Some cover properties of spaces

by

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Let X be a regular topological space. We say that X is a *Hurewicz space* if for every sequence $\gamma_1, \gamma_2, \ldots$ of open covers of X there exists a cover α of X such that $\alpha = \alpha_1 \cup \alpha_2 \cup \ldots$ where $\alpha_i \subset \gamma_i$ and α_i is finite for $i = 1, 2, \ldots$ This class of spaces has been introduced by W. Hurewicz (see [10], property E*). Clearly, each Hurewicz space is a Lindelöf space. The aim of the present paper is to give several characterizations of Hurewicz spaces under the assumption of metrizability.

Let X be a metrizable topological space. Given a metrization of X, we say that a collection α of subsets of X is a *zero sequence* if α is countable and the diameters of elements of α are real numbers converging to zero. The following conditions describe some properties of X in terms of zero sequences. They will be shown to be equivalent to each other (see Theorem 1 below).

(i) *For every metrization of X there exists a cover α of X such that α is a zero sequence.*

(ii) *For every metrization of X there exists an open basis β in X such that β is a zero sequence.*

(iii) *There exists a metrization of X for which every open basis in X contains a cover α of X such that α is a zero sequence.*

(iv) *For every metrization of X every open basis in X contains a cover α of X such that α is a zero sequence.*

(v) *For every metrization of X every open basis in X contains an open basis β in X such that β is a zero sequence.*

It has been stated without proof by W. Hurewicz that properties (i) and (iii) are equivalent for metrizable spaces (see [11], p. 204). A proof of this theorem is given in our Theorem 1.

Metric spaces possessing property (iii) are called *strongly Lindelöf* [17] provided the metrizations whose existence is required by (iii) coincide with those already given in metric spaces. The class of strongly Lindelöf spaces has been distinguished by K. Menger (see [19], property E). Belonging to this class might depend on a metric, but since (iii) and (iv) turn out to be equivalent, being a strongly Lindelöf space is a topological invariant. Moreover, the metrizability yields the equivalence of all properties involved, including (E) and (E*), and strongly Lindelöf spaces coincide with Hurewicz metric spaces. It should be noted that some metric analogues of properties described in (i) and (ii) are not topological invariants [7]. An essentially smaller class of spaces has been examined by W. Hurewicz (see [11], property E**). We say that a regular topological space X is *strongly Hurewicz* if for every sequence $\gamma_1, \gamma_2, \ldots$ of open covers of X there exist finite collections $\alpha_i \subset \gamma_i$ $(i = 1, 2, \ldots)$ satisfying the equality

$$X = \bigcup_{i=1}^{\infty} \bigcap_{j=0}^{\infty} |\alpha_{i+j}| ,$$

where $|\alpha|$ denotes the union of elements of α. It has been shown by W. Sierpiński that there exist Hurewicz separable metric spaces which are not strongly Hurewicz (see [11], p. 196). A theorem of W. Hurewicz [11] says that a metrizable topological space X is strongly Hurewicz if and only if for every metrization of X there exists a countable cover of X consisting of totally bounded sets. It seems to be still an unsettled question whether or not each strongly Hurewicz metric space is an absolute F_σ (see [11], pp. 200 and 204). If this question has an affirmative answer, absolute F_σ can be characterized in terms of open covers. An analogous characterization of locally compact spaces has recently been done by R. Telgársky [22]. However, it has been proved by W. Hurewicz [10] that each absolute analytic Hurewicz metric space is an absolute F_σ. Thus, for instance, the space $\mathfrak{N}$ of irrational numbers with the natural topology is not Hurewicz. There exists an example of an open basis β in $\mathfrak{N}$ (see [4], this example is due to N. N. Konstantinov) such that every cover $\gamma \subset \beta$ contains a strictly increasing sequence of elements, i.e. there are sets $G_1 \subset G_2 \subset \ldots$ where $G_i \in \gamma$ and $G_i \neq G_{i+1}$ for $i = 1, 2, \ldots$ Such a basis cannot contain a cover which is a zero sequence, for any metrization of $\mathfrak{N}$. Independently, C. G. Lekkerkerker [16] has found another example of a basis having the latter property. On the other hand, some rather pathological properties can be possessed by Hurewicz separable metric spaces, e.g. the property of F. Rothberger (see [20], property C″). A space X fulfils (C″) if for every collection of open sets $G_i(x) \subset X$ satisfying $x \in G_i(x)$ for $x \in X$ $(i = 1, 2, \ldots)$ there exist points $x_i \in X$ such that the sets $G_i(x_i)$ where $i = 1, 2, \ldots$ constitute a cover of X. Property (C″) is related to some problems of the theory of measure. Each metric space with property (C″) is totally imperfect and the existence of uncountable metric spaces having property (C″) follows from the continuum hypothesis (see [15], § 40). Since (C″) implies (E*), these spaces are examples of Hurewicz metric spaces none of which is an absolute F_σ (see also [21], p. 48). It remains an open question whether or not one can construct a Hurewicz metric space which is not an absolute F_σ without using the continuum hypothesis.

