..., $v_{n(i)} \leqslant s_j + 2r_0$ and $\mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)})) \equiv \mathfrak{P}_i(v_1, \ldots, v_{n(i)})$. Thus, under the changed circumstances, we find another, possibly different, answer to the value of $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$.

If we think of our procedure as assigning the value T to $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$ at the outset and only changing its mind when CAT terminates in a value different from the previously accepted value, then, since different CAT answers result only from rejections in VERT of supposed ideal numbers, we see that there can be at most $n(i)$ changes of mind for the value of $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$. In other words, our effective procedure will eventually give us the correct answer (and keep repeating it) at some generally unknown time with at most $n(i)$ intervening changes of mind. Hence $\mathfrak{Q}_i \in \Sigma_1^*$ for each $i = 1, \ldots, m$. This ends the proof of the main theorem 1.1.

In closing, it should be remarked that the proof of theorem 1.1 goes through with practically no changes if T_0 contains an at most denumerable number of individual constants. Also it is obvious from the actual use made of property (I) that T_3 could have been taken to be T_2 augmented only by the extra indiscernibility axioms provided by *atomic* ϵ-wffs.

References

[1] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Vol. II, Berlin 1939.

[2] S. C. Kleene, *Introduction to metamathematics*, Princeton (Van Nostrand) 1952.

[3] E. Mendelson, *Introduction to mathematical logic*, Princeton (Van Nostrand) 1964.

[4] A. Mostowski, *The present state of investigations on the foundations of mathematics*, Rozprawy Matematyczne 9, Warszawa 1955.

[5] H. Putnam, *Trial and error predicates and the solution to a problem of Mostowski*, J. Symbolic Logic 30 (1965), pp. 49-57.

[6] F. P. Ramsey, *The foundations of mathematics*. London, 1931.

CATHOLIC UNIVERSITY
HARVARD UNIVERSITY

Reçu par la Rédaction le 1. 12. 1967

On equational classes of abstract algebras defined by regular equations

by

J. Płonka (Wrocław)

0. Introduction. In [1], theorem I it was shown that the operation of the sum of a direct system of algebras of the same similarity class preserves all regular equations and only such equations are preserved by it. (For the definitions of these notions, see below.)

It follows that if the algebras $\mathfrak{A}_i$ belong to an equational class K_E defined by the set E of equations, then the sum of any direct system of algebras consisting of the algebras $\mathfrak{A}_i$ belongs to the equational class $K_{R(E)}$ defined by the set $R(E)$ of all equations which are consequences of the set E and are regular. The question can be asked whether the converse is true, i.e. whether every algebra of the class $K_{R(E)}$ can be represented as a sum of a direct system of algebras from K_E. It turns out (see below), that in many important cases, e.g. for lattices, Boolean algebras and groups this is the case, but in general the answer is negative. However, below we shall give a full description of algebras from $K_{R(E)}$ using the class K_E.

At first we shall recall some definitions and results from [1], for convenience of the reader.

Let

$$\mathcal{A} = \langle I, \langle \mathfrak{A}_i \rangle_{i \in I}, \langle \varphi_{ij} \rangle_{i,j \in I, i \leqslant j} \rangle$$

be a direct system of similar algebras, without nullary fundamental operations, indexed by a poset I with the least upper bound property. Let $\langle F_t \rangle_{t \in T}$ be the set of fundamental operations of the algebras in $\mathcal{A}$, and let A_i be the carrier of $\mathfrak{A}_i$. The *sum of the system* $\mathcal{A}$ is an algebra $S(\mathcal{A}) = \langle A; \langle F_t \rangle_{t \in T} \rangle$ where A is the disjoint sum of the carriers A_i ($i \in I$), and the fundamental operations F_t are defined by

$$F_t(a_1, \ldots, a_n) = F_t(\varphi_{i_1 i_0}(a_1), \ldots, \varphi_{i_n i_0}(a_n))$$

where $a_j \in A_{i_j}$ and i_0 is the least upper bound of $i_1, \ldots, i_n$.

An equation $f = g$ where f and g are terms in an algebra we shall call *regular* if on both sides of it the same free variables occur.

