Since n is arbitrary, we have

$$\sum_i |f(a_i-)-f(\beta_i+)| \leqslant \tfrac{2}{3}\varepsilon < \varepsilon .$$

This proves the theorem.

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Normal models and the field Σ_1^*

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It is known ([2], theorem 35, p. 394) that every axiomatizable, consistent, first-order theory has a model in $\Sigma_2 \cap \Pi_2$. Putnam [5] has shown that such theories, based on a finite number of predicates, have models in Σ_1^*, where Σ_1^* denotes the field of predicates generated by the recursively enumerable predicates.

The purpose of this paper is to extend this result to the case of an axiomatizable, consistent, first-order theory with identity built on a finite number of predicates. More precisely, we shall show that such a theory, if it possesses an infinite normal model, has a normal model in Σ_1^*. The model exhibited will be the simplest possible, in the sense that it will contain Ramsey indiscernibles and only those extra elements needed for completion. This answers completely the open question of Mostowski in [4], p. 39.

§ 1. The theory T_0 and the main theorem. As mentioned previously, we shall employ the symbol Σ_1^* to stand for the smallest field of number-theoretic predicates (of all orders, 1-ary, 2-ary, etc.) which includes the recursively enumerable predicates and is closed under the truth functions (e.g. closed under $\neg$ (not) and $\vee$ (or)).

Let T_0 stand for an axiomatizable, consistent, first-order theory with equality which is based on the predicates $P_0^{n(0)}, \ldots, P_m^{n(m)}$. Here the superscripts denote the order of the predicate symbol, and we shall usually omit them. P_0 will be taken to be the *equality symbol*. All models of T_0 are hence of the form $(A; \mathfrak{R}_0, \ldots, \mathfrak{R}_m)$ where $A \neq \emptyset$ and $\mathfrak{R}_j \subset A^{n(j)}$. If $\mathfrak{R}_0$ is the identity relation on A, then the model is said to be *normal*.

THEOREM 1.1. (MAIN THEOREM). *If T_0 has an infinite normal model, then T_0 also has a normal model $\mathfrak{Q} = (N; \mathfrak{Q}_0, \ldots, \mathfrak{Q}_m)$ where N is the set of natural numbers and $\mathfrak{Q}_j \in \Sigma_1^*$ for all $j = 1, \ldots, m$.*

To prove this theorem it will be necessary to work with models of theories stronger than T_0. But before defining these new theories we shall need a result due to Ramsey.

§ 2. A lemma due to Ramsey. In Ramsey [6], theorem A, p. 82, is proved an interesting combinatorial theorem, a corollary of which will be useful in the sequel.

THEOREM 2.1 (Ramsey). *Let Γ be an infinite set and m, n positive integers; and let all those subsets of Γ which have exactly n members (the n-subsets) be partitioned in any manner into m mutually exclusive classes C_i, $i = 1, \dots, m$. Then, assuming the axiom of choice, Γ must contain an infinite subset Δ such that all the n-subsets of Δ belong to the same C_i.*

The corollary will be stated in considerably less generality than possible. Before stating it let us introduce the following definition.

DEFINITION 2.2. Two n-tuples of natural numbers $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ will be called *similar* ($(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$) iff, for all i, j, $a_i < a_j$, $a_i = a_j$, $a_i = a_j$ or $a_i > a_j$ according as $b_i < b_j$, $b_i = b_j$ or $b_i > b_j$ respectively. [For example, $(3, 7, 3, 18, 4) \sim (5, 14, 5, 17, 9)$.]

It is easy to check that similarity is an equivalence relation. Also it is obvious that there are only a finite number of equivalence classes. Let $\sim[a_1, \dots, a_n]$ stand for the class determined by $(a_1, \dots, a_n)$.

COROLLARY 2.3. *Let $\mathfrak{R}(x_1, \dots, x_n)$ be an n-ary predicate on an infinite subset A of the natural numbers. Then there exists an infinite subset $\Delta \subset A$ such that, for all n-tuplets $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in \Delta^n$, if $(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$ then $\mathfrak{R}(a_1, \dots, a_n) \equiv \mathfrak{R}(b_1, \dots, b_n)$.*

Proof. Let $\sim[x_1, \dots, x_n] = D_1$ be the first class in some enumeration, $D_1, \dots, D_r$, of the finite number of similarity classes over A. To any $(x_1, \dots, x_n) \in D_1$ associate the set $\{u_1, \dots, u_{n'}\}$ of distinct elements in the n-tuple $(x_1, \dots, x_n)$. $n' \leqslant n$ and n' is well-defined for the class D_1. Also this association is a (1-1) correspondence between D_1 and the set of n'-subsets of A.