Let X be a topological space. We say that a collection α of subsets of X is *almost point finite* (or *almost locally finite*) if for every open subset G of X the collection $\{A \in \alpha\colon A \backslash G \neq \emptyset\}$ is point finite (or locally finite, respectively) at each point of G. The following conditions describe properties of X in terms of various types of finiteness. They will be shown to be equivalent to each other in the case of separable metric spaces (see Theorem 2 below).

(vi) *Every open basis in X contains a cover α of X such that α is almost point finite.*

(vii) *Every open basis in X contains a cover α of X such that α is almost locally finite.*

(viii) *Every open basis in X contains a cover α of X such that α is point finite.*

(ix) *Every open basis in X contains a cover α of X such that α is locally finite.*

(x) *Every open basis in X contains an open basis β in X such that β is almost point finite.*

(xi) *Every open basis in X contains an open basis β in X such that β is almost locally finite.*

It is not difficult to check that each open cover being a zero sequence in a metric space contains a locally finite cover. Consequently, property (iii) implies property (ix). This has been observed by R. M. Ford who has raised the question as to whether the converse is true under the assumption of metrizability and separability. An affirmative answer to this question is given in our Theorem 2.

Eleven conditions listed above will not be complete if we do not mention the twelfth one. Condition (xii) which follows is, however, much more restrictive than either of conditions (i)-(xi) since, for instance, (xii) is not satisfied by the space of rationals [22].

(xii) *Every open basis in X contains a cover α of X such that α is star finite.*

Topological spaces possessing property (ix) are called *totally paracompact* [9], while those satisfying (viii) or (xii) might be called *totally metacompact* or *totally hypocompact*, respectively. Each discrete space is totally hypocompact. Thus totally paracompact metric spaces need not be strongly Lindelöf if the separability is not assumed. The concept of total paracompactness can be generalized in order to include hypocompact metric spaces, i.e. metric spaces with the star finite property [8]. It can also be compared with some singularities for subsets of complete metric spaces [18]. Open bases are called *point regular* or *regular* [2] if they are almost point finite or almost locally finite, respectively. It has been proved by A. V. Arhangel'skiĭ [4] that a T_1-space X is metrizable if and only if X admits a regular basis. The metrizability can also be derived from the existence of a point regular basis [1] provided the space is Hausdorff and paracompact. Consequently, in most cases a space satisfying either (x) or (xi) is metrizable. On the other hand, each compact space is totally paracompact. Thus properties (ix) and (xi) are not equivalent in general. An open basis β is called *fine* [6] if every open cover has a locally finite refinement contained in β. Clearly, each open basis in a totally paracompact space is fine. It is rather easy to verify that each regular basis is fine, and an open basis β in a metrizable space is fine if and only if β contains a regular basis [4]. Hence properties (ix) and (xi) are equivalent for metrizable spaces. Fine bases play a role in some uniformity problems (see [12], p. 144). An open basis β is called *coarse* [6] if β contains no locally finite cover. It has been known [6] that the space of irrationals $\mathfrak{N}$ admits a coarse basis. The example of N. N. Konstantinov mentioned above is such a basis in $\mathfrak{N}$. Observe that there can exist a countable open basis β in a metric space such that β is not coarse and β contains no cover being a zero sequence. In fact, a basis satisfying the latter conditions can be found in the remetrized space $\mathfrak{N}$. If $\mathfrak{J}$ denotes the space of integers with the usual discrete topology, $\mathfrak{N}$ is homeomorphic to the Baire space $\mathfrak{J}\times\mathfrak{J}\times\ldots$ with the metric defined by

