[14] J. L. Kelley, *General topology*, Princeton 1955.
[15] K. Kuratowski, *Topology*, vol. I, Warszawa 1966.
[16] C. G. Lekkerkerker, *On metric properties of bases for a separable metric space*, Nieuw Arch. Wisk. 13 (1965), pp. 192-199.
[17] A. Lelek, *Some problems in metric topology*, Louisiana State University (Lecture Notes), Baton Rouge, Louisiana, 1965.
[18] — *On totally paracompact metric spaces*, Proc. Amer. Math. Soc. 19 (1968), pp. 168-170.
[19] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*. Sitzungsberichte Wiener Akad. 133 (1924), pp. 421-444.
[20] F. Rothberger, *Eine Verschärfung der Eigenschaft C*, Fund. Math. 30 (1938), pp. 50-55.
[21] W. Sierpiński, *Hypothèse du continu*, New York 1956.
[22] R. Telgársky, *Star-finite coverings and local compactness*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 625-628.

Reçu par la Rédaction le 15. 11. 1967

Some results on AC-ω functions

by

M. C. Chakrabarty (West Bengal, India)

1. Introduction. Let $\omega(x)$ be non-decreasing on the closed interval $[a, b]$. Outside the interval, $\omega(x)$ is defined by $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and let $D = [a, b] - S$. Let S_0 denote the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$ on each of which $\omega(x)$ is constant,

$$S_1 = \{a_1, b_1, a_2, b_2, \ldots\}, \quad S_2 = SS_1 \quad \text{and} \quad S_3 = [a, b]\cdot S - (S_0 + S_2).$$

R. L. Jeffery [4] has denoted by $\mathfrak{U}$ the class of functions $f(x)$ defined as follows.

$f(x)$ is defined on the set $S\cdot[a, b]$ such that $f(x)$ is continuous at each points of $S\cdot[a, b]$ with respect to S. If a point $x_0 \in D$, $f(x)$ tends to a limit (finite or infinite) as x tends to x_0+ and x_0- over the points of the set S. These limits will be denoted by $f(x_0+)$ and $f(x_0-)$, respectively. When $x < a$, $f(x) = f(a+)$ and $f(x) = f(b-)$ for $x > b$. $f(x)$ may or may not be defined at the points of the set D.

In [4] Jeffery has introduced the following definitions.

DEFINITION 1.1. A function $f(x)$ defined on $[a, b]$ and in the class $\mathfrak{U}$ is *absolutely continuous* relative to ω, AC-ω, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') on $[a, b]$ with $\sum_i\{\omega(x_i'+) - \omega(x_i-)\} < \delta$ the relation $\sum_i |f(x_i'+) - f(x_i-)| < \varepsilon$ is satisfied.

DEFINITION 1.2. Let $f(x)$ belong to the class $\mathfrak{U}$. For any x and any $h \neq 0$ with $x + h \in S$, the function $\psi(x, h)$ is defined by

$$\psi(x, h) = \begin{cases} \dfrac{f(x+h) - f(x-)}{\omega(x+h) - \omega(x-)}, & h > 0, \quad \omega(x+h) - \omega(x-) \neq 0, \\ \dfrac{f(x+h) - f(x+)}{\omega(x+h) - \omega(x+)}, & h < 0, \quad \omega(x+h) - \omega(x+) \neq 0, \\ 0, & \qquad\qquad \omega(x+h) - \omega(x\pm) = 0. \end{cases}$$

If $\psi(x, h)$ tends to a limit as $h \to 0$, this limit is called the *ω-derivative of $f(x)$ at x* and is denoted by $f'_\omega(x)$. The upper and lower limits of $\psi(x, h)$

on the right and on the left are the corresponding *upper* and *lower* ω-*derivatives of* $f(x)$ *at* x.

Let $\omega(a) = y_0 < y_1 < y_2 < \ldots < y_n = \omega(b)$ be any subdivision of $[\omega(a), \omega(b)]$ where $y_i \in \omega(I)$, $I = [a, b]$. For any y_i there is an $x_i \in I$ for which $\omega(x_i) = y_i$. If for an y_i there exist more than one x_i such that $y_i = \omega(x_i)$, we take any one x_i. The set of points $x_0, x_1, x_2, \ldots, x_n$ is called a ω-*subdivision* ([1], [2]) of $[a, b]$. In [1] the following definition has been introduced.

DEFINITION 1.3. Let $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}$. The least upper bound of the sums

$$V = \sum_i |f(x_i+) - f(x_{i-1}-)|$$

for all possible ω-subdivisions $x_0, x_1, \ldots, x_n$ of $[a, b]$ is called the *total* ω-*variation*, $V_\omega(f; a, b)$, of $f(x)$ on $[a, b]$. If $V_\omega(f; a, b) < +\infty$, then $f(x)$ is said to be of *bounded variation relative to* ω, BV-ω, on $[a, b]$.

