Infinite complementation in the lattice of topologies *

by

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1. Introduction. This paper attempts to be self-contained, but for basic facts and backround see the author's previous paper [4] in this journal. In that paper a theorem of Hartmanis [3] was strengthened to show that every proper (i.e., neither trivial nor discrete) topology on a finite set X with $n \geqslant 2$ elements has at least $n-1$ complements in the lattice of all topologies on X. This lower bound is best possible as was shown by an example. Utilizing this result, together with other results in that paper, it was shown that every proper topology on an infinite set has infinitely many principal (1) complements. This answered affirmatively a strengthened form of a question of Berri [1] (implicit in Hartmanis [3]).

The further questions naturally arise. Is it possible to estimate the cardinality of the set of complements (respectively, principal complements) for a proper topology on an infinite set? In particular, can one obtain a (best possible) lower bound on the cardinality of the set of complements for a proper topology on an *infinite* set X analogous to that obtained if the set is *finite*? A natural candidate for such a lower bound would be $|X|$.

The purpose of this paper is to answer these questions affirmatively. Specifically, the following result is established.

THEOREM. *Every proper topology on an infinite set X has at least $|X|$ complements* (resp., *principal complements*) *and at most $2^{2^{|X|}}$ complements* (resp., $2^{|X|}$ *principal complements*). *Moreover, these bounds are best possible.*

Remark. This provides an alternate way of establishing the main result of [4] which follows as an immediate corollary.

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(1) A topology is *principal* iff it is closed under the formation of arbitrary intersections [5].

In the next section appears a proof that $|X|$ is a lower bound. In section 3 the theorem will be completed and examples will be given to show that these estimates are the best possible.

2. Proof that $|X|$ is a lower bound. The proof of this, the difficult part of the theorem, although elementary, involves a series of reductions resulting from a fairly complicated case (and subcase) analysis. For convenience the proof is broken down into several propositions. Quite heavy use is made of the following lemma which appeared in an equivalent form in [4]. The details of its proof are omitted here. Throughout this section let t be a proper topology on an infinite set X.

LEMMA 2.1. *Let $X_1 \subset X$. Then, t has at least as many (principal) complements as does $t|X_1$, the restriction of t to X_1.*

Proof. Let $X_2 = X\backslash X_1$ and suppose without loss of generality that $X_i \neq \emptyset$, $i = 1, 2$. Let $t_i = t|X_i$, $i = 1, 2$. Suppose that $\{t_1^j\colon j \in J$ and $|J| = K\}$ is a set of K distinct (principal) complements for t_1. Let t_2' be a fixed principal complement for t_2. Such exists by Steiner's Theorem ([5], p. 397). Define K distinct (principal) complements for t: $\{t^j\colon j \in J\}$ as follows. Let

$$S(a, \mathcal{U}(b)) = \{U \subset X\colon a \in U \rightarrow b \in U\} \quad \text{and} \quad t_1^j * t_2' = \{U \cup V\colon U \in t_1^j \text{ and } V \in t_2'\}\ (^2)\,.$$

Next fix: $x_i \in X_i$, $i = 1, 2$.

Case 1. $X_i \notin t$ for $i = 1, 2$. Let $t^j = t_1^j * t_2'$.

Case 2. $X_1 \in t$ and $X_2 \notin t$. Let $t^j = (t_1^j * t_2') \wedge S(x_1, \mathcal{U}(x_2))$.

Case 3. $X_1 \notin t$ and $X_2 \in t$. Let $t^j = (t_1^j * t_2') \wedge S(x_2, \mathcal{U}(x_1))$.

Case 4. $X_i \in t$, $i = 1, 2$. Let $t^j = (t_1^j * t_2') \wedge S(x_1, \mathcal{U}(x_2)) \wedge S(x_2, \mathcal{U}(x_1))$. ∎

Note that in Cases 2, 3, and 4 of the proof above, the topology t^j depends not only on t_1^j and the choice of t_2', but also on the choice of $x_i \in X_i$, $i = 1, 2$. Since in what follows we will vary the x_i's in an attempt to obtain different (principal) complements, let us for a fixed t_2' denote the complement for t obtained in Case 2, 3, and 4 from t_1^j for the arbitrary points $x_i \in X_i$, $i = 1, 2$ by $t^j = t(x_1, x_2, j)$.