$$\varrho[(i_1, i_2, \ldots), (j_1, j_2, \ldots)] = [\operatorname{Min}\{k\colon\ i_k \neq j_k\}]^{-1}$$

for $(i_1, i_2, \ldots) \neq (j_1, j_2, \ldots)$. It is known [16] that the Baire space does not admit any countable cover consisting of sets whose diameters are less than one and converge to zero. Coarse bases have been also exhibited in the Hilbert space l^2 which is neither strongly Lindelöf nor totally paracompact. In particular, H. H. Corson [5] has proved that no reflexive infinite-dimensional Banach space admits a locally finite cover consisting of bounded convex sets. Since l^2 is homeomorphic to the infinite product of countably many copies of the real line [3], both l^2 and $\mathfrak{N}$ show that the infinite product of Hurewicz spaces needs not be Hurewicz. At the end of the present paper we give an example of a Hurewicz space X such that $X\times X$ is not Hurewicz. Roughly, our example (whose idea has been proposed by J. R. Isbell) is an uncountable set possessing the Lusin property L [15] and carrying the half-open interval topology [14]. The latter topology yields, by a result due to F. B. Jones [13], the non-normality of the product, thus also the non-metrizability. It is an unsettled question whether or not each product of two Hurewicz metric spaces is a Hurewicz space. Also, construction of uncountable sets with the Lusin property requires a well-ordering technique. It remains an open question whether or not one can construct two Hurewicz spaces whose product is not a Hurewicz space without using the continuum hypothesis.

LEMMA 1.0. *If X is a Lindelöf regular space and $\gamma_0, \gamma_1, \ldots$ are open covers of X, then there exists a pseudo-metric p in X and open covers $\chi_0, \chi_1, \ldots$ of X such that χ_i refines γ_i $(i = 0, 1, \ldots)$ and, for every subset $Y\subset X$ satisfying*

$$\operatorname{diam}_p Y < \infty \quad \text{or} \quad \operatorname{diam}_p Y < 2^{-i},$$

Y intersects only finitely many elements of χ_0 or χ_i, respectively $(i = 1, 2, \ldots)$.

Proof. First recall that Lindelöf regular spaces are paracompact and normal (see [14], pp. 159 and 172). Since X is Lindelöf and paracompact, there exists a countable locally finite open cover $\{U_{i1}, U_{i2}, \ldots\}$ of X which refines γ_i $(i = 0, 1, \ldots)$. Since X is normal, there exists an open cover

$$\chi_i = \{G_{i1}, G_{i2}, \ldots\}$$

of X such that $\operatorname{cl} G_{ij}\subset U_{ij}$ for $j = 1, 2, \ldots$ (see [14], p. 171). Thus χ_i refines γ_i. Let f_{ij} be a real-valued continuous function on X such that $f_{ij}(x) = j$ for $x\in G_{ij}$ and $f_{ij}(x) = 0$ for $x \in X\setminus U_{ij}$. Then the function p defined by

$$p(x, y) = \sum_{j=1}^{\infty} |f_{0j}(x) - f_{0j}(y)| + \sum_{i=1}^{\infty} 2^{-i}\operatorname{Min}\Big\{1, \sum_{j=1}^{\infty} |f_{ij}(x) - f_{ij}(y)|\Big\}$$

