ and $f(A_n) = A_{n+1}$, $n = 1, 2, \ldots$

It is easy to see that (a)-(f) give a well defined continuous mapping $f\colon X \to X$ without a fixed point.

Indeed, let $\pi_i(a_1, a_2, a_3) = a_i$ be the projections, $i = 1, 2, 3$. Then

$$\pi_1 f(a) = -\pi_1(a) \neq \pi_1(a) \quad \text{for any } a \in X \setminus (B \cup C \cup D),$$
$$\operatorname{sgn} \pi_1 f(a) \neq \operatorname{sgn} \pi_1(a) \quad \text{for any } a \in B \setminus \{(0,0,0)\} = B \setminus C,$$
$$\pi_3 f(a) > \pi_3(a) \quad \text{for any } a \in C,$$

and

$$D \cap f(D) = D \cap (X_1 \cup A_1) = \{(1,0,1)\} \in X_1 ,$$

whence

$$f(1,0,1) \neq (1,0,1) .$$

(7.3) Example. There exists a 1-dimensional hereditarily unicoherent arcwise connected separable metric space X which does not contain any one-to-one continuous image of the ray $[0, \infty)$ as a closed subset, and which does not possess the fixed-point property even for homeomorphisms onto itself.

Indeed, let $X = \mathfrak{R}$ be the set of real numbers and let $d_a\colon \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, where a is an irrational number, be a metric given by

$$d_a(x,y) = \max\big(\min([x-y], 1-[x-y]), \min([ax-ay], 1-[ax-ay])\big),$$

where $[x]$ is a fraction part of x (evidently $d_a = d_{-a}$ and the metric spaces $(\mathfrak{R}, d_a)$ and $(\mathfrak{R}, d_{1/a})$ are isometric). Then $(\mathfrak{R}, d_a)$ is a metric group, i.e.

$$d_a(x+z, y+z) = d_a(x,y)$$

(and it is totally bounded as a metric space). The following lemma shows that the space $(\mathfrak{R}, d_a)$ is hereditarily unicoherent.

(7.4) Lemma. *If A is a proper connected closed subset of space $(\mathfrak{R}, d_a)$, then A is a simple arc.*

Proof. Let Z be a group of integers and let $p\colon \mathfrak{R}^2 \to \mathfrak{R}^2/Z^2$ be a canonical homomorphism. We can define in the quotient group $\mathfrak{R}^2/Z^2$ a metric d by

$$d\big(p(x,y), p(u,v)\big) = \max\big(\min([x-u], 1-[x-u]), \min([y-v], 1-[y-v])\big) .$$

Then the mapping p is an open continuous mapping of the Euclidean plane $\mathfrak{R}^2$ onto the metric space $(\mathfrak{R}^2/Z^2, d)$. Next, the mapping

$$p \circ f_a\colon \mathfrak{R} \to \mathfrak{R}^2/Z^2, \quad \text{where } f_a\colon \mathfrak{R} \to \mathfrak{R}^2 \quad \text{is given by } f_a(x) = (x, ax),$$

is an isometric imbedding of $(\mathfrak{R}, d_a)$ into $(\mathfrak{R}^2/Z^2, d)$.

Now let $x \in A$, where A is a proper connected closed subset of $(\mathfrak{R}, d_a)$. Then

$$(-\infty, x] \not\subseteq A \quad \text{and} \quad [x, \infty) \not\subseteq A,$$

as half-lines are dense in $(\mathfrak{R}, d_a)$. Let $u < x < v$ and $u, v \notin A$. The set $(\mathfrak{R}^2 Z^2) \setminus p \circ f_a(A)$ is contained in an open subset of $(\mathfrak{R}^2/Z^2, d)$ which containes $p \circ f_a(\mathfrak{R} \setminus A)$. Hence, for a certain positive integer $4n$, we obtain

(7.5)
$$p\left(\left\{(u,y)\colon au-\frac{1}{n} < y < au+\frac{1}{n}\right\}\right) \cup p\left(\left\{(v,y)\colon av-\frac{1}{n} < y < av+\frac{1}{n}\right\}\right) \subseteq (\mathfrak{R}^2/Z^2) \setminus p \circ f_a(A) .$$

Next

(7.6)
$$p\left[\left\{\left(y, ay+\frac{1}{n}\right)\colon y \in \mathfrak{R}\right\} \cup \left\{\left(y, ay-\frac{1}{n}\right)\colon y \in \mathfrak{R}\right\}\right] \subseteq (\mathfrak{R}^2 R^2/Z^2) \setminus p \circ f_a(\mathfrak{R}) .$$

