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Expansive homeomorphisms on homogeneous spaces

by

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1. In [2] Murray Eisenberg has shown how an expansive homeomorphism may be constructed from a positively expansive map. In this note we show that a positively expansive map on a compact connected manifold must be a covering map of the manifold on itself, and that a manifold admitting such a map cannot be simply connected. Furthermore, if the manifold is triangulable, its Euler characteristic must be zero. Next, positively expansive maps of various manifolds are exhibited; and using Eisenberg's technique, we infer:

A. In every finite dimension greater than one there is a compact connected space, fibered over a manifold by the Cantor set, which admits an expansive homeomorphism and which is not an abelian group space. In every finite dimension greater than 2 there are countably many different such spaces.

Finally we prove:

B. In every finite dimension greater than 3 there is a compact, connected manifold, not an abelian group space which admits an expansive homeomorphism.

In previous examples of expansive homeomorphisms on compact, perfect, homogeneous spaces, the space has been a group space nad the homeomorphism conjugate (in the homeomorphism group) to an automorphism of the topological group carried by the space ([2], [3], [7], [9]). T. S. Wu [10] has shown that compact connected finite dimensional topological groups which admit expansive automorphisms are abelian. It follows that the expansive homeomorphisms constructed here cannot be conjugate to an automorphism of a topological group.

2. Let X be a metric (d) space. A map f of X onto itself is called *expansive* provided that there exists a positive constant c such that to each pair (x, y) of distinct points of X there corresponds an integer n with $d[f^n(x), f^n(y)] > c$. The number c is called the *expansive constant.* The distance $d[f^n(x), f^n(y)]$ is to be interpreted as the usual distance between sets. If, to each pair (x, y) of distinct points of X, there corresponds a positive integer with the required property, it will be said that f is *possitively expansive.*

The set of points at which f fails to be a local homeomorphism is denoted by B_f. If B_f is empty and each point of X has exactly k counter-images, f is said to be a *k-to-one covering map*. If $f(U)$ is open whenever U is, f is called *open*. If $f^{-1}(x)$ is totally disconnected for each x in X, f is called *light*. If f is light and open, and if $X-f(B_f)$ is connected, and the restriction of f to $(X-B_f)$ is a k-to-one covering map, we say f is a *pseudo-covering map*.

Throughout this note, f will denote a map of the compact metric (d) space X onto itself. M will denote a compact connected manifold.

Consider the inverse limit sequence in which each X_i is a canonically chosen copy of X and each bonding map the corresponding copy of f. Call the resulting space X^*. If f is a k-to-one covering map, $k>1$, it is easy to see that X^* is fibered by the Cantor set over X. If X is compact and connected, so is X^*. If f is positively expansive, the map $f^*\colon X^*\to X^*$ given by $f^*(x_0, x_1, x_2, \ldots) = (f[x_0], x_0, x_1, \ldots)$ is known to be an expansive homeomorphism of X^* onto itself ([2], Theorem 3).

3. In this paragraph we will prove

THEOREM 1. *If X is a compact, connected manifold and f is positively expansive, then f is a k-to-1 covering map, with $k>1$.*

Proof. First we observe that $B_f=\emptyset$. For if $a\in B_f$, then there are pairs (x,y) of distinct points of X arbitrarily close to a with $f(x) = f(y)$. But then f is not positively expansive. Next we observe that f is light. Indeed, there exists a positive integer N such that no point x in X has more than N counterimages. For if not, since X is totally bounded, we can find pairs (x,y) of distinct points arbitrarily near each other with $f(x)=f(y)$, contrary to the hypothesis that f is positively expansive. It now follows by a theorem of Church and Hemmingsen ([1], 2.4) that f is a pseudo-covering map. Since $B_f=\emptyset$, f is a covering map. If f were one-to-one, it would have a pair of positively asymptotic points ([4], 10.36). Then we could find pairs (x,y) of distinct points arbitrarily near each other with $d[f^n(x), f^n(y)]$ arbitrarily small for all $n>0$, a situation which cannot occur for a positively expansive map. Hence f is a k-to-one covering map with $k>1$.

We remark that, in case two isometric copies of M can be triangulated so that f is represented as a simplicial map, it follows from Tucker's formulas [8] that the Euler characteristic of M is zero. This will restrict considerations to toruses and Klein bottles in the two dimensional case.

If M is a manifold, triangulable or not, it follows from [1], 5.5, that $\pi_1(M)\neq 0$. Hence, in particular, S^1 is the only sphere admitting a positively expansive map.

Not every covering map with $k>1$ is positively expansive. To see this, let X be the product of two circles and let f be the product of a 2-to-1 covering map and the identity.

4. THEOREM 2: *Let $\varphi\colon Y\to X$ be a covering map of Y onto X, where Y is a metric space with metric ϱ. Suppose that there exists a number $\eta>0$ such that whenever $\varphi(y_1)=\varphi(y_2)$, then $\varrho(y_1. y_2)>\eta$. Let g be a map of Y onto Y such that $\varphi g = f\varphi$. Then g is positively expansive if and only if f is positively expansive.*

Proof. Minor modifications of [5], 3.4, case 1, and 3.6 suffice.