is easily seen to be a pseudo-metric in X. If a subset $Y\subset X$ intersects infinitely many elements of χ_i, there exist points $y_{ik}\in Y\cap G_{ij_k}$ where $j_k < j_{k+1}$ for $k = 1, 2, \ldots$ Since y_{i1} (together with any point from a neighbourhood) belongs to finitely many sets U_{ij} $(j = 1, 2, \ldots)$, we have $f_{ij_k}(y_{i1}) = 0$ for k sufficiently large. Thus

$$\lim_{k\to\infty} |f_{ij_k}(y_{ik}) - f_{ij_k}(y_{i1})| = \lim_{k\to\infty} j_k = \infty$$

whence $\operatorname{diam}_p Y = \infty$ or $\operatorname{diam}_p Y \geqslant 2^{-i}$ depending on whether $i = 0$ or $i > 0$, respectively.

THEOREM 1. *Let X be a metrizable space. Then X is a Hurewicz space if and only if X satisfies either of conditions* (i)-(v).

Proof. A scheme of the proof is given by Diagram 1 in which only three implications are non-trivial. We are going to prove them.

Suppose that X satisfies (i). Let ϱ be a metric in X and let $\gamma_0, \gamma_1, \ldots$ be open covers of X. Let p be a pseudo-metric in X and let $\chi_0, \chi_1, \ldots$ be open covers of X such that p and χ_i fulfil all requirements from Lemma 1.0. Then $\sigma = \varrho + p$ is a metric in X and, by (i), there exists a cover $\{A_1, A_2, \ldots\}$ of X such that $\operatorname{diam}_\sigma A_i < \infty$ for $i = 1, 2, \ldots$ and $\operatorname{diam}_\sigma A_i$ converges to zero when i tends to the infinity. Thus there exist positive integers i_k such that $i_k < i_{k+1}$, and $i_k \leqslant i \leqslant i_{k+1}$ implies $\operatorname{diam}_\sigma A_i < 2^{-k}$, for $k = 1, 2, \ldots$ Let $i_0 = 1$ and

$$\delta_k = \{G \in \chi_k \colon G \cap \bigcup_{i=i_k}^{i_{k+1}} A_i \neq \emptyset\}$$

for $k = 0, 1, \ldots$ Since χ_k and $\{A_1, A_2, \ldots\}$ are covers of X, so is the union $\delta_0 \cup \delta_1 \cup \ldots$ It follows from the inequality

$$\operatorname{diam}_p A_i \leqslant \operatorname{diam}_\sigma A_i$$

that A_i intersects only finitely many elements of χ_k provided $i_k \leqslant i \leqslant i_{k+1}$. Consequently, δ_k is finite for $k = 0, 1, \ldots$ But $\delta_k \subset \chi_k$ and χ_k refines γ_k, whence each element of δ_k is contained in an element of γ_k. In this way we get a finite collection $a_k \subset \gamma_k$ such that $|\delta_k| \subset |a_k|$ for $k = 0, 1, \ldots$ Thus $a_0 \cup a_1 \cup \ldots$ is a cover of X. We have proved that X is a Hurewicz space.

Now, suppose that X satisfies (iii). Let $\gamma_1, \gamma_2, \ldots$ be open covers of X. Given $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball having the center at x and the radius equal to r with respect to a metric ϱ in X whose existence is claimed by (iii). Since γ_i is an open cover of X, there exist numbers $r_i(x) > 0$ such that $r_i(x) < 2^{-i}$ and $B[x, r_i(x)]$ is contained in an element $G_i(x)$ of γ_i for $x \in X$. Let

$$C_i = \{(x, y) \in X \times X \colon 2^{-i} \leqslant \varrho(x, y) \leqslant 2^{1-i}\}$$

for $i = 1, 2, \ldots$ We denote by $I(X)$ the set of isolated points of X. Then the collection