We introduce the following definition.

DEFINITION 1.4. Let $f(x)$ be defined on $[a, b]$. $f(x)$ is said *to have the property* (N_ω) if for every set $e \subset [a, b]$ with ω-measure [4] zero the Lebesgue measure of the map $f(e)$ is zero.

The purpose of the present paper is to study some properties of AC-ω functions and to show that if $f(x)$ is BV-ω on $[a, b]$ and possesses property (N_ω), then $f(x)$ is AC-ω on $[a, b]$.

We require the following known results.

THEOREM 1.1. ([3], Th. 4.1.) *If $f(x)$ belongs to the class $\mathfrak{U}$, then all the four ω-dervatives of $f(x)$ are ω-measurable* [4].

THEOREM 1.2. ([3], Th. 6.2.) *If $f(x)$ is* BV-ω *on $[a, b]$, then $f'_\omega(x)$ exists and is finite at all points of $[a, b]$ except a set of ω-measure zero.*

THEOREM 1.3. ([3], Th. 6.3.) *If $f(x)$ is* BV-ω *on $[a, b]$, then $f'_\omega(x)$ is summable in Lebesgue-Stieltjes sense* [4], *summable* (LS), *on* $[a, b]$.

The *outer* ω-measure [4] and the ω-*measure* [4] of a set E will be denoted by $\omega^*(E)$ and $|E|_\omega$, respectively.

2. Preliminary lemmas.

LEMMA 2.1. *Let E be a subset of S_3 and let $f(x)$ belong to the class $\mathfrak{U}$. If $f'_\omega(x)$ exists at each point of E and $|f'_\omega(x)| \leqslant k$ on E, then*

$$m^*f(E) \leqslant k\omega^*(E).$$

Proof. Choose $\varepsilon > 0$ arbitrarily. For each positive integer n denote by E_n the set of points x of E such that

$$|f(x) - f(y)| \leqslant (k+\varepsilon)|\omega(x) - \omega(y)| \quad \text{whenever} \quad y \in S \text{ and } |x-y| \leqslant 1/n.$$

Then clearly $E_1 \subset E_2 \subset \ldots$ and $E = \sum_1^\infty E_n$. So

$$f(E_1) \subset f(E_2) \subset \ldots \quad \text{and} \quad f(E) = \sum_1^\infty f(E_n).$$

Therefore $\lim m^*f(E_n) = m^*f(E)$. We find a positive integer N such that $m^*f(E_N) > m^*f(E) - \varepsilon$. We now choose a sequence $\{I_n\}$ of pairwise disjoint intervals I_n with the properties

(i) $m(I_n) \leqslant 1/N$ for each n,

(ii) $E_N \subset \sum_1^\infty I_n$, and

(iii) $\sum_1^\infty |I_n|_\omega < \omega^*(E_N) + \varepsilon$.

From the definition of the set E_N we see that for every pair of elements x_1, x_2 of $I_n \cdot E_N$ we have

$$|f(x_1) - f(x_2)| \leqslant (k+\varepsilon)|\omega(x_1) - \omega(x_2)| \leqslant (k+\varepsilon)|I_n|_\omega$$

which gives that $m^*f(I_n \cdot E_n) \leqslant (k+\varepsilon)|I_n|_\omega$. Since $E_N = \sum_1^\infty I_n \cdot E_N$, we have

$$m^*f(E_N) \leqslant \sum_{n=1}^\infty m^*f(I_n \cdot E_N) \leqslant (k+\varepsilon) \sum_1^\infty |I_n|_\omega < (k+\varepsilon)[\omega^*(E_N) + \varepsilon].$$

So

$$m^*f(E) - \varepsilon < (k+\varepsilon)[\omega^*(E) + \varepsilon].$$

Since $\varepsilon > 0$ is arbitrary, we obtain $m^*f(E) \leqslant k\omega^*(E)$.