Now there is obviously another approach that might yield new complements for t, namely, to vary the subset X_1. We shall also use this method in the following propositions.

(2) Note: $S(a, \mathcal{U}(b))$ is a maximal proper principal topology, when $a \neq b$.

PROPOSITION 2.2 *Suppose that X has a subspace Y with $t|Y$ proper and $|Y| = |X|$. Suppose further that there exists a point $y_0 \in Y$ such that $\{y_0\} \notin t|Y$ and for $|X|$ y's in $Y\backslash\{y_0\}$, $\{y_0\} \in t|\{y_0, y\}$. Then t has at least $|X|$ principal complements.*

Proof. By hypothesis there exist $|X|$ y's in $Y\backslash\{y_0\}$ such that $\{y_0\} \in t|\{y_0, y\}$ and $\{y_0\} \notin t|Y$. By excluding at most one such y we may assume further that $\{y_0, y\} \notin t|Y$. For each such y let $X_1 = \{y_0, y\}$. Use Lemma 2.1, Case 1 or Case 3 to construct a principal complement $t'(y)$ for $t|Y$ from $(t|\{y_0, y\})'$, the unique complement for $t|\{y_0, y\}$, and a fixed principal complement for $t|(Y\backslash\{y_0, y\})$. By construction we note that $\{y_0, y\} \in t'(y)$. Suppose that $y_1 \neq y_2$. We claim that $t'(y_1) \neq t'(y_2)$ for otherwise $\{y_0\} = \{y_0, y_1\} \cap \{y_0, y_2\} \in t'(y_1)$. Hence, $\{y_0\} \in (t|\{y_0, y_1\})'$. This is a contradiction since by hypothesis $\{y_0\} \in t|\{y_0, y_1\}$. It follows that $t|Y$ has at least $|X|$ principal complements. By Lemma 2.1 t has at least $|X|$ principal complements. ∎

COROLLARY 2.3. *If X has a non-discrete T_1 subspace Y with $|Y| = |X|$, then t has at least $|X|$ principal complements.*

PROPOSITION 2.4. *Suppose that (X, t) has no isolated points. Then t has at least $|X|$ principal complements.*

Proof. Fix a proper open subset $W \subset X$ such that no non-empty open subset has smaller cardinality. We consider the following possibilities: $|X\backslash W| = |X|$ and $|X\backslash W| < |X|$.

First assume that $|X\backslash W| = |X|$. Fix an arbitrary point $w_0 \in W$. Since there are no isolated points in (X, t), we have $\{w_0\} \notin t$. Since $|X\backslash W| = |X|$, there exist $|X|$ v's in $X\backslash W$ such that $\{w_0\} = W \cap \{w_0, v\} \in t|\{w_0, v\}$. By Proposition 2.2, t has at least $|X|$ principal complements.

Next assume $|X\backslash W| < |X|$. Then $|W| = |X|$. We note that $t|W$ is not discrete. Thus, if $t|W$ is T_1, then by Corollary 2.3 t has at least $|X|$ principal complements. So we assume further that $t|W$ is not T_1. Then there exists $w_0 \in W$ with $\{w_0\}$ not closed in $t|W$. By excluding at most one point of $W\backslash\{w_0\}$, there exist $|X|$ w's in $W\backslash\{w_0\}$ such that $\{w_0, w\}$ is not closed (or open) in $t|W$.

Case 1. For $|X|$ such w's $t|\{w_0, w\}$ is not trivial. For each of these use Lemma 2.1, Case 1 to define a principal complement $t'(w)$ for $t|W$ from $(t|\{w_0, w\})'$ and a principal complement for $t|(W\backslash\{w_0, w\})$. If $w_1 \neq w_2$ but $t'(w_1) = t'(w_2)$, then $\{w_0\} = \{w_0, w_1\} \cap \{w_0, w_2\} \in t'(w_1)$. Since $\{w_0, w_2\}$ is not closed in $t|W$, then $W\backslash\{w_0, w_2\} \in t'(w_2) = t'(w_1)$. Hence, $\{w_1\} = \{w_0, w_1\} \cap (W\backslash\{w_0, w_2\}) \in t'(w_1)$. Thus, $(t|\{w_0, w_1\})'$ is discrete contradicting the assumption that $t|\{w_0, w_1\}$ is not trivial. Hence, $t'(w_1) \neq t'(w_2)$ and $t|W$ has at least $|X|$ principal complements. Therefore, t has at least $|X|$ principal complements by Lemma 2.1.