(i) ([1], theorem I). *If $\mathcal{A}$ is a non-trivial direct system of algebras (i.e. I contains at least two elements), then in $S(\mathcal{A})$ all regular equations satisfied in all algebras of the system are true, whereas every other equation is false.*

Let $\mathfrak{A} = (A; \langle F_t\rangle_{t\in T})$ be an algebra without nullary fundamental operations. A function $f\colon A^2 \to A$ will be called a *partition function* for $\mathfrak{A}$, or shortly, a *P-function*, if the following equalities are satisfied:

(0.1) $$f(f(x, y), z) = f(x, f(y, z)),$$

(0.2) $$f(x, x) = x,$$

(0.3) $$f(x, f(y, z)) = f(x, f(z, y)),$$

(0.4) $$f(F_t(x_1, \ldots, x_n), y) = F_t(f(x_1, y), \ldots, f(x_n, y)),$$

(0.5) $$f(y, F_t(x_1, \ldots, x_n)) = f(y, F_t(f(y, x_1), \ldots, f(y, x_n))),$$

(0.6) $$f(F_t(x_1, \ldots, x_n), x_k) = F_t(x_1, \ldots, x_n) \quad (k = 1, 2, \ldots, n),$$

(0.7) $$f(y, F_t(y, \ldots, y)) = y.$$

The following result forms a part of theorem II of [1]:

(ii) *To every P-function for an algebra $\mathfrak{A}$ there corresponds a representation of $\mathfrak{A}$ as a sum of a direct system of its subalgebras $\mathfrak{A}_t$. Two elements a, b of the carrier of $\mathfrak{A}$ belong to the carrier of the same algebra $\mathfrak{A}_t$ if and only if $f(a, b) = a$ and $f(b, a) = b$.*

1. Let K_E be the equational class of algebras of the type τ, defined by the set E of equations. By $R(E)$ we shall denote the set of all equations which are consequences of E and which are regular, and by $C(E)$ we shall denote the set of all consequences of E. Clearly $R(E) = R(C(E)) = C(R(E))$. If E is the void set, then $R(E) = E$ by definition. In the sequel *we shall consider exclusively algebras without nullary fundamental operations* and we shall not repeat this assumption.

LEMMA. *For every set E of equations, one of the following possibilities holds:*

(1) $E \subset R(E)$.

(2) *The equation $x = y$ belongs to $C(E)$.*

(3) *An equation of the form*

(a) $$f(x, y) = x$$

(where $f(x, y)$ is a term in two variables) belongs to $C(E)$.

(4) *An equation of the form*

(b) $$g(x) = h(y)$$

(where g and h are terms on one variable) belongs to $C(E)$ and, moreover, all algebras of the class K_E are unary.

(5) *An equation of the form*

(c) $$f(x, y) = f(x, x)$$

(where f is a term in two variables) belongs to $C(E)$.

Proof. If $E \subset R(E)$, then we have case (1). Assume thus that E contains a non-regular equation, say $F = G$, and let x_1 be a free variable occuring in F but not in G. Put now $x_1 = y$, $x_i = x$ $(i \neq 1)$ and consider the resulting equation. It must have one of the following forms:

$$y = f(x), \quad f(y) = x, \quad f(x, y) = g(x), \quad g(y) = h(x).$$

In the first two cases the equation $x = y$ belongs to $C(E)$, and so we have case (2). In the third case we get $f(x, x) = g(x)$ and $f(x, y) = f(x, x)$, hence we get (5) or (3). In the last case assume first that all algebras of K_E are unary. This implies (4). Finally, if in the last case $p(x, y)$ is a term of two variables (which exists in every non-unary algebra), then we obtain $g(p(x, y)) = h(x)$, which implies in turn

$$g(p(x, x)) = h(x) \quad \text{and} \quad g(p(x, y)) = g(p(x, x)),$$

i.e. an an equation of the form (c).

THEOREM I. *If E is a set of equations, $\mathfrak{A} = \langle X; \langle F_t\rangle_{t\in T}\rangle$ belongs to $K_{R(E)}$ and one of cases (1), (2), (3) of the lemma holds. then the algebra $\mathfrak{A}$ can be represented as the sum of a direct system of algebras from the class K_E.*

Proof. In case (1) the assertion is trivial, just consider the trivial direct system consisting of $\mathfrak{A}$ alone.

In case (2) every equation of the form $F(x) = x$ holds in $\mathfrak{A}$, and so $\mathfrak{A}$ is a disjoint sum of one-element subalgebras. This implies in the case of unary $\mathfrak{A}$ the possibility of a well ordering of those subalgebras (we assume of course the axiom of choice) which together with the mappings $\varphi_{a,b}(a) = b$ (for $a \leqslant b$) give the desired representation.