$$\beta = \{\{x\} \colon x \in I(X)\} \cup \bigcup_{i=1}^{\infty} \{B[x, r_i(x)] \cup B[y, r_i(y)] \colon (x, y) \in C_i\}$$

is an open basis in X and, by (iii), there exist a sequence of positive integers i_j and a sequence of pairs $(x_j, y_j) \in C_{i_j}$ such that

$$X \setminus I(X) \subset \bigcup_{j=1}^{\infty} B[x_j, r_{i_j}(x_j)] \cup \bigcup_{j=1}^{\infty} B[y_j, r_{i_j}(y_j)]$$

and $\operatorname{diam}_\varrho B[x_j, r_{i_j}(x_j)] \cup B[y_j, r_{i_j}(y_j)]$ converges to zero when j tends to the infinity. Thus $\varrho(x_j, y_j)$ also converges to zero and, consequently, the set C_i can contain only finitely many pairs (x_j, y_j). It follows that the collection

$$\varphi_i = \{G_{i_j}(x_j) \colon i_j = i\} \cup \{G_{i_j}(y_j) \colon i_j = i\}$$

is finite for $i = 1, 2, \ldots$ Moreover, $X \setminus I(X)$ is contained in the union $|\varphi_1| \cup |\varphi_2| \cup \ldots$ and since, according to (iii), X is separable, $I(X)$ must be countable. Let the points of $I(X)$ be ordered in a sequence and let G_i be an element of γ_i which contains the ith point from this sequence. Then setting $a_i = \varphi_i \cup \{G_i\}$ we get finite collections $a_i \subset \gamma_i$ such that $a_1 \cup a_2 \cup \ldots$ is a cover of X. We have proved that X again is a Hurewicz space.

Finally, suppose that X is a Hurewicz space. Let ϱ be a metric in X and let γ_i be the collection composed of all elements G of an arbitrarily given open basis in X such that $\operatorname{diam}_\varrho G < 2^{-i}$ $(i = 1, 2, \ldots)$. Since X is Hurewicz, there exist covers a_j of X $(j = 1, 2, \ldots)$ such that $a_j = a_{1j} \cup a_{2j} \cup \ldots$ where $a_{ij} \subset \gamma_{i+j}$ and a_{ij} are finite. Then each a_j is a zero sequence and $\operatorname{diam}_\varrho G < 2^{-j}$ for $G \in a_j$. Thus $\beta = a_1 \cup a_2 \cup \ldots$ is a zero sequence too, and β is an open basis in X. We have proved that X satisfies (v).

COROLLARY 1.1. *Each metric space being the continuous image of a strongly Lindelöf metric space is strongly Lindelöf.*

COROLLARY 1.2. *Each metric space being the union of countably many strongly Lindelöf subspaces is strongly Lindelöf.*

COROLLARY 1.3. *Each F_σ in a strongly Lindelöf metric space is strongly Lindelöf.*

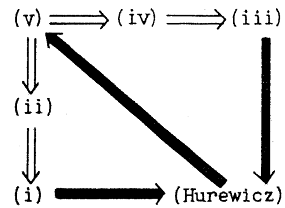

Diagram 1

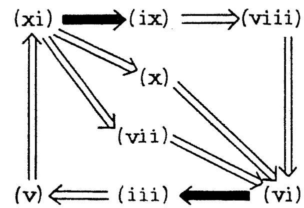

Diagram 2

THEOREM 2. *Let X be a separable metrizable space. Then X is a Hurewicz space if and only if X satisfies either of conditions* (i)-(xi).

Proof. A scheme of the proof is given by Diagram 2 which has two items in common with Diagram 1. These are conditions (iii) and (v), and we know that (iii) implies (v) according to Theorem 1. Among the remaining implications in Diagram 2 only two are non-trivial. One of them, namely that (xi) implies (ix), has already been discussed (see [4], p. 590). It suffices to prove that (vi) implies (iii).