LEMMA 2.2. *Let $f(x)$ be defined on $[a, b]$, be in class $\mathfrak{U}$, and have property* (N_ω). *If E is the set of points in $[a, b]$ where $f'_\omega(x)$ exists and $|f'_\omega(x)| \leqslant k$, then*

$$m^*f(E) \leqslant (k\omega^*(E)).$$

Proof. We have $[a, b] = S_0 + S_2 + S_3 + D$ where $|S_0|_\omega = 0$, $|S_2|_\omega = 0$ and D is at most enumerable. Since $f(x)$ possesses property (N_ω),

$$m^*f(ES_0) = 0, \quad m^*f(ES_2) = 0.$$

Also $m^*f(ED) = 0$ since $f(ED)$ is at most enumerable. Since $E = ES_0 + ES_2 + ES_3 + ED$, we have

$$\begin{aligned} m^*f(E) &\leqslant m^*f(ES_3), \\ &\leqslant k\omega^*(ES_3) \quad \text{(by lemma 2.1)}, \\ &\leqslant k\omega^*(E). \end{aligned}$$

icm©

LEMMA 2.3. *Let $f(x)$ be defined on $[a, b]$, be in class $\mathfrak{U}$ and possess property (N_ω). If E is a ω-measurable set on $[a, b]$ where $f'_\omega(x)$ exists finitely, then*

$$m^*f(E) \leqslant (\mathrm{LS}) \int_E |f'_\omega(x)|\, d\omega .$$

Proof. If $|f'_\omega(x)|$ is not summable (LS) on E, the result is trivial. So we suppose that $|f'_\omega(x)|$ is summable (LS) on E. Let ε be any positive number. For any positive integer n, let E_n denote the set of points of E for which $(n-1)\varepsilon \leqslant |f'_\omega(x)| < n\varepsilon$. From theorem 1.1 it follows that $f'_\omega(x)$ and therefore $|f'_\omega(x)|$ is ω-measurable on E. So the sets $E_1, E_2, \ldots$ are ω-measurable, they are pairwise disjoint and $E = \sum_1^\infty E_n$. Therefore we have

$$m^*f(E) \leqslant \sum_1^\infty m^*f(E_n) \leqslant \sum_1^\infty n\varepsilon\cdot |E_n|_\omega \quad \text{(by lemma 2.2)},$$

$$\leqslant \sum_1^\infty \Big(\int_{E_n} |f'_\omega(x)|\, d\omega + \varepsilon\cdot |E_n|_\omega$$

$$\leqslant \int_E |f'_\omega(x)|\, d\omega + \varepsilon\cdot |E|_\omega .$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$m^*f(E) \leqslant \int_E |f'_\omega(x)|\, d\omega .$$

LEMMA 2.4. *Let $f(x)$ be defined on $[a, b]$ and let be in class $\mathfrak{U}$. If $f(x)$ is bounded on S, then for any two points $\alpha, \beta\ (>\alpha)$ of S in $[a, b]$,*

$$|f(\alpha) - f(\beta)| \leqslant m^*f(E) + \sum_i |f(x_i+) - f(x_i-)| ,$$

where $E = [\alpha, \beta]\cdot S$ and $x_1, x_2, \ldots$ are the points of D which lie in $[\alpha, \beta]$.

Proof. If the series $\sum_i |f(x_i+) - f(x_i-)|$ is divergent, the result is trivial. So we suppose that $\sum_i |f(x_i+) - f(x_i-)|$ is finite. Let A, B be the lower and upper bounds of $f(x)$ on E. Denote by A_i, B_i the minimum and maximum of $f(x_i+)$, $f(x_i-)$ and write $D_0 = \{x_1, x_2, \ldots\}$. We show that

(1) $$(A, B) \subset f(E) + f(D_0) + \sum_i [A_i, B_i] .$$

Let $y \in (A, B)$. If there is a point $x \in [\alpha, \beta] = E + D_0$ such that $f(x) = y$, then $y \in f(E) + f(D_0)$. Suppose that there is no point x in $[\alpha, \beta]$ for which $y = f(x)$. Since $A < y < B$, there exist two points c, d of E such that $A \leqslant f(c) < y < f(d) \leqslant B$. Without loss of generality we may assume that $c < d$. Let

$$P = \{x;\ x \in E\cdot[c, d] \text{ and } f(x) < y\} .$$

Denote by ξ the upper bound of P. It is easy to see that $c < \xi < d$ and $f(\xi-) \leqslant y \leqslant f(\xi+)$. Since $y \neq f(x)$ for any x in $[\alpha, \beta]$, it follows that $\xi \in D_0$. So $\xi = x_i$ for some i which shows that $y \in [A_i, B_i]$. This proves (1). Since D_0 is at most enumerable, $m^*f(D_0) = 0$. So from (1) we have

$$|f(\alpha) - f(\beta)| \leqslant B - A \leqslant m^*f(E) + \sum_i |f(x_i+) - f(x_i-)| .$$

LEMMA 2.5. *If $f(x)$ has property (N_ω) and is BV-ω on $[a, b]$, then for any two points $\alpha, \beta\ (>\alpha)$ in $[a, b]$,*

$$|f(\beta+) - f(\alpha-)| \leqslant \int_{[\alpha,\beta]} |f'_\omega(x)|\, d\omega .$$

Proof. The following cases come up for consideration:

(i) $a < \alpha$, $\beta < b$,

(ii) $a = \alpha$, $\beta < b$,

(iii) $a < \alpha$, $\beta = b$ and $a = \alpha$, $b = \beta$.