In theorem 2.1 take $m = 2$ and $n = n'$ and put $\{u_1, \dots, u_{n'}\}$ in C_1 if $\mathfrak{R}(x_1, \dots, x_n)$ is true or in C_2 if $\mathfrak{R}(x_1, \dots, x_n)$ is false, where $(x_1, \dots, x_n)$ is associated with $\{u_1, \dots, u_{n'}\}$ as described above. Then theorem 2.1 gives us an infinite subset $\Delta_1 \subset A$ such that for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in D_1 \cap \Delta_1^n$, $\mathfrak{R}(x_1, \dots, x_n) \equiv \mathfrak{R}(y_1, \dots, y_n)$.

Exactly the same argument can be repeated, replacing A by Δ_1 and D_1 by $D_2 \cap \Delta_1^n$, to get an infinite set $\Delta_2 \subset \Delta_1$ such that, for all $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in D_2 \cap \Delta_2^n$, $\mathfrak{R}(x_1, \dots, x_n) \equiv \mathfrak{R}(y_1, \dots, y_n)$. And, since $\Delta_2^n \subset \Delta_1^n$, we still have $\mathfrak{R}(x_1, \dots, x_n) \equiv \mathfrak{R}(y_1, \dots, y_n)$ for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in D_1 \cap \Delta_2^n$. Repeating this argument for all classes, $D_1, \dots, D_r$, we shall eventually obtain a set Δ $(= \Delta_r)$ with the property mentioned in the theorem.

This set Δ will be called a *set of indiscernibles* for $\mathfrak{R}$.

§ 3. Indiscernibles and the theory T_1. Let $\mathfrak{M} = (N, \mathfrak{M}_0, \dots, \mathfrak{M}_m)$ be a normal model for T_0, where N is the set of natural numbers. Applying corollary 2.3 to $\mathfrak{M}_0$ over N we get a set, say Δ_0, of indiscernibles for $\mathfrak{M}_0$. Repeating 2.3 for $\mathfrak{M}_1$ over Δ_0, we get a set, say Δ_1, which is a set of indiscernibles, simultaneously, for both $\mathfrak{M}_0$ and $\mathfrak{M}_1$. Hence applying 2.3 in this fashion $m+1$ times, we get a set Δ $(= \Delta_m)$ which is a set of indiscernibles for all of the relations $\mathfrak{M}_0, \dots, \mathfrak{M}_m$, simultaneously.

On the basis of this remark the theory T_0 can be consistently extended by adding an infinite list of new constants, $a_0, \dots, a_n, \dots$, and the following new axioms:

$$\ulcorner \neg P_0(a_i, a_j) \urcorner \quad \textit{for all } i, j \textit{ with } i \neq j;$$

$$\ulcorner P_k(a_{i_1}, \dots, a_{i_{n(k)}}) \equiv P_k(a_{j_1}, \dots, a_{j_{n(k)}}) \urcorner \textit{ for all } k,\ 0 \leqslant k \leqslant m, \textit{ and for all } (i_1, \dots, i_{n(k)}), (j_1, \dots, j_{n(k)}) \textit{ such that } (i_1, \dots, i_{n(k)}) \sim (j_1, \dots, j_{n(k)}).$$

It is obvious that $\mathfrak{M}$ discussed above would be a model for this new theory if a_i were taken to be the ith member of Δ, according to ordinary $<$. This new consistent, axiomatizable theory will be called T_1.

§ 4. ϵ-terms, the theory T_2 and more indiscernibility. The theory T_1 will now be strengthened, as to its naming power, by the addition of the ϵ-terms and the ϵ-axiom schema ([1]; pp. 9-18). That is, for every ϵ-wff $A(x)$, which does not contain any variable in both free and bound occurrences, we add the axiom $\ulcorner \exists x A(x) \supset A(\epsilon_x A(x)) \urcorner$. [It is convenient to think of individual variables divided into two classes, the free variables and the bound variables.]