Suppose that X satisfies (vi). Since X is separable and metrizable, X can be imbedded in the Hilbert cube. Let ϱ be the metric in X which comes from the Hilbert cube and let β be an open basis in X. Since ϱ is totally bounded, there exist finite sets $F_i \subset X$ such that $\varrho(x, F_i) < 2^{-i}$ for $x \in X$ $(i = 1, 2, \ldots)$. Let

$$\gamma_i = \{G \in \beta \colon 2^{-i} \leqslant \operatorname{diam}_\varrho G \leqslant 2^{1-i}\}$$

for $i = 1, 2, \ldots$ Given $G \in \gamma_i$, we take points $x \in G$, $y \in F_i$ such that $\varrho(x, y) < 2^{-i}$ and an element $U_i(G) \in \beta$ such that $y \in U_i(G)$ and $\operatorname{diam}_\varrho U_i(G) < 2^{-i}$. We have

$$\operatorname{diam}_\varrho [G \cup U_i(G)] < 2^{1-i} + 2^{-i} + 2^{-i} = 2^{2-i}$$

for $G \in \gamma_i$ and $i = 1, 2, \ldots$ Then the collection

$$\beta' = \{\{x\}: x \in I(X)\} \cup \bigcup_{i=1}^{\infty} \{G \cup U_i(G): G \in \gamma_i\}$$

is an open basis in X and, by (vi), there exists a sequence of collections $\varphi_i \subset \gamma_i$ such that

$$\alpha' = \bigcup_{i=1}^{\infty} \{G \cup U_i(G): G \in \varphi_i\}$$

is almost point finite and $X \setminus I(X) \subset |\alpha'|$. It can be assumed that every element of α' has exactly one representation $G \cup U_i(G)$ where $G \in \varphi_i$ and $i = 1, 2, \ldots$

We claim that φ_i is finite for $i = 1, 2, \ldots$ Suppose on the contrary that φ_{i_0} is infinite. Since $U_{i_0}(G)$ meets F_{i_0} for $G \in \varphi_{i_0}$, and F_{i_0} is finite, there exists a point $y_0 \in F_{i_0}$ such that y_0 belongs to infinitely many sets $G \cup U_{i_0}(G)$ where $G \in \varphi_{i_0}$. Since $\varphi_{i_0} \subset \gamma_{i_0}$, these sets have diameters not less than 2^{-i_0}, and thus the collection

$$\alpha_0 = \{A \in \alpha': A \setminus B(y_0, 2^{-2-i_0}) \neq \emptyset\}$$

contains all of them. Consequently, α_0 is not point finite at y_0; this contradicts the fact that α' is almost point finite.

The space X being separable, the set $I(X)$ of its isolated points is countable, and $x \in I(X)$ implies $\{x\} \in \beta$. It follows that the collection

$$\alpha = \{\{x\}: x \in I(X)\} \cup \bigcup_{i=1}^{\infty} \varphi_i \cup \bigcup_{i=1}^{\infty} \{U_i(G): G \in \varphi_i\}$$

is contained in β, and $|\alpha| = I(X) \cup |\alpha'| = X$. Moreover, since φ_i are finite $(i = 1, 2, \ldots)$, the cover α is a zero sequence. We have proved that X satisfies (iii).

COROLLARY 2.1. *Each metric space being the continuous image of a totally paracompact separable metric space is totally paracompact.*

COROLLARY 2.2. *Each separable metric space being the union of countably many totally paracompact subspaces is totally paracompact.*

COROLLARY 2.3. *Each separable F_σ in a totally paracompact metric space is totally paracompact.*