Case (i). Since $f(x)$ is BV-ω on $[a, b]$ by theorem 1.3, $f'_\omega(x)$ is summable (LS) on $[a, b]$. So for any ω-measurable set $e \subset [a, b]$,

(2) $$\int_e |f'_\omega(x)|\, d\omega \to 0 \quad \text{as} \quad |e|_\omega \to 0 .$$

Let $\varepsilon > 0$ be arbitrary. Choose two points ξ, η of S with $a < \xi < \alpha$, $\beta < \eta < b$. Let $A = [\xi, \eta]\cdot S$ and $B = [\xi, \eta] - S$. Since B is at most enumerable, we can take its elements as $x_1, x_2, \ldots$ Then by lemma 2.4 we have

$$|f(\xi) - f(\eta)| \leqslant m^*f(A) + \sum_i |f(x_i+) - f(x_i-)| .$$

Let A_1 denote the set of points of A where $f'_\omega(x)$ exists finitely and let $A_2 = A - A_1$. Then by theorem 1.2, $|A_2|_\omega = 0$. Since $f(x)$ possesses property (N_ω), $m^*f(A_2) = 0$. Therefore $m^*f(A) = m^*f(A_1)$. Hence using lemma 2.3 we get

$$|f(\xi) - f(\eta)| \leqslant m^*f(A_1) + \sum_i |f(x_i+) - f(x_i-)|$$

$$\leqslant \int_{A_1} |f'_\omega(x)|\, d\omega + \int_B |f'_\omega(x)|\, d\omega = \int_{[\xi,\eta]} |f'_\omega(x)|\, d\omega$$

$$\leqslant \int_{[\xi,\alpha)} |f'_\omega(x)|\, d\omega + \int_{[\alpha,\beta]} |f'_\omega(x)|\, d\omega + \int_{(\beta,\eta]} |f'_\omega(x)|\, d\omega .$$

Letting $\xi \to \alpha -$ and $\eta \to \beta +$ over the points of S, we obtain

$$(3) \qquad |f(\beta+)-f(\alpha-)| \leqslant \int_{[\alpha,\beta]} |f'_\omega(x)|\,d\omega .$$

Proceeding as above we can prove (3) in other cases.

LEMMA 2.6. *If $f(x)$ is AC-ω on $[a,b]$, then for every set of pairwise disjoint intervals (α_i,β_i) on $[a,b]$ with $\alpha_i,\beta_i \in S$ we have*

$$\sum_i |f(\alpha_i)-f(\beta_i)| \leqslant V_\omega(f;a,b) .$$

Proof. Let n be any positive integer. Without loss of generality we may assume that $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)$ are in the order of increasing end points. If the points $a \leqslant \alpha_1,\beta_1,\alpha_2,\beta_2,\ldots,\alpha_n,\beta_n \leqslant b$ form a ω-subdivision of $[a,b]$, then clearly

$$\sum_i |f(\alpha_i)-f(\beta_i)| \leqslant V_\omega(f;a,b) .$$

Otherwise $\omega(x)$ has the same value at two or more of the consecutive end points of the intervals at one or more stages. For simplicity let us suppose that $\omega(\alpha_1)=\omega(\beta_1)$ and $\omega(\beta_3)=\omega(\alpha_4)$ but at all points of the set $\{\alpha_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\beta_4,\alpha_5,\beta_5,\ldots,\alpha_n,\beta_n\}$ $\omega(x)$ has distinct values. Then the points

$$a \leqslant \alpha_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\beta_4,\alpha_5,\beta_5,\ldots,\alpha_n,\beta_n \leqslant b$$

form a ω-subdivision of $[a,b]$. So

$$|f(\alpha_2)-f(\beta_2)|+|f(\alpha_3|-f(\beta_3|+|f(\beta_3)-f(\beta_4)+\sum_{i=5}^{n}|f(\alpha_i)-f(\beta_i)| \leqslant V_\omega(f;a,b) .$$

By theorem 4 [1], $f(\alpha_1)=f(\beta_1)$ and $f(\beta_3)=f(\alpha_4)$. So we have

$$\sum_{i=1}^{n}|f(\alpha_i)-f(\beta_i)| \leqslant V_\omega(f;a,b) .$$

Since n is arbitrary, we obtain

$$\sum_i |f(\alpha_i)-f(\beta_i)| \leqslant V_\omega(f;a,b) .$$

This proves the lemma.