By the second ϵ-theorem ([1]; pp. 130-149) this augmented theory is also a consistent, axiomatizable theory. We shall denote it by T_2. It is well known by the upward Skolem-Löwenheim theorem ([3]; pp. 64-67) that the set of constant terms of T_2, call it $\mathfrak{T}$, forms the universe of a non-normal model for a certain complete, consistent extension of T_2. Let us call this new model

$$\mathfrak{N} = (\mathfrak{T}; \mathfrak{N}_0, \dots, \mathfrak{N}_m; a_0, \dots, a_n, \dots).$$

In $\mathfrak{N}$ the constants, $a_0, \dots, a_n, \dots$, form a set of indiscernibles for $\mathfrak{N}_0, \dots, \mathfrak{N}_m$, simultaneously. Obviously (cf. § 3) what is meant here and elsewhere is indiscernibility with respect to the subscripts. This indiscernibility, however, is not strong enough for our purposes. What we desire is the following more general type of indiscernibility. If $A(x_1, \dots, x_k)$ is an ϵ-wff of T_2 with *no* occurrences of the constants, $a_0, \dots, a_n, \dots$, and occurrences of exactly the distinct free variables $x_1, \dots, x_k$ (taken in increasing subscript order) then this ϵ-wff determines a k-ary predicate over $\mathfrak{T}$. And we would like to have indiscernibility over this predicate also. We shall use the same symbol A to stand for the k-ary predicate over $\mathfrak{T}$.

Of course we can again apply corollary 2.3 to A over N, which is the set of subscripts of the a_i's, to get an infinite subset $\Delta \subset N$ such that $\{a_i: i \in \Delta\}$ is a set of indiscernibles for $\mathfrak{N}_0, \ldots, \mathfrak{N}_m$ and A. Certainly for any *finite* number of ϵ-wffs like A above we could apply the same procedure. That is, if $A_1, \ldots, A_p$ are any ϵ-wffs, each having at least one free variable and satisfying the conditions as A above, then there exists a subset $\Delta_p \subset N$ such that $\{a_i: i \in \Delta_p\}$ is a set of indiscernibles for $\mathfrak{N}_0, \ldots, \mathfrak{N}_m, A_1, \ldots, A_p$, where again A_i is also used to stand for the associated predicate over $\mathfrak{T}$.

§ 5. The theory T_3. Using the notation of the previous section, let $\{a_i: i \in \Delta_p\}$ be a set of indiscernibles for $\mathfrak{N}_0, \ldots, \mathfrak{N}_m, A_1, \ldots, A_p$. Since the ϵ-terms provide names for the Skolem functions needed to satisfy the axioms of T_0, it is easy to see by the downward Skolem–Löwenheim theorem that $\mathfrak{N}' = (\mathfrak{T}'; \mathfrak{N}'_1, \ldots, \mathfrak{N}'_m)$ is also a model for T_0, where $\mathfrak{T}'$ is the collection of all constant terms of T_2 which contain no occurrences of constants a_i for $i \notin \Delta_p$ and where $\mathfrak{N}'_i$ is the restriction of $\mathfrak{N}_i$ to the subset $\mathfrak{T}'$.

But now from the foregoing, we see that the theory T_2 can be consistently extended by adding the following extra axioms: (where the above wffs, $A_1, \ldots, A_p$, are of orders $m(1), \ldots, m(p)$, respectively)

$$\ulcorner A_k(a_{i_1}, \ldots, a_{i_{m(k)}}) \equiv A_k(a_{j_1}, \ldots, a_{j_{m(k)}})\urcorner \textit{ for all } k,\ 1 \leqslant k \leqslant p \textit{ and for all } (i_1, \ldots, i_{m(k)}),\ (j_1, \ldots, j_{m(k)}) \textit{ such that } (i_1, \ldots, i_{m(k)}) \sim (j_1, \ldots, j_{m(k)}).$$

It is obvious that $\mathfrak{N}'$ is a model for this augmented theory if a_i is taken to be the ith member of Δ_p in the ordinary increasing order of subscripts.

Now, however, since $A_1, \ldots, A_p$ were arbitrary ϵ-wffs, we see by the compactness property of first-order theories that the theory T_2 can be consistently extended to a new theory T_3 by the addition of *all* formulas of the form:

$$\ulcorner A(a_{i_1}, \ldots, a_{i_k}) \equiv A(a_{j_1}, \ldots, a_{j_k})\urcorner,$$

where $A(x_1, \ldots, x_k)$ is an ϵ-wff containing occurrences of exactly the distinct free variables $x_1, \ldots, x_k$, listed in increasing subscript order, and no occurences of the constants a_i, and where $(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k)$.