Case 2. For $|X|$ such w's, $t|\{w_0, w\}$ is trivial. We claim that there exists a subset $A \subset W$ with $|A| = |X|$ and $t|A$ trivial. Let $A = \{w \in W: \{w_0, w\}$ is neither open nor closed in W and $t|\{w_0, w\}$ is trivial$\}$. By hypothesis in this case $|A| = |X|$. To see that $t|A$ is trivial, we suppose that for some proper subset $B \subset A$ we have $B \in t|A$. If $w_0 \in B$, then there exists $w \in A \backslash B$ such that $\{w_0\} \in t|\{w_0, w\}$, a contradiction. If $w_0 \notin B$, then there exists $w \in B$ such that $\{w\} \in t|\{w_0, w\}$, also a conditradiction. Thus, $t|A$ is trivial.

If $A \in t$, then fix a point $b \in X \backslash A$. Let t_1^1 be the discrete topology on A and t_2' a fixed principal complement on $t|(X \backslash A)$. For each $a \in A$ we now use Lemma 2.1, Case 2 or Case 4 to construct a principal complement for X: $t^1(a) = t(a, b, 1)$ from a, b, t_1^1 and t_2'. By construction if $a_1 \neq a_2$, then $\{a_1\} \in t^1(a_2) \backslash t^1(a_1)$. Consequently, there are at least $|X|$ principal complements for t.

If $A \notin t$, then $A \notin t|W$. We now use Lemma 2.1, Case 1 or Case 3 to construct a principal complement t_1^1 for $t|W$ from the discrete topology on A and a principal complement for $t|(W \backslash A)$. We note that if $a \in A$, then $\{a\} \in t_1^1$. Fixing a point $b \in X \backslash W$ and a principal complement for $t|(X \backslash W)$, we now use Lemma 2.1, Case 2 or Case 4 to construct $t^1(a) = t(a, b, 1)$ a principal complement for t. By construction $\{a\} \notin t^1(a)$ but if $a' \in A \backslash \{a\}$, then $\{a'\} \in t^1(a)$. Hence, if a, $a' \in A$ and $a \neq a'$, then $t^1(a) \neq t^1(a')$. Consequently, t has at least $|X|$ principal complements. ■

In the next two propositions I denotes the set of isolated points in (X, t).

PROPOSITION 2.5. *If $|X \backslash I| = |X|$, then t has at least $|X|$ principal complements.*

Proof. If $t|(X \backslash I)$ has no isolated points and is a proper topology, then by Proposition 2.4 $t|(X \backslash I)$ has at least $|X|$ principal complements. Consequently, by Lemma 2.1, t has at least $|X|$ principal complements.

If $t|(X \backslash I)$ is not proper and has no isolated points, then $t|(X \backslash I)$ is trivial. Using Lemma 2.1, Case 3 or Case 4, for each $y \in X \backslash I$ one can construct a principal complement $t^1(y) = t(y, i_0, 1)$ for t. Here $i_0 \in I$ is fixed; t_1^1 is the discrete topology on $X \backslash I$; and the fixed complement, t_2', for $t|I$ is the trivial topology on I. Now $I \cup \{y\} \in t^1(y)$ but if $y' \in X \backslash I$ and $y' \neq y$, then $I \cup \{y'\} \notin t^1(y)$ and $t^1(y) \neq t^1(y')$. Consequently, since $|X \backslash I| = |X|$, t has at least $|X|$ principal complements.

Suppose, then, that $t|(X \backslash I)$ has an isolated point y_0. Then, $\{y_0\} \notin t$ and for every $y \in X \backslash I$ distinct from y_0, $\{y_0\} \in t|\{y_0, y\}$. Thus, by Proposition 2.2, t has at least $|X|$ principal complements. ■

PROPOSITION 2.6. *If $|X \backslash I| < |X|$, then t has at least $|X|$ principal complements.*

Proof. Corollary 2.3 takes care of the case where (X, t) is T_1. So let us suppose that (X, t) is not T_1.