3. Results on AC-ω functions.

THEOREM 3.1. *If $f(x)$ is AC-ω on $[a,b]$, then*

$$f(x)=f(a+)+\int_a^x f'_\omega(t)\,d\omega \quad \textit{for} \quad x \in [a,b]\cdot S$$

where $\int_\alpha^\beta \varphi(t)\,d\omega$ denotes the (LS) *integral of $\varphi(t)$ over the closed interval $[\alpha,\beta]$.*

Proof. Since $f(x)$ is AC-ω on $[a,b]$, by theorem 5 [1], it is BV-ω on $[a,b]$. So by theorems 1.2 and 1.3 the ω-derivative $f'_\omega(x)$ of $f(x)$ exists and is finite at all points of $[a,b]$ except a set of ω-measure zero and $f'_\omega(x)$ is summable (LS) on $[a,b]$. We define the function $g(x)$ by

$$g(x)=\begin{cases} f(a+) & \text{for} \quad x<a, \\ f(a+)+\int_a^x f'_\omega(t)\,d\omega & \text{for} \quad a \leqslant x \leqslant b, \\ f(b-) & \text{for} \quad x>b. \end{cases}$$

Then clearly $g(x)$ belongs to the class $\mathfrak{U}$ and AC-ω on $[a,b]$. By theorem 1 [4], $f(x)-g(x)=k$ (constant) on S. Letting $x \to a+$ over the points of S we see that $f(a+)-g(a+)=k$. Now, for $x \in [a,b]$,

$$(4) \qquad g(x)=f(a+)+\int_A f'_\omega\,d\omega+\int_{(a,x]} f'_\omega\,d\omega , \quad \text{where} \quad A=\{a\} .$$

If

$$|A|_\omega=\omega(a+)-\omega(a-)=\omega(a+)-\omega(a)\neq 0 ,$$

then $f'_\omega(a)=0$ which gives that in any case the first integral of (4) is zero. So, letting $x \to a+$ over the points of S in (4) we get $g(a+)=f(a+)$. So $k=0$. Thus

$$f(x)=f(a+)+\int_a^x f'_\omega(t)\,d\omega \quad \text{for all } x \in [a,b]\cdot S .$$

THEOREM 3.2. *If $f(x)$ is AC-ω on $[a,b]$, then $f(x)$ has property* (N_ω).

Proof. By the previous theorem we have

$$f(x)=f(a+)+\int_a^x f'_\omega(t)\,d\omega \quad \text{for all } x \in [a,b]\cdot S .$$

Let E be any set on $[a,b]$ with ω-measure zero. Then $E \subset S$. Write $E'=E(a,b)$ and $E''=E-E'$. The set E'' contains at most two points and therefore so does $f(E'')$ which gives that $m^*f(E'')=0$. Hence $m^*f(E)=m^*f(E')$. Choose $\varepsilon>0$ arbitrarily. Since $|f'_\omega(x)|$ is summable (LS) on $[a,b]$, we can find a $\delta>0$ such that for any ω-measurable set $e \subset [a,b]$,

$$\int_e |f'_\omega|\,d\omega<\varepsilon \quad \text{whenever} \quad |e|_\omega<\delta.$$

There exists an open set $A \subset (a,b)$ such that $E' \subset A$ and $|A|_\omega<\delta$. Let $A=\sum_i(\alpha_i,\beta_i)$, where the intervals (α_i,β_i) are pairwise disjoint. Write

$$A_i=(\alpha_i,\beta_i) \quad \text{and} \quad E_i=E'\cdot(\alpha_i,\beta_i) \quad (i=1,2,\ldots) .$$

If $\xi, \eta\ (>\xi)$ be any two points of E_i, then

$$f(\eta)-f(\xi)=\int_{\xi}^{\eta} f'_\omega(t)\,d\omega\,.$$

So,

$$|f(\xi)-f(\eta)|\leqslant\int_{\xi}^{\eta}|f'_\omega|\,d\omega\leqslant\int_{A_i}|f'_\omega|\,d\omega$$

which gives that $m^*f(E_i)\leqslant\int_{A_i}|f'_\omega|\,d\omega$. Therefore

$$m^*f(E)=m^*f(E')\leqslant\sum_i m^*f(E_i)\leqslant\sum_i\int_{A_i}|f'_\omega|\,d\omega=\int_A|f'_\omega|\,d\omega<\varepsilon\,.$$

Since $\varepsilon>0$ is arbitrary, $m^*f(E)=0$. This proves the theorem.