The new theory T_3 is certainly still axiomatizable. Also it might be remarked that the condition, that $A(x_1, \ldots, x_k)$ contain exactly the distinct free variables $x_1, \ldots, x_k$, is only for notational precision and could have been replaced by other conventions. Finally, as will be realized later, we could have restricted our additional axioms by demanding that A be an atomic ϵ-wff.

§ 6. The model $\mathfrak{P}$, characterizations of $\Sigma_2 \cap \Pi_2$ and Σ_1^*, and the ideal numbers. By the upward Skolem–Löwenheim theorem we can complete T_3 to get a model $\mathfrak{P} = (\mathfrak{T}; \mathfrak{P}_0, \ldots, \mathfrak{P}_m; a_0, \ldots, a_m, \ldots)$ of T_3, where again $\mathfrak{T}$ is the set of all constant terms of T_3. The most important property of $\mathfrak{P}$, resulting from the fact that $\mathfrak{P}$ is a model of T_3, is the following indiscernibility property:

(I) *If $A(x_1, \ldots, x_k)$ is an ϵ-wff containing occurrences of exactly the distinct free variables $x_1, \ldots, x_k$, listed in increasing subscript order, and no occurrences of the constants a_i and if $(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k)$, then*

$$\models_{\mathfrak{P}} A(a_{i_1}, \ldots, a_{i_k}) \equiv A(a_{j_1}, \ldots, a_{j_k}).$$

The use of the semantical symbol "$\models_{\mathfrak{P}}$" is justified here, since $\mathfrak{T}$ contains all the ϵ-terms needed for interpretation of an ϵ-wff.

The model $\mathfrak{P}$ can be turned into a numerical model over N by first arithmetizing T_3 in a standard, effective way. Listing the elements of $\mathfrak{T}$ in the order of their Gödel numbers, say $t_0, \ldots, t_n, \ldots$, we define $\varrho = \{(n, t_n): n \in N\}$. For technical reasons which will appear latter, we use the standard listing ϱ to define another listing, θ, as follows:

$$\begin{aligned} \theta(i) &= \textit{the first } t_n \textit{ in the list } \varrho \textit{ such that, for all } j,\ 0 \leqslant j < i,\\ &\quad\ t_n \neq \theta(j), \textit{ and } t_n \textit{ of the form } a_i, \textit{ if } i \textit{ is odd};\\ &= \textit{the first } t_n \textit{ in the list } \varrho \textit{ such that for all } j,\ 0 \leqslant j < i,\\ &\quad\ t_n \neq \theta(j), \textit{ if } i \textit{ is even}. \end{aligned}$$

Obviously θ is also an effective listing of $\mathfrak{T}$. From now on we shall consider θ as an identification between N and $\mathfrak{T}$; and, using θ, we can change $\mathfrak{P}$ into an isomorphic numerical model (i.e. with universe N). We shall still use $\mathfrak{P} = (N; \mathfrak{P}_0, \ldots, \mathfrak{P}_m; a_0, \ldots, a_n, \ldots)$ as notation for the numerical model. Finally, we shall assume that T_3 has been arithmetized in such a way that

(∗) (i) *if $i < j$, then $\theta^{-1}(a_i) < \theta^{-1}(a_j)$, and*

(ii) *if b is an ϵ-term and a_i occurs in b, then $\theta^{-1}(a_i) < \theta^{-1}(b)$.*

Certainly ϱ would have properties (∗) for most standard arithmetizations and, in passing from ϱ to θ, these properties are, if anything, accentuated.

It is well known from Kleene ([2]; pp. 394, 395) that if $\mathfrak{P}$ is obtained by one of the standard completion procedures, then the predicates of $\mathfrak{P}$ (under either of the identifications, ϱ or θ) are in $\Sigma_2 \cap \Pi_2$.

Putnam ([5], theorems 1 and 2, pp. 51, 52) provides the following alternate characterizations of the classes $\Sigma_2 \cap \Pi_2$ and Σ_1^*. These characterizations will be used from now on in changing $\mathfrak{P}$ into a model in Σ_1^*.

THEOREM 6.1. *A number-theoretic predicate $\mathfrak{R}(x_1, \ldots, x_n) \in \Sigma_2 \cap \Pi_2$ iff there exists a recursive $(n+1)$-ary characteristic function $f(x_1, \ldots, x_n, y)$ (i.e. f takes only the values 0, 1) such that, for all $(x_1, \ldots, x_n)$,*

(i) $\lim\limits_{y\to\infty} f(x_1, \ldots, x_n, y)$ *exists and*

(ii) $\mathfrak{R}(x_1, \ldots, x_n)$ *is true iff* $\lim\limits_{y\to\infty} f(x_1, \ldots, x_n, y) = 1$.