If $\mathfrak{A}$ is not unary and $F(x_1, \ldots, x_n)$ is a fundamental operation with $n > 1$, then define $f(x, y) = F(x, y, \ldots, y)$. Equations (0.1)-(0.7) are in $R(E)$ as well as $f(x, y) = f(y, x)$ and $F(x_1, \ldots, x_n) = f(x_1, f(x_2, \ldots, f(x_{n-1}, x_n)\ldots))$. Hence f is a P-function and so by (ii) A can be represented as the sum of a direct system of its subalgebras. Since in those subalgebras $f(x, y) = x$ holds, we obtain $x = f(x, y) = f(y, x) = y$, and so they belong to K_E.

In case (3) consider the operation $f(x, y)$ occuring in (a). Equations (0.1)-(0.7) are in $R(E)$, hence are fulfilled in the algebra $\mathfrak{A}$ and so the mapping $f\colon X^2 \to X$ induced by $f(x, y)$ is a P-function for $\mathfrak{A}$. It follows from (ii) that A is the sum of a direct system of subalgebras in which equation (a) is satisfied. In those subalgebras all regular equations from E

are satisfied. Let $F = G$ be a non-regular equation from E, and let $x_1, \ldots, x_p$ be the free variables occuring in F but not in G, and similarly, let $y_1, \ldots, y_q$ be the free variables occuring in G but not in F. The equation

$$f\Big(F, f\big(y_1, f(y_2, \ldots, f(y_{q-1}, y_q)\ldots)\big)\Big) = f\Big(G, f\big(x_1, f(x_2, \ldots, f(x_{p-1}, x_p)\ldots)\big)\Big)$$

belongs to $R(E)$, and so is satisfied in the said subalgebras, but there we have $f(x, y) = x$, and the equation $F = G$ follows, which shows that our subalgebras belong to K_E.

THEOREM II. *If E is a set of equations, $\mathfrak{A} = \langle X; \langle F_t \rangle_{t \in T}\rangle$ belongs to $K_{R(E)}$ and case* (4) *of the lemma holds, then $\mathfrak{A}$ is a disjoint sum of its subalgebras, belonging to K_E.*

Proof. We shall show that every $a \in X$ is contained in a unique maximal subalgebra of $\mathfrak{A}$ belonging to K_E. As $\mathfrak{A}$ is unary, this will imply the theorem. Consider the set of all subalgebras of $\mathfrak{A}$ belonging to K_E and containing a. This set is non-void, as the subalgebra generated by a belongs to it. In fact, every element of this subalgebra is of the form $p(a)$, where $p(x)$ is a term in one free variable. If $g(x) = h(y)$ is a non-regular equation from E, then the regular equation $g\big(p_1(x)\big) = h\big(p_2(x)\big)$ belongs to $R(E)$, hence is satisfied in A for every pair $p_1(x), p_2(x)$ of terms in one free variable. It results that in the subalgebra generated by a the equation $g(x) = h(y)$ is satisfied.

A straightforward application of Zorn's lemma shows now the existence of a maximal subalgebra with the properties needed. It remains to prove its uniqueness.

Assume thus that B_1 and B_2 are two distinct maximal subalgebras of A containing a and belonging to $K_{R(E)}$. Let $g(x) = h(y)$ be an arbitrary non-regular equation from E, and let $b \in B_1 \backslash B_2$, $c \in B_2 \backslash B_1$. We have $g(b) = h(a) = g(a) = h(c)$, thus the equation $g(x) = h(y)$ is satisfied in $B_1 \cup B_2$, which is an algebra in view of the unarity of $\mathfrak{A}$ and belongs to K_E. This contradicts the maximality of B_1 and B_2.

THEOREM III. *If E is a set of equations, $\mathfrak{A} = \langle X; \langle F_t \rangle_{t \in T}\rangle$ belongs to $K_{R(E)}$ and condition* (5) *of the lemma is satisfied, then $\mathfrak{A}$ is a semilattice of algebras from K_E, i.e. there exists a semilattice I and a family of subalgebras of $\mathfrak{A}$ indexed by I, with mutually disjoint carriers, such that if a_{i_j} belongs to the subalgebra indexed by $i_j \in I$ for $j = 1, 2, \ldots, k$ then for all $t \in T$, $F_t(a_{i_1}, \ldots, a_{i_k})$ belongs to the subalgebra indexed by $i =$ l.u.b. $(i_1, \ldots, i_k)$.*

Proof. For shortness let us denote the operation $f(x, y)$ occuring in (c) by xy, so (c) takes the form

$$xy = x^2. \tag{c'}$$

In the set X we introduce a relation S by putting aSb if and only if $ab = a^2$. Observe that xSx. We prove now that S is also transitive. As $\mathfrak{A} \in K_{R(E)}$, we may in the proofs use all regular consequences of (c').