THEOREM 3.3. *If $f(x)$ is AC-ω on $[a,b]$, then $f(x)=f_1(x)-f_2(x)$ for all $x\in[a,b]\cdot S$ where $f_1(x)$ and $f_2(x)$ are non-decreasing on $[a,b]$.*

Proof. Let A denote the set of points in $[a,b]$ where $f'_\omega(x)$ exists and is finite. We define the functions $p(x)$ and $q(x)$ on $[a,b]$ as follows:

$$p(x)=\begin{cases}f'_\omega(x) & \text{if } f'_\omega(x)\geqslant 0\\ 0 & \text{if } f'_\omega(x)<0\end{cases},\qquad x\in A\,,$$

$$q(x)=\begin{cases}0 & \text{if } f'_\omega(x)<0\\ -f'_\omega(x) & \text{if } f'_\omega(x)\geqslant 0\end{cases},\qquad x\in A\,,$$

and $p(x)=q(x)=0$ for $x\in[a,b]-A$. Then $p(x)$ and $q(x)$ are non-negative on $[a,b]$ and $f'_\omega(x)=p(x)-q(x)$ for $x\in A$. Each of $p(x)$ and $q(x)$ is summable (LS) on $[a,b]$. We now define the functions $P(x)$ and $Q(x)$ by

$$P(x)=\begin{cases}0 & \text{for } x<a\,,\\ \int_a^x p(t)\,d\omega & \text{for } a\leqslant x\leqslant b\,,\\ P(b-) & \text{for } x>b\,,\end{cases}$$

$$Q(x)=\begin{cases}0 & \text{for } x<a\,,\\ \int_a^x q(t)\,d\omega & \text{for } a\leqslant x\leqslant b\,,\\ Q(b-) & \text{for } x>b\,.\end{cases}$$

Then clearly $P(x)$ and $Q(x)$ belong to the class $\mathfrak{U}$ and are AC-ω on $[a,b]$. Since $p(x)$ and $q(x)$ are non-negative, the functions $P(x)$ and $Q(x)$ are non-decreasing on $[a,b]$. If $x\in[a,b]\cdot S$, then

$$\begin{aligned}f(x)&=f(a+)+\int_a^x f'_\omega(t)\,d\omega\\ &=f(a+)+\int_{E_x} f'_\omega(t)\,d\omega\,, \qquad E_x=[a,x]\cdot A\,,\\ &=f(a+)+\int_{E_x}[p(t)-q(t)]\,d\omega\\ &=f(a+)+\int_a^x p(t)\,d\omega-\int_a^x q(t)\,d\omega\\ &=f(a+)+P(x)-Q(x)\,.\end{aligned}$$

This proves the theorem.

The following example illustrates the extent of Theorem 3.3:

EXAMPLE. Let the functions $\omega(x)$ and $f(x)$ be defined in the interval $[0,2]$ as follows:

$$\omega(x)=\begin{cases}0\,, & 0\leqslant x\leqslant 1\,,\\ x-1 & 1\leqslant x\leqslant 2\,,\end{cases}$$

and

$$f(x)=\begin{cases}x\sin(1/x)\,, & 0<x\leqslant 2\,,\\ 0 & x=0\,.\end{cases}$$

Clearly $f(x)$ belongs to the class $\mathfrak{U}$. Let $0\leqslant x_0<x_1<\ldots<x_n\leqslant 2$ be any ω-subdivision of $[0,2]$. Then $0\leqslant x_0\leqslant 1$, $x_1>1$. We have

$$\begin{aligned}V&=\sum_{i=1}^{n}|f(x_i+)-f(x_{i-1}-)|\\ &=\sum_{i=1}^{n}|f(x_i)-f_{i-1}(x)|\\ &\leqslant|f(x_0)|+|f(x_1)|+\sum_{i=2}^{n}|f(x_i)-f(x_{i-1})|\\ &\leqslant 3+\overset{2}{\underset{1}{V}}(f)=\text{a finite quantity}\,,\end{aligned}$$

because $f(x)$ is BV on $[1,2]$. Thus $f(x)$ is BV-ω on $[0,2]$.

But it is well known that $f(x)$ is not BV on $[0,2]$. This example shows that every BV-ω function cannot be expressed as the difference of two non-decreasing functions.

THEOREM 3.4. *If $f(x)$ is AC-ω on $[a,b]$, then*

$$\int_a^b|f'_\omega(t)|\,d\omega\leqslant V_\omega(f;a,b)\leqslant\int_a^b|f'_\omega(t)|\,d\omega+\sum_i|f(\xi_i+)-f(\xi_i-)|$$

where $\xi_1,\xi_2\ldots$ are the points in $[a,b]$ for which $f(\xi_i+)\neq f(\xi_i-)$.