THEOREM 6.2. *A number-theoretic predicate* $\mathfrak{R}(x_1, \ldots, x_n) \in \Sigma_1^*$ *iff there exists a recursive* $(n+1)$*-ary characteristic function* $f(x_1, \ldots, x_n, y)$ *and a natural number* k *such that, for all* $(x_1, \ldots, x_n)$,

(i) *there are at most* k *integers* y *such that* $f(x_1, \ldots, x_n, y) \neq f(x_1, \ldots, x_n, y+1)$ *and*

(ii) $\mathfrak{R}(x_1, \ldots, x_n)$ *is true iff* $\lim_{y\to\infty} f(x_1, \ldots, x_n, y) = 1$.

DEFINITION 6.3. If $\mathfrak{R} \in \Sigma_2 \cap \Pi_2$ (Σ_1^*) and f has the properties mentioned in theorem 6.1 (6.2), then f will be called a (k-) *trial* and *error function* for $\mathfrak{R}$. And, for each $i \in N$, any number m with the property: for all $x_1, \ldots, x_n$, $y \in N$, if $y \geqslant m$ and $x_1, \ldots, x_n \leqslant i$, then $f(x_1, \ldots, x_n, y) = f(x_1, \ldots, x_n, m)$: will be called an (f, i)*-modulus of convergence.*

In giving the next definition we shall use r_0 to stand for the maximum order of the predicate symbols $P_0, \ldots, P_m$. That is, using our previous notation, $r_0 = \max[n(0), \ldots, n(m)]$. Also from now on let $f_0, \ldots, f_m$ be trial and error functions for the predicates $\mathfrak{P}_0, \ldots, \mathfrak{P}_m$, respectively. That these functions exist follows from theorem 6.1 and the previously mentioned fact that $\mathfrak{P}_i \in \Sigma_2 \cap \Pi_2$ for $i = 0, \ldots, m$.

DEFINITION 6.4. A number of the form 2^s3^t is said to be an *ideal number* if

(i) s (i.e. $\theta(s)$) is an ϵ-term,

(ii) t is the smallest integer which is simultaneously an $(f_i, s+2r_0)$-modulus of convergence for $i = 0, \ldots, m$, and

(iii) for all $j < s$, $f_0(j, s, t) = 0$.

Let us remark that if 2^s3^t is ideal then $\mathfrak{P}_0(j, s)$ is false for all j such that j is a constant a_i. If $j < s$, then this is what is given as condition (iii) of (6.4) (even if j is not an a_i). Now suppose that there is a $j_1 > s$ where j_1 is a constant a_i such that $\mathfrak{P}_0(j_1, s)$ is true. Pick $j_2 \neq j_1$ such that $j_2 > s$ and j_2 is a constant a_i. Then by (⁎) of this section, j_1 and j_2 would both be greater than any a_i's in the ϵ-term s. Hence it is easy to see by (I) of this section that

$$\mathfrak{P}_0(j_1, s) \equiv \mathfrak{P}_0(j_2, s).$$

Therefore, since $\mathfrak{P}_0$ is an equivalence relation and $\mathfrak{P}_0(j_1, s)$ is supposed true, $\mathfrak{P}_0(j_1, j_2)$ would also be true. But, since $\mathfrak{P}$ is a model of T_3 and $\ulcorner \neg \mathfrak{P}_0(\theta(j_1), \theta(j_2)) \urcorner$ is an axiom of T_3, $\mathfrak{P}_0(j_1, j_2)$ would be false. This contradiction shows that $\mathfrak{P}_0(j, s)$ is false for all j such that j is a constant a_i.

§ 7. The injection φ and the induced model $\mathfrak{Q}$ .We define a function π: $N \to N$ as follows:

$\pi(i)$ *is the* ith, *natural number* n, *in the usual ordering, such that* n *is a constant* a_k (i.e. *such that* $\theta(n) = a_k$ *for some* k).

And the function φ: $N \to N$ is defined as follows:

$$\varphi(2^s3^t) = s \quad \textit{if} \quad 2^s3^t \textit{ is an ideal number},$$

$$\varphi(j) = \pi(j-k) \quad \textit{if } j \textit{ is not an ideal number and there are } k \textit{ ideal numbers less than } j.$$

Obviously π is recursive and injective and φ is injective. Also it follows from definition 6.4 and its accompanying remark and the distinctness of the constants a_i (cf. theory T_1 in § 3) that in every equivalence class of N under $\mathfrak{P}_0$ there is an element of the form $\varphi(i)$ for exactly one natural number i.