If now $ab = a^2$, $bc = b^2$ then $ac = a(ac) = a(a^2c) = a[(ab)c] = a[a(bc)] = a(ab^2) = a(ab) = ab = a^2$, proving thus the transitivity.

Now we define another relation by means of aQb if and only if aSb and bSa. It is clear that Q is an equivalence, and we shall prove that it is a congruence in $\mathfrak{A}$. In fact, for every $t \in T$, $a_k Q b_k$ $(k = 1, 2, \ldots, n)$ implies

$$\begin{aligned} F_t(a_1, \ldots, a_n)F_t(b_1, \ldots, b_n) &= F_t(a_1, \ldots, a_n)F_t(a_1b_1, \ldots, a_nb_n) \\ &= F_t(a_1, \ldots, a_n)F_t(a_1^2, \ldots, a_n^2) = \big(F_t(a_1, \ldots, a_n)\big)^2 \end{aligned}$$

thus $F_t(a_1, \ldots, a_n) S F_t(b_1, \ldots, b_n)$. Similarly one gets $F_t(b_1, \ldots, b_n) S F_t(a_1, \ldots, a_n)$. It follows that Q is a congruence in $\mathfrak{A}$. Let us denote by A_i $(i \in I)$ the classes $(\bmod Q)$ in $\mathfrak{A}$. Observe that each A_i is a subalgebra of $\mathfrak{A}$, as from $a_1, \ldots, a_n \in A_i$ follows

$$F_t(a_1, \ldots, a_n)a_1 = \big(F_t(a_1, \ldots, a_n)\big)^2$$

and

$$a_1F_t(a_1, \ldots, a_n) = a_1F_t(a_1a_1, a_1a_2, \ldots, a_1a_n) = a_1F_t(a_1^2, \ldots, a_1^2) = a_1^2.$$

We show now that $A_i \in K_E$. Clearly every regular equation from E is satisfied in A_i. Let

$$F(x_1, \ldots, x_n, y_1, \ldots, y_p) = G(x_1, \ldots, x_n, z_1, \ldots, z_q)$$

be an arbitrary non-regular equation from E. We may freely assume that $p \neq 0$. Let $a_1, \ldots, a_n, b_1, \ldots, b_p, c_1, \ldots, c_q \in A_i$ and let d be an arbitrary element of A_i. We have then

$$\begin{aligned} F(a_1, \ldots, a_n, b_1, \ldots, b_p) &= F(a_1, \ldots, a_n, b_1^2, \ldots, b_p^2) \\ &= F(a_1, \ldots, a_n, b_1d, \ldots, b_pd) = F(a_1, \ldots, a_n, db_1, \ldots, db_p) \\ &= F(a_1, \ldots, a_n, d^2, d^2, \ldots, d^2) = G(a_1, \ldots, a_n, d^2, \ldots, d^2) \\ &= G(a_1, \ldots, a_n, dc_1, dc_2, \ldots, dc_q) = G(a_1, \ldots, a_n, c_1d, \ldots, c_qd) \\ &= G(a_1, \ldots, a_n, c_1^2, \ldots, c_q^2) = G(a_1, \ldots, a_n, c_1, \ldots, c_q). \end{aligned}$$

In the set I of indices we define the relation $\leqslant$ in the following way: $i_1 \leqslant i_2$ if for some $a \in A_{i_1}$, $b \in A_{i_2}$ one has bSa. It is clear that this definition does not depend on the choice of a and b. Moreover, the set I is by this definition turned into a semilattice $\langle I, \leqslant \rangle$. Clearly, I is a poset. Let $a \in A_{i_1}$, $b \in A_{i_2}$, $ab \in A_{i_0}$. We prove that $i_0 =$ l.u.b.(i_1, i_2). We have $(ab)a = (ab)^2$, hence $i_0 \geqslant i_1$, and similarly the inequality $i_0 \geqslant i_2$ follows. Let $j \geqslant i_1, i_2$ and $c \in A_j$. Then

$$c(ab) = c[(ca)(cb)] = c(c^2c^2) = c^2, \quad \text{hence} \quad j \geqslant i_0.$$

Finally from $a_k \in A_{i_k}$ $(k = 1, 2, \ldots, n)$, $i_0 = \text{l.u.b.}(i_1, \ldots, i_n)$ and $F_t(a_1, \ldots, a_n) \in A_j$ follows

$$F_t(a_1, \ldots, a_n)a_k = (F_t(a_1, \ldots, a_n))^2,$$

thus $j \geqslant i_0$, and on the other hand from $c \in A_{i_0}$ follows

$$cF_t(a_1, \ldots, a_n) = c(F_t(ca_1, \ldots, ca_n)) = cF_t(c^2, \ldots, c^2) = c^2,$$

hence $i_0 \geqslant j$, thus $j = i_0$, proving all assertions of the theorem.