The set

$$G_n = \left\{(y,z) \in \mathfrak{R}^2\colon u < y < v \text{ and } ay-\frac{1}{n} < z < ay+\frac{1}{n}\right\}$$

is open in a Euclidean plane and $p(G_n)$ is open in $\mathfrak{R}^2/Z^2$. Furthermore,

$$f_a(A) \cap \operatorname{Fr}(G_n) = \emptyset$$

and consequently, by (7.5) and (7.6),

$$p \circ f_a(A) \cap \operatorname{Fr}\big(p(G_n)\big) = \emptyset ,$$

since for any bounded set $G \subseteq \mathcal{R}^2$ we have

$$\mathrm{Fr}\big(p(G)\big) \subseteq p(\mathrm{Fr}\, G)\,.$$

Thus

$$p(G_n) \cap p \circ f_a(A)$$

is a closed-open set in the subspace $p \circ f_a(A)$ for $n = 1, 2, \ldots$, and, since $p \circ f_a(A)$ is connected,

$$p \circ f_a(A) \subseteq \bigcap_{n=1}^{\infty} p(G_n)\,.$$

Hence $A \subseteq u, v$. This proves Lemma (7.4).

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Reçu par la Rédaction le 28. 8. 1967

Rank theory of modules

by

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1. Preliminaries. The application [2] of the general algebraic dependence scheme of [1] to modules resulted in obtaining some basic information on dependence over modules. The aim of the present paper is to extend these investigations and build up a rank theory of modules parallel to that of abelian groups (cf. e.g. L. Fuchs [6]). In particular, the theory offers a generalization of some results of A. W. Goldie [8] and, when applied to injective modules, it enables us to generalize some results of E. Matlis [9]. In the latter, invariants $r_{\mathfrak{J}}(M)$ are derived which coincide with the invariants of P. Gabriel and U. Oberst in [7]. The value of our approach rests on the fact that, in contrast to [7], we define $r_{\mathfrak{J}}(M)$ for an R-module M without any reference to its injective hull $H(M)$ and can then use these cardinals $r_{\mathfrak{J}}(M)$ to characterize $H(M)$.

Throughout the paper, R denotes a fixed (associative) ring with unity, $\mathfrak{L}$ — the family of all its proper (i.e. $\neq R$) left ideals and $\mathfrak{J} \subseteq \mathfrak{L}$ — the subfamily of all (meet —) irreducible ideals. For $L \in \mathfrak{L}$ and $\varrho \in R$, the symbol $L\colon \varrho$ stands for the (left) ideal consisting of all $\chi \in R$, such that $\chi\varrho \in L$. Following [3], a subfamily $\mathfrak{K}$ of $\mathfrak{L}$ is said to be a Q-*family* if

$$\text{(Q)} \qquad \forall L, \varrho\, (L \in \mathfrak{K} \wedge \varrho \in R \backslash L \rightarrow L\colon \varrho \in \mathfrak{K})\,.$$

Denote the least Q-family containing a given ideal $L \in \mathfrak{L}$ by $\mathfrak{Q}_L$; thus, $\mathfrak{Q}_L = \{L\colon \varrho | \varrho \in R \backslash L\}$. Define in the set $\boldsymbol{Q}$ of all Q-families $\mathfrak{K}$ the "*duality*" map ∂ by

$$(\partial) \qquad L \in \partial\mathfrak{K} \leftrightarrow L \in \mathfrak{L} \wedge \mathfrak{Q}_L \cap \mathfrak{K} = \emptyset\,.$$

Thus, ∂ defines in $\boldsymbol{Q}$ a Galois connection (cf. O. Ore [10]). In particular,

$$\mathfrak{K}_1 \subseteq \mathfrak{K}_2 \rightarrow \partial\mathfrak{K}_1 \supseteq \partial\mathfrak{K}_2\,,$$

and ∂^2 is an (idempotent) closure operator; in fact, $\partial^{2n+i}\mathfrak{K} = \partial^i\mathfrak{K}$ for any two positive integers n and i. Making use of ∂, we can introduce the symmetric relation $\nabla \subseteq \boldsymbol{Q} \times \boldsymbol{Q}$ by

$$(\nabla) \qquad [\mathfrak{K}^1, \mathfrak{K}^2] \in \nabla \leftrightarrow \partial^2\mathfrak{K}^1 = \partial\mathfrak{K}^2 (\leftrightarrow \partial\mathfrak{K}^1 = \partial^2\mathfrak{K}^2)\,.$$