A new structure $\mathfrak{Q} = (N; \mathfrak{Q}_0, \ldots, \mathfrak{Q}_m)$ can be induced using $\mathfrak{R}$ and the injection φ. For arbitrary $i = 0, \ldots, m$ and an arbitrary $n(i)$-tuplet $(m_1, \ldots, m_{n(i)})$ of natural numbers we define

$$\mathfrak{Q}_i(m_1, \ldots, m_{n(i)}) \equiv \mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)})).$$

By the fact that $\mathfrak{P}_0$ is compatible with $\mathfrak{P}_1, \ldots, \mathfrak{P}_m$ (by the equality axioms) and the fact that the range of φ represents all classes of $N/\mathfrak{P}_0$ we see that $\mathfrak{Q}$ is also a model of T_0. And, since φ picks out exactly one member from each class of $N/\mathfrak{P}_0$, the model $\mathfrak{Q}$ is normal, i.e. $\mathfrak{Q}_0$ is identity.

In conjunction with the above definition of $\mathfrak{Q}$, there is one more indiscernibility property of $\mathfrak{P}$ that will be essential in § 8. Suppose that $A(x_1, \ldots, x_k)$ is a wff with no ϵ-terms and no constants a_i, where $x_1, \ldots, x_k$ are the distinct free variables occuring in A. Let $(u_1, \ldots, u_k)$ be a k-tuplet of elements of N and let $p_1, \ldots, p_r$ be an enumeration of all those places in the tuplet where the term occurring is a constant a_i which is greater than any ϵ-term of the tuplet. Let $(v_1, \ldots, v_k)$ be another k-tuplet from N such that (i) $p_1, \ldots, p_r$ is still an enumeration for this tuplet also of all those places where the term occurring is a constant a_i which is greater than any ϵ-term of the tuplet, (ii) $u_i = v_i$ for all $i \notin [p_1, .., p_r]$, and (iii) $(u_{p_1}, \ldots, u_{p_r}) \sim (v_{p_1}, \ldots, v_{p_r})$.

By examining carefully the k-tuples $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ and the general indiscernibility property (I) of § 6, it can be seen that

$$\models_{\mathfrak{P}} A(\theta(u_1), \ldots, \theta(u_k)) \equiv A(\theta(v_1), \ldots, \theta(v_k)).$$

In § 8 this latter fact is used in the special case that A is an atomic wff of the form $P_i(x_1, \ldots, x_{n(i)})$. Then we get the result that, for $(u_1, \ldots, u_{n(i)})$, $(v_1, \ldots, v_{n(i)})$ related as above, $\mathfrak{P}_i(u_1, \ldots, u_{n(i)}) \equiv \mathfrak{P}_i(v_1, \ldots, v_{n(i)})$.

§ 8. $\mathfrak{Q}$ is in Σ_1^*. We end the proof of the main theorem (1.1) by showing that $\mathfrak{Q}_i \in \Sigma_1^*$ for all $i = 1, \ldots, m$. Actually it can be shown that $\mathfrak{Q}_i$ has an $n(i)$-trial and error function. Rather than enter into details of formalization, we shall use Church's Thesis and definition 6.2 to show that to decide $\mathfrak{Q}_i$ for a given $n(i)$-tuplet of natural numbers we have an effective, but non-terminating, procedure which eventually will give us the correct answer (and keep repeating it) after *at most* $n(i)$ "changes of mind". Of course, we shall never be in a position to know if our "latest answer" is the correct answer, unless the procedure has already changed its mind $n(i)$ times, cf. [5], p. 49.

Let i be arbitrary, such that $1 \leqslant i \leqslant m$, and let $(m_1, \ldots, m_{n(i)})$ be an arbitrary $n(i)$-tuplet of natural numbers. It might happen that none of the m's is of the form 2^s3^t. In this case all of the m_j's are a_i's, i.e. $\varphi(m_j)$ is an a_i. Now, since φ is not necessarily recursive, we might never know the exact value of $\varphi(m_j)$; but this is not important since obviously, by definition of φ, $(m_1, \ldots, m_{n(i)}) \sim (\varphi(m_1), \ldots, \varphi(m_{n(i)}))$. That is $(m_1, \ldots, m_{n(i)})$ already gives us the similarity class of $(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$. Hence, since $\mathfrak{P}_i$ is well determined on the similarity classes by property (I) of § 6 and since there are only a *finite* number of similarity classes, we can determine $\mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$ and hence $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$. It can be assumed, of course, that we are given, at the outset of our procedure, the value of $\mathfrak{P}_i$ on each of the finite number of similarity classes.