Proof. Let $x_0, x_1, x_2, \ldots, x_n$ be any ω-subdivision of $[a, b]$. Then

$$V = \sum_{i=1}^{n} |f(x_i+)-f(x_{i-1})|$$
$$\leqslant \sum_{i=1}^{n} |f(x_i-)-f(x_{i-1}+)| + \sum_{i=0}^{n-1} |f(x_i+)-f(x_i-)|$$
$$\leqslant \sum_{i=1}^{n} \int_{(x_{i-1},x_i)} |f'_\omega(t)|\,d\omega + \sum_i |f(\xi_i+)-f(\xi_i)|$$
$$\leqslant \int_a^b |f'_\omega(t)|\,d\omega + \sum_i |f(\xi_i+)-f(\xi_i-)| .$$

Because $x_0, x_1, x_2, \ldots, x_n$ is an arbitrary ω-subdivision of $[a, b]$, we have

$$V_\omega(f; a, b) \leqslant \int_a^b |f'_\omega(t)|\,d\omega + \sum_i |f(\xi_i+)-f(\xi_i-)| . \tag{5}$$

Choose $\varepsilon > 0$ arbitrarily. Since $|f'_\omega(t)|$ is summable (LS) on $[a, b]$, there exists a $\delta > 0$ such that for any ω-measurable set $e \subset [a, b]$

$$\int_e |f'_\omega(t)|\,d\omega < \varepsilon \quad \text{whenever} \quad |e|_\omega < \delta .$$

Let A denote the set of points in (a, b) where $f'_\omega(x)$ exists and is finite. Write

$$A_1 = \{x;\ x \in A \text{ and } f'_\omega(x) \geqslant 0\} \quad \text{and} \quad A_2 = A - A_1 .$$

Choose closed sets B_1 and B_2 with $B_1 \subset A_1$, $B_2 \subset A_2$ such that $|A_1 - B_1|_\omega < \delta$ and $|A_2 - B_2|_\omega < \delta$. By theorem 2 of [5], p. 46 there exist open sets G_1 and G_2 contained in (a, b) with $B_1 \subset G_1$, $B_2 \subset G_2$, $G_1 G_2 = 0$ and $|G_1 - B_1|_\omega < \delta$, $|G_2 - B_2|_\omega < \delta$. We choose sets

$$P_1 = \sum_{i=1}^{m} [a_i, \beta_i] , \quad P_2 = \sum_{i=1}^{n} [a'_i, \beta'_i] \quad \text{with} \quad a_i, \beta_i, a'_i, \beta'_i \in S$$

and $P_1 \subset G_1$, $P_2 \subset G_2$ such that

$$|G_i - P_i|_\omega < \delta \quad (i = 1, 2) .$$

Now,

$$\int_a^b |f'_\omega(t)|\,d\omega = \int_{(a,b)} |f'_\omega(t)|\,d\omega = \int_{A_1} f'_\omega(t)\,d\omega - \int_{A_2} f'_\omega(t)\,d\omega \tag{6}$$
$$< \int_{B_1} f'_\omega(t)\,d\omega - \int_{B_2} f'_\omega(t)\,d\omega + 2\varepsilon$$

$$< \int_{G_1} f'_\omega(t)\,d\omega - \int_{G_2} f'_\omega(t) + 4\varepsilon$$
$$< \int_{P_1} f'_\omega(t)\,d\omega - \int_{P_2} f'_\omega(t) + 6\varepsilon$$
$$< \sum_{i=1}^{m} \int_{a_i}^{\beta_i} f'_\omega(t)\,d\omega - \sum_{i=1}^{n} \int_{a'_i}^{\beta'_i} f'_\omega(t)\,d\omega + 6\varepsilon$$
$$< \sum_{i=1}^{m} |f(\beta_i) - f(a_i)| + \sum_{i=1}^{n} |f(\beta'_i) - f(a'_i)| + 6\varepsilon .$$

Since the intervals $(a, \beta_1), (a_2, \beta_2), \ldots, (a_m, \beta_m), (a'_1, \beta'_1), (a'_2, \beta'_2), \ldots, (a'_n, \beta'_n)$ are pairwise disjoint, we get from (6) and lemma 2.6

$$\int_a^b |f'_\omega(t)|\,d\omega < V_\omega(f; a, b) + 6\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\int_a^b |f'_\omega(t)|\,d\omega \leqslant V_\omega(f; a, b) . \tag{7}$$

The theorem follows from (5) and (7).