Now we show that case (5) is essentially more general than the case (3), i.e. that not every semilattice of algebras from K_E is a sum of a direct system of algebras from K_E.

Let E consist of the following two equations:

(1.1) $$xy = x^2,$$

(1.2) $$x^8 = x^4,$$

where the abbreviation x^{2^n} is defined by induction:

$$x^2 = xx, \quad x^{2^n} = x^{2^{n-1}}x^{2^{n-1}}.$$

Let $\mathfrak{A}$ be the free algebra with 2 free generators x, y in the class K_E, and let $\mathfrak{B}$ be the free algebra with two free generators x, y in the class $K_{R(E)}$. Clearly the elements of $\mathfrak{A}$ are x, x^2, x^4, y, y^2 and y^4, and similarly, the elements of $\mathfrak{B}$ are $x, x^2, x^4, y, y^2, y^4, xy, (xy)^2, yx$ and $(yx)^2$.

We shall show that the algebra $\mathfrak{B}$ cannot be represented as the sum of a direct system of algebras from the class K_E. Let

$$B_1 = \{x, x^2, x^4\}, \quad B_2 = \{y, y^2, y^4\}, \quad B_3 = \{xy, (xy)^2\}, \quad B_4 = \{yx, (yx)^2\}.$$

Clearly there are only two possible decompositions of $\mathfrak{B}$ into the disjoint sum of subalgebras from K_E, namely

$$B_1 \cup B_2 \cup B_3 \cup B_4 \quad \text{and} \quad B_1 \cup B_2 \cup (B_3 \cup B_4).$$

If the subalgebras B_i $(i = 1, 2, 3, 4)$ would form a direct system, then we would have $B_1 < B_3$ and $B_2 < B_4$, but from $(xy)(yx) = (xy)^2 \in B_3$ and $(yx)(xy) = (yx)^2 \in B_4$ it would follow $B_3 < B_4$ and $B_4 < B_3$, a contradiction. If the subalgebras B_1, B_2 and $B_3 \cup B_4$ would form a direct system, then $B_1 < B_3 \cup B_4$, $B_2 < B_3 \cup B_4$, and the algebras B_1 and B_2 would be incomparable. But in this case a trivial checking of all possibilities shows that it is impossible to define the homomorphisms of the direct system.

From theorems 1-3 and (i) we get the following corollaries:

COROLLARY 1. *Every algebra of $K_{R(E)}$ is decomposable into a disjoint sum of subalgebras belonging to K_E.*

COROLLARY 2. *If K_E is idempotent, i.e. for all fundamental operations F_t one has $F_t(x, x, \ldots, x) = x$, or either if $C(E)$ contains an equation of the form $f(x, y) = x$, then the sums of direct systems of algebras from K_E form an equational class, namely $K_{R(E)}$.*

COROLLARY 3. *The sums of direct systems of Boolean algebras, lattices, or groups form equational classes.*

(This follows from corollary 2, if one defines those algebras using the equation $x + xy = x$ in the first two cases, and the equation $xyy^{-1} = x$ in the case of groups, and not using algebraic constants.

EXAMPLE 1. An algebra $\mathfrak{A} = (X; +, \cdot)$ is a sum of a direct system of lattices if and only if the fundamental operations $+$ and $\cdot$ are idempotent, commutative, associative and, moreover, the following equations are satisfied:

$$x + xy = x(x + y),$$

$$x + y + (x + y)(u + v) = x + y + xu + xv + yu + yv,$$

$$xy(xy + uv) = xy(x + u)(x + v)(y + u)(y + v).$$

EXAMPLE 2. If we define a group as an algebra $(X; \cdot, ^{-1})$ in which the following equations are satisfied:

$$(xy)z = x(yz), \quad y^{-1}yx = x = xyy^{-1},$$

then from our theorems we obtain that the algebra $(X; \cdot, ^{-1})$ is a sum of a direct system of groups if and only if the following equations are satisfied:

$$(xy)z = x(yz), \quad (xy)^{-1} = y^{-1}x^{-1}, \quad (x^{-1})^{-1} = x, \quad xxx^{-1} = x,$$

and

$$y^{-1}yx = xyy^{-1}.$$

The proof of those facts follows from (i) and (ii). In the first example the operation $x + xy$ and in the second example the operation xyy^{-1} are the P-functions needed.

References

[1] J. Płonka, *On a method of construction of abstract algebras*, Fund. Math. 61 (1967), pp. 183-189.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 1. 12. 1967