If there is an m_j in $(m_1, \ldots, m_{n(i)})$ which is of the form $2^{s_j}3^{t_j}$, then among such m_j's we pick that one such that s_j is the largest, and if there are several such then that one with the largest t_j. It is provisionally assumed that this m_j is ideal. The procedure now splits into two main parts. Half of the time, VERT (verification time), we check on m_j to see if it is really ideal. In VERT we simply check conditions (i), (ii), (iii) of definition 6.4. Conditions (i) and (iii) are checked effectively, since the functions θ and f_0 are recursive. The word "smallest" in condition (ii) is obviously checked effectively. But to verify that t_j is an (f_i, s_j+2r_0)-modulus of convergence for each $i = 0, \ldots, m$, is a process which, although it obviously proceeds effectively, will never terminate unless there does exist an $i = 0, \ldots, m$ such that t_j is *not* an (f_i, s_j+2r_0)-modulus of convergence. In other words, if m_j is in fact ideal, then VERT will proceed effectively forever; whereas if m_j is not ideal, then VERT will eventually discover this, report the fact and stop.

The other half of the time, CAT (construct and answer time), we use m_j in its guise of ideal number to locate the other ideal numbers, $m = 2^s3^t$, such that $m < m_j$ (in which case, since m_j is considered ideal, $s < s_j$ and $t \leqslant t_j$). Obviously by the definition of ideal number and the assumption that m_j is ideal, these numbers $m < m_j$ can be found effectively. Hence, due to the effectiveness of θ and π, we can effectively calculate $\varphi(n)$ for all $n \leqslant m_j$. Therefore, since $m_j \geqslant 2s_j$ and all odd numbers v are such that $\varphi(v)$ is an a_i, we can actually do the following:

(†) (i) effectively calculate $\varphi(m_k)$ for all $m_k \leqslant m_j$ in $(m_1, \ldots, m_{n(i)})$,

(ii) effectively decide which of the terms in $(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$ are ϵ-terms and which are a_i's,

(iii) effectively pick out those terms in $(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$ which are a_i's larger than the ϵ-term s_j (i.e., larger in terms of the identification θ).

Let us suppose that $p_1, \ldots, p_r$ is an enumeration of those places in $(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$ where the term occurring is a constant a_i which is greater than the ϵ-term s_j. As mentioned above, as longs as m_j is being retained as an ideal number in VERT, we can effectively determine this enumeration. By definition of r_0 $(= \max[n(0), \ldots, n(m)])$ we see that $r < r_0$. Hence, since there are at least r distinct odd numbers v such that $s_j < v \leqslant s_j+2r_0$ and since v is an a_i for odd v (cf. definition of θ), we can effectively pick a new $n(i)$-tuplet $(v_1, \ldots, v_{n(i)})$ such that

(i) $p_1, \ldots, p_r$ is still an enumeration for this tuplet also of all those places where the term occurring is a constant a_i which is greater than s_j,

(ii) $\varphi(m_i) = v_i$ for all $i \notin [p_1, \ldots, p_r]$,

(iii) $(\varphi(m_{p_1}, \ldots, \varphi(m_{p_r})) \sim (v_{p_1}, \ldots, v_{p_r})$, and finally

(iv) $s_j < v_{p_1}, \ldots, v_{p_r} \leqslant s_j+2r_0$.

Now by the indiscernibility discussion in § 7, (⁎) in § 6, and (†) (iii) above, we see that

$$\mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)})) \equiv \mathfrak{P}_i(v_1, \ldots, v_{n(i)}) .$$

And, since $0 \leqslant v_1, \ldots, v_{n(i)} \leqslant s_j+2r_0$, we can use f_i and the (f_i, s_j+2r_0)-modulus t_j to give us effectively the value of $\mathfrak{P}_i(v_1, \ldots, v_{n(i)})$ (i.e. T or F according as $f_i(v_1, \ldots, v_{n(i)}, t_j) = 1$ or 0 respectively). Hence we effectively obtain the value of $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$.