THEOREM 3.5. *If $f(x)$ is BV-ω on $[a, b]$ and has the property* (N_ω), *then $f(x)$ is AC-ω on $[a, b]$.*

Proof. Let ε be any positive number. By theorem 1.3, $f'_\omega(x)$ is summable (LS) on $[a, b]$. So there is a $\delta > 0$ such that for any ω-mesurable set $e \subset [a, b]$ with $|e|_\omega < \delta$ we have

$$\int_e |f'_\omega|\,d\omega < \tfrac{1}{3}\varepsilon . \tag{8}$$

Let $\{(a_i, \beta_i)\}$ be any set of pairwise disjoint open intervals with $\sum_i \{\omega(\beta_i+) - -\omega(a_i-)\} < \delta$. Choose any positive integer n. Without loss of generality we may suppose that the intervals $(a_1, \beta_1), (a_2, \beta_2), \ldots, (a_n, \beta_n)$ are in the order of increasing end points. We divide the set $\{1, 2, \ldots, n\}$ into two parts A and B such that A consists of odd integers and B consists of even integers. Let

$$e_1 = \sum_{i \in A} [a_i, \beta_i] , \quad e_2 = \sum_{i \in B} [a_i, \beta_i] .$$

The intervals $[a_i, \beta_i]$ $(i \in A)$ are pairwise disjoint; the intervals $[a_i, \beta_i]$ $(i \in B)$ are also pairwise disjoint. Further $|e_1|_\omega < \delta$ and $|e_2|_\omega < \delta$. Using lemma 2.5 and formula (8) we get

$$\sum_{i=1}^{n} |f(a_i-) - f(\beta_i+)| \leqslant \sum_{i=1}^{n} \int_{a_i}^{\beta_i} |f'_\omega|\,d\omega \leqslant \int_{e_1} |f'_\omega|\,d\omega + \int_{e_2} |f'_\omega|\,d\omega < \tfrac{2}{3}\varepsilon .$$

Since n is arbitrary, we have

$$\sum_i |f(a_i-)-f(\beta_i+)| \leqslant \tfrac{2}{3}\varepsilon < \varepsilon\,.$$

This proves the theorem.

I am grateful to Dr. P. C. Bhakta for his kind help and suggestions in the preparation of the paper.

References

[1] P. C. Bhakta, *On functions of bounded ω-variation*, Rev. Mat. Univ. Parma (2) 6 (1965).

[2] — *On functions of bounded ω-variation, II*, J. Aust. Math. Soc. Vol. V, part 3 (1965), pp. 380-387.

[3] M. C. Chakrabarty, *Some results on ω-derivatives and* BV-ω *functions*, to appear in J. Aust. Math. Soc.

[4] R. L. Jeffery, *Generalised integrals with respect to functions of bounded variation*, Canad. J. Math. 10 (1958), pp. 617-628.

[5] I. P. Natanson, *Theory of functions of a real variable*, Vol. I, New York ,1955.

SURI VIDYASAGAR COLLEGE

Reçu par la Rédaction le 18. 11. 1967

Normal models and the field Σ_1^*

by

Gustav Hensel and Hilary Putnam (Cambridge, Mass.)

It is known ([2], theorem 35, p. 394) that every axiomatizable, consistent, first-order theory has a model in $\Sigma_2 \cap \Pi_2$. Putnam [5] has shown that such theories, based on a finite number of predicates, have models in Σ_1^*, where Σ_1^* denotes the field of predicates generated by the recursively enumerable predicates.

The purpose of this paper is to extend this result to the case of an axiomatizable, consistent, first-order theory with identity built on a finite number of predicates. More precisely, we shall show that such a theory, if it possesses an infinite normal model, has a normal model in Σ_1^*. The model exhibited will be the simplest possible, in the sense that it will contain Ramsey indiscernibles and only those extra elements needed for completion. This answers completely the open question of Mostowski in [4], p. 39.

§ 1. The theory T_0 and the main theorem. As mentioned previously, we shall employ the symbol Σ_1^* to stand for the smallest field of number-theoretic predicates (of all orders, 1-ary, 2-ary, etc.) which includes the recursively enumerable predicates and is closed under the truth functions (e.g. closed under $\neg$ (not) and $\vee$ (or)).

Let T_0 stand for an axiomatizable, consistent, first-order theory with equality which is based on the predicates $P_0^{n(0)}, \ldots, P_m^{n(m)}$. Here the superscripts denote the order of the predicate symbol, and we shall usually omit them. P_0 will be taken to be the *equality symbol*. All models of T_0 are hence of the form $(A; \mathfrak{R}_0, \ldots, \mathfrak{R}_m)$ where $A \neq \emptyset$ and $\mathfrak{R}_j \subset A^{n(j)}$. If $\mathfrak{R}_0$ is the identity relation on A, then the model is said to be *normal*.

THEOREM 1.1. (MAIN THEOREM). *If T_0 has an infinite normal model, then T_0 also has a normal model $\mathfrak{Q} = (N; \mathfrak{Q}_0, \ldots, \mathfrak{Q}_m)$ where N is the set of natural numbers and $\mathfrak{Q}_j \in \Sigma_1^*$ for all $j = 1, \ldots, m$.*

To prove this theorem it will be necessary to work with models of theories stronger than T_0. But before defining these new theories we shall need a result due to Ramsey.