Case 1. There exists a point $y_0 \in X \backslash I$ such that $\{y_0\}$ is not closed in (X, t). Since $|I| = |X|$, then there exist $|X|$ i's in I such that $\{y_0, i\}$ is neither open nor closed in (X, t). By using Lemma 2.1, Case 1 we can construct a principal complement $t'(i)$ for t from $(t|\{y_0, i\})'$ and a principal complement for $t|(X \backslash \{y_0, i\})$. If we assume that $t'(i_1) = t'(i_2)$ for $i_1 \neq i_2$, then $\{i_1\} \in \{y_0, i_1\} \cap X \backslash \{y_0, i_2\} \in t'(i_1)$. But i_1 is isolated in (X, t) and $\{i_1\} \in t$. Since t and $t'(i_1)$ are complements, this is a contradiction. Thus, $t'(i_1) \neq t'(i_2)$ for $i_1 \neq i_2$. Hence, there exist at least $|X|$ principal complements for t.

Case 2. Every point of $X \backslash I$ is closed in (X, t). Since (X, t) is not T_1, there exists a point $i_0 \in I$ such that i_0 is not closed in (X, t). Hence, there exist $|X|$ i's in $I \backslash \{i_0\}$ such that $\{i_0, i\}$ is not closed in (X, t). Fix $y_0 \in X \backslash I$ and using Lemma 2.1, Case 2 define for each such i a principal complement $t^1(i) = t(i, y_0, 1)$ for t where t_1^1 is the trivial topology on $\{i_0, i\}$. If $i_1 \neq i_2$, we claim that $t^1(i_1) \neq t^1(i_2)$ for suppose the contrary. Then, $X \backslash \{i_0\} = X \backslash \{i_0, i_1\} \cup X \backslash \{i_0, t_2\} \in t^1(i_1)$. But then, $\{i\} = X \backslash \{i_0\} \cap \{i_0, i\} \in t_1^1$. This contradicts the fact that t_1^1 is trivial. Thus $t^1(i_1) \neq t^1(i_2)$. Since $|I| = |X|$, then t has at least $|X|$ principal complements. ■

This completes the proof that the cardinal number of the set of complements (resp., principal complements) for a proper topology on an infinite set has $|X|$ as a lower bound.

3. Completion of the proof and examples.

LEMMA 3.1. *On an infinite set X, there are $2^{2^{|X|}}$ topologies and $2^{|X|}$ principal topologies.*

Proof. Let X be an infinite set. It is well known that there are $2^{2^{|X|}}$ topologies on X. It is easily seen that there are $|X|$ principal ultraspaces ([3]) on X. Since each principal topology is the intersection of principal ultraspaces [5], there are at most $2^{|X|}$ principal topologies. But for each subset $A \subset X$ the topology $\{\emptyset, A, X\}$ is principal. Hence X has exactly $2^{|X|}$ principal topologies. ■

COROLLARY 3.2. *A proper topology on an infinite set X has at most $2^{2^{|X|}}$ complements and $2^{|X|}$ principal complements.*

We now give examples to show that the bounds obtained are best possible.

([3]) An *ultraspace* is a maximal proper topology. A topology t is an ultraspace iff $t = S(x, \mathcal{U}) = \{U \subset X: x \in U \to U \in \mathcal{U}\}$, where $\mathcal{U}$ is an ultrafilter on X different from $\mathcal{U}(x)$, the principal ultrafilter generated by x [2]. The ultraspace $S(x, \mathcal{U})$ is principal iff $\mathcal{U} = \mathcal{U}(y)$ for some $y \in X \backslash \{x\}$.

THEOREM 3.3. *Let $t = S(x, \mathfrak{U})$ be an ultraspace on the infinite set X. Then t has exactly $2^{2^{|X|}}$ complements and $2^{|X|}$ principal complements.*

Proof. First we shall prove that there exists a subset $Y \subset X \setminus \{x\}$ with $|Y| = |X|$ and $Y \notin \mathfrak{U}$. If $\mathfrak{U}$ is principal, say $\mathfrak{U} = \mathfrak{U}(y)$ where $x \neq y$, then let $Y = X \setminus \{x, y\}$. If $\mathfrak{U}$ is non-principal, we can express $X \setminus \{x\}$ as the disjoint union of two subsets both with cardinality $|X|$. Since exactly one of them belongs to $\mathfrak{U}$, let Y be the other one.