EXAMPLE. We give an example of a Hurewicz space X such that $X \times X$ is not normal. Let X be an uncountable dense subset of the real line R such that if $A \subset R$ is nowhere dense in R, then $A \cap X$ is countable (see [15], p. 525). We provide X with the topology for which intersectiones of X with half-open intervals

$$[a, b) = \{r \in R: a \leqslant r < b\}$$

constitute an open basis $(a, b \in R)$. The space X is regular (see [14], p. 133). To see that X is a Hurewicz space let us consider a sequence $\gamma_1, \gamma_2, \ldots$ of open covers of X. Since X is dense in R, there exist points $x_i \in X$ such that $\{x_1, x_2, \ldots\}$ is dense in R. Since γ_i is an open cover of X, there exists a number $r_i > x_i$ such that $X \cap [x_i, r_i)$ is contained in an element G_i of γ_i. Then the set

$$Y = R \setminus \bigcup_{i=1}^{\infty} [x_i, r_i)$$

is nowhere dense in R, and thus $Y \cap X$ is countable. We can write $Y \cap X = \{y_1, y_2, \ldots\}$, and let $\alpha_i \subset \gamma_i$ be the two-element collection consisting of G_i and an element of γ_i which contains y_i $(i = 1, 2, \ldots)$. It follows that $\alpha_1 \cup \alpha_2 \cup \ldots$ is a cover of X, and X is Hurewicz. To see that $X \times X$ is not normal let us consider subsets

$$C = \{(x, -x): x \in X\}, \quad D = \{(x_i, x_j): i, j = 1, 2, \ldots\}$$

of $X \times X$. Clearly, C is an uncountable closed discrete subspace of $X \times X$. On the other hand, D is a countable dense subspace of $X \times X$. A normal space cannot contain subspaces C and D which possesse the latter properties (see [13], p. 671). It follows that $X \times X$ is not normal.

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Some results on AC-ω functions

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1. Introduction. Let $\omega(x)$ be non-decreasing on the closed interval $[a, b]$. Outside the interval, $\omega(x)$ is defined by $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and let $D = [a, b] - S$. Let S_0 denote the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$ on each of which $\omega(x)$ is constant,

$$S_1 = \{a_1, b_1, a_2, b_2, \ldots\}, \quad S_2 = SS_1 \quad \text{and} \quad S_3 = [a, b]\cdot S - (S_0 + S_2).$$

R. L. Jeffery [4] has denoted by $\mathfrak{U}$ the class of functions $f(x)$ defined as follows.

$f(x)$ is defined on the set $S\cdot[a, b]$ such that $f(x)$ is continuous at each points of $S\cdot[a, b]$ with respect to S. If a point $x_0 \in D$, $f(x)$ tends to a limit (finite or infinite) as x tends to x_0+ and x_0- over the points of the set S. These limits will be denoted by $f(x_0+)$ and $f(x_0-)$, respectively. When $x < a$, $f(x) = f(a+)$ and $f(x) = f(b-)$ for $x > b$. $f(x)$ may or may not be defined at the points of the set D.

In [4] Jeffery has introduced the following definitions.

DEFINITION 1.1. A function $f(x)$ defined on $[a, b]$ and in the class $\mathfrak{U}$ is *absolutely continuous* relative to ω, AC-ω, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') on $[a, b]$ with $\sum_i\{\omega(x_i'+) - \omega(x_i-)\} < \delta$ the relation $\sum_i |f(x_i'+) - f(x_i-)| < \varepsilon$ is satisfied.

DEFINITION 1.2. Let $f(x)$ belong to the class $\mathfrak{U}$. For any x and any $h \neq 0$ with $x + h \in S$, the function $\psi(x, h)$ is defined by

$$\psi(x, h) = \begin{cases} \dfrac{f(x+h) - f(x-)}{\omega(x+h) - \omega(x-)}, & h > 0, \quad \omega(x+h) - \omega(x-) \neq 0, \\ \dfrac{f(x+h) - f(x+)}{\omega(x+h) - \omega(x+)}, & h < 0, \quad \omega(x+h) - \omega(x+) \neq 0, \\ 0, & \qquad\quad \omega(x+h) - \omega(x\pm) = 0. \end{cases}$$

If $\psi(x, h)$ tends to a limit as $h \to 0$, this limit is called the *ω-derivative of $f(x)$ at x* and is denoted by $f'_\omega(x)$. The upper and lower limits of $\psi(x, h)$