If it happens above that $r = 0$, that is that any a_i's in $(\varphi(m_1), \ldots \ldots, \varphi(m_{n(i)}))$ are less than the ϵ-term s_j, then of course we simply evaluate $\mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)}))$ directly as $f(\varphi(m_1), \ldots, \varphi(m_{n(i)}), t_j)$. Since $0 \leqslant \varphi(m_1), \ldots, \varphi(m_{n(i)}) \leqslant s_j < s_j+2r_0$, we thus get the correct answer for $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$, assuming as always that m_j is ideal.

All of this work in CAT is effective and terminates with the correct answer for $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$—or rather keeps repeating the answer—on the assumption that $m_j = 2^{s_j}3^{t_j}$ is actually ideal. If in reality m_j is not ideal, then sooner or later VERT will tell us that. In such a case, we stop CAT and begin our work all over again, this time realizing that m_j is not an ideal number, or equivalently that $\varphi(m_j)$ is a constant a_i. That is, among the remaining possible ideal numbers we again pick the largest as before and, assuming that it is ideal and letting VERT check it, we again find another $n(i)$-tuplet $(v_1, \ldots, v_{n(i)})$ in CAT such that $0 \leqslant v_1, \ldots$

$\ldots, v_{n(i)} \leqslant s_j + 2r_0$ and $\mathfrak{P}_i(\varphi(m_1), \ldots, \varphi(m_{n(i)})) \equiv \mathfrak{P}_i(v_1, \ldots, v_{n(i)})$. Thus, under the changed circumstances, we find another, possibly different, answer to the value of $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$.

If we think of our procedure as assigning the value T to $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$ at the outset and only changing its mind when CAT terminates in a value different from the previously accepted value, then, since different CAT answers result only from rejections in VERT of supposed ideal numbers, we see that there can be at most $n(i)$ changes of mind for the value of $\mathfrak{Q}_i(m_1, \ldots, m_{n(i)})$. In other words, our effective procedure will eventually give us the correct answer (and keep repeating it) at some generally unknown time with at most $n(i)$ intervening changes of mind. Hence $\mathfrak{Q}_i \in \Sigma_1^*$ for each $i = 1, \ldots, m$. This ends the proof of the main theorem 1.1.

In closing, it should be remarked that the proof of theorem 1.1 goes through with practically no changes if T_0 contains an at most denumerable number of individual constants. Also it is obvious from the actual use made of property (I) that T_3 could have been taken to be T_2 augmented only by the extra indiscernibility axioms provided by *atomic* ϵ-wffs.

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On equational classes of abstract algebras defined by regular equations

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0. Introduction. In [1], theorem I it was shown that the operation of the sum of a direct system of algebras of the same similarity class preserves all regular equations and only such equations are preserved by it. (For the definitions of these notions, see below.)

It follows that if the algebras $\mathfrak{A}_i$ belong to an equational class K_E defined by the set E of equations, then the sum of any direct system of algebras consisting of the algebras $\mathfrak{A}_i$ belongs to the equational class $K_{R(E)}$ defined by the set $R(E)$ of all equations which are consequences of the set E and are regular. The question can be asked whether the converse is true, i.e. whether every algebra of the class $K_{R(E)}$ can be represented as a sum of a direct system of algebras from K_E. It turns out (see below), that in many important cases, e.g. for lattices, Boolean algebras and groups this is the case, but in general the answer is negative. However, below we shall give a full description of algebras from $K_{R(E)}$ using the class K_E.

At first we shall recall some definitions and results from [1], for convenience of the reader.

Let

$$\mathcal{A} = \langle I, \langle \mathfrak{A}_i \rangle_{i \in I}, \langle \varphi_{ij} \rangle_{i,j \in I, i \leqslant j} \rangle$$

be a direct system of similar algebras, without nullary fundamental operations, indexed by a poset I with the least upper bound property. Let $\langle F_t \rangle_{t \in T}$ be the set of fundamental operations of the algebras in $\mathcal{A}$, and let A_i be the carrier of $\mathfrak{A}_i$. The *sum of the system* $\mathcal{A}$ is an algebra $S(\mathcal{A}) = \langle A; \langle F_t \rangle_{t \in T} \rangle$ where A is the disjoint sum of the carriers A_i ($i \in I$), and the fundamental operations F_t are defined by

$$F_t(a_1, \ldots, a_n) = F_t(\varphi_{i_1 i_0}(a_1), \ldots, \varphi_{i_n i_0}(a_n))$$

where $a_j \in A_{i_j}$ and i_0 is the least upper bound of $i_1, \ldots, i_n$.

An equation $f = g$ where f and g are terms in an algebra we shall call *regular* if on both sides of it the same free variables occur.