Now, given any topology s on Y, define the following topology on X: $\bar{s} = \{U \cup \{x\}: U \in s\} \cup \{\emptyset, X\}$. Clearly, if s is principal so is $\bar{s}$, and if $s_1 \neq s_2$, then $\bar{s}_1 \neq \bar{s}_2$.

We claim that each $\bar{s}$ is a complement for t. Since $2^{X \setminus \{x\}} \subset t$ and $\{x\} = \emptyset \cup \{x\} \in \bar{s}$, then $t \vee \bar{s} = 1$. We shall now show that $t \wedge \bar{s} = 0$. Suppose that there exists $U \in s$ such that $U \cup \{x\} \neq X$ and $U \cup \{x\} \in t = S(x, \mathfrak{U})$. Then, $U \cup \{x\} \in \mathfrak{U}$. Since $\{x\} \notin \mathfrak{U}$, then $U \in \mathfrak{U}$ and, consequently, $Y \in \mathfrak{U}$, a contradiction. Thus, $t \wedge \bar{s} = 0$.

We have just shown that distinct (principal) topologies on Y yield distinct (principal) complements for t on X. Since $|Y| = |X|$, it follows from Lemma 3.1 and Corollary 3.2 that t has $2^{2^{|X|}}$ complements and $2^{|X|}$ principal complements. ■

THEOREM 3.4. *There exists, on every infinite set X, a topology with exactly $|X|$ complements all of which are principal.*

Proof. Let $t = \{\emptyset, \{x\}, X \setminus \{x\}, X\}$. Each complement for t is of the form $t'(y) = S(x, \mathfrak{U}(y)) \wedge S(y, \mathfrak{U}(x))$, where $y \in X \setminus \{x\}$. Clearly, each $t'(y)$ is principal, and there are $|X \setminus \{x\}| = |X|$ such complements. ■

Remark. The author [4] has shown that a non-discrete T_1 topology never possesses a maximal complement or a maximal principal complement. However, the example used in the proof of Theorem 3.4 shows that for every positive cardinal number K there exists a principal topological space with exactly K maximal (principal) complements.

If we assume the generalized continuum hypothesis (GCH), then the statement of our main result can be somewhat sharpened.

THEOREM 3.5. *Suppose that $|X| = \aleph_\alpha$. Assuming* GCH *every proper topology on X has $\aleph_\alpha$, $\aleph_{\alpha+1}$, or $\aleph_{\alpha+2}$ complements and $\aleph_\alpha$ or $\aleph_{\alpha+1}$ principal complements. Moreover, each of these values is attained.*

Proof. The only thing not already established is the existence of a proper topology on X with $2^{|X|} = \aleph_{\alpha+1}$ complements. This is accomplished in the following theorem. ■

THEOREM 3.6. *The minimal T_1 (cofinite) topology on an infinite set X has exactly $2^{|X|}$ complements* ([4]).

([4]) The author thanks P. Nanzetta for help in proving Theorem 3.6.

Proof. Partition X into $|X|$ disjoint subsets each of cardinality $|X|$, say $X = \bigcup \{X_a: a \in A\}$, where $|A| = |X| = |X_a|$ for each $a \in A$ and $X_a \cap X_{a'} = \emptyset$ for $a \neq a'$. Let t denote the minimal T_1 topology on X and $t_a = t|X_a$ for each $a \in A$. Observe that t_a is the minimal T_1 topology on X_a. By our main Theorem each t_a has at least $|X|$ complements. Suppose, for each $a \in A$, that t'_a is a complement for t_a. Then $t' = *\{t'_a: a \in A\} = \{\bigcup \{U_a: a \in A\}: U_a \in t'_a\}$ is a complement for t as is easily verified. Suppose that $t'' = *\{t''_a: a \in A\}$ and for some $a_0 \in A$ $t'_{a_0} \neq t''_{a_0}$. Then $t' \neq t''$. Consequently, t has at least $|X|^{|X|} = 2^{|X|}$ complements.

Suppose that t' is any complement for t. Then, given $x \in X$, there exists $U \in t$ and $V \in t'$ such that $\{x\} = U \cap V$. But, since U is cofinite, V is finite. It follows that each point $x \in X$ is a member of a minimal open neighborhood in (X, t'). Thus, t' is principal. By Lemma 3.1, t has at most $2^{|X|}$ complements. ■

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