Now let Y be the cylinder in E_3 given by $r=1$ in cylindrical coordinates (r,θ,h). By identifying points which differ in h by an integer and in θ not at all, we induce a covering map onto the torus. By identifying (r,θ,h) to $(r,-\theta,h+1)$ we induce a covering map onto the Klein bottle. Both these covering maps commute with the map $g\colon Y\to Y$ given by $g(r,\theta,h) = (r, 2\theta, 3h)$ which is clearly positively expansive. Therefore there exist positively expansive maps on the torus and Klein bottle, call the one on the Klein bottle f. The associated inverse limit systems therefore admit expansive homeomorphisms. Let (K,ψ) denote the expansive homeomorphism induced by the mapping on the Klein bottle. It can be seen by Theorem 1 that K is the inverse limit space of a sequence of Klein bottles with binding maps f. It follows from the continuity of Čech theory that $H_1(K, R)$ is a copy of the reals R. Suppose that K admits an abelian topological group structure such that ψ is conjugate to a group automorphism θ. By a theorem of Kodaira and Abe [6]. K is an inverse limit space over tori with each binding map a finite-to-one covering map. Hence, K has the Čech homology of the torus. Then $H_1(K,R)$ is $R\times R$, a contradiction.

It is easy to see that the mapping $\theta\to 2\theta$ is a positively expansive mapping of the circle, and that the product of finitely many positively expansive maps is positively expansive on the corresponding product space. Therefore, by taking positively expansive maps on the product of a finite collection of Klein bottles and circles, one can find a space X of any finite dimension greater than one and an expansive homeomorphism $g\colon X\to X$ where the Künneth formula shows that X is not an abelian group space.

5. In this section we consider spaces having dimension greater than 2. In dimension two the restriction of attention to manifolds having Euler characteristic zero is a serious inconvenience. In higher dimensions, of course, there is much more freedom. Since it is easy to construct positively expansive maps of the 3-torus on itself, it is natural to attempt to construct such maps on other three-dimensional nilmanifolds. We follow the construction in [0], pp. 46-48. We consider the topological group T_3 represented by all real matrices of the form

$$\begin{Vmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{Vmatrix}.$$

We may consider the underlying space to be E_3. For each $k = 1, 2, 3, \ldots$ we consider the uniform, discrete, non normal subgroup $D(k)$ of T_3 consisting of all matrices of the form

$$\begin{Vmatrix} 1 & a & ab + c/k \\ 0 & 1 & b \\ 0 & 0 & 1 \end{Vmatrix}$$

where a, b, and c are integers. It is easy to check that the positively expansive map $f(x, y, z) = (2x, 2y, 4z)$ respects the right cosets of $D(k)$ and so induces a positively expansive map of the compact manifold $T_3/D(k) = M(k)$ onto itself. It follows that the shift on the corresponding inverse limit space, $M^*(k)$ is an expansive homeomorphism. Now by the Hurewicz homomorphism, $\pi_1\big(M(k)\big) = D(k)$ ([0], p. 47). No $D(k)$ is abelian, and no $M^*(k)$ has the Čech homology of the 3 torus. So $M^*(k)$ is not an abelian group space. $D(k')$ is isomorphic to $D(k)$ if and only if $k = k'$, so that no two $M(k)$. $k = 1, 2, 3, \ldots$ are homeomorphic. If n is a positive integer, it is easy to see that the product of the factor space $T_3/D(k)$ with n-circles is an $(n+3)$-manifold admitting a positively expansive map.

6. Let B be the total space of a fiber bundle with fiber Y, base space X, a family $\{V_j\}$ of coordinate neighborhoods, and corresponding coordinate functions φ_j: $V_j \times Y \to p^{-1}(V_j)$. Suppose that f is a homeomorphism of Y that commutes with elements of the structure group and that g is a homeomorphism of X such that for each point x in X there is at least one coordinate neighborhood containing both x and $g(x)$. To define a homeomorphism h: $B \to B$, let $b \in B$, let V_i be a coordinate neighborhood containing both $p(b)$ and $gp(b)$, and let $h(b)$ be defined by

$$h(b) = \varphi_i\big(gp(b), f\varphi^{-1}_{i,p(b)}(b)\big).$$

The commutativity of f with the elements of the structure group guarantees that $h(b)$ is independent of the choice of V_j. If the action of the structure group is uniformly equicontinuous and the covering $\{V_j\}$ has a positive Lebesgue number, then h is expansive when both f and g are expansive.

Consider $T_n \times (T_m \times I)$ where T_n is the n-torus, $n \geqslant 2, m \geqslant 1$, and I is the unit interval. Identify the points $(p, q, 0)$ and $\big(\psi(p), q, 1\big)$ where ψ is a group automorphism reversing the orientation of an odd number of factor circles in T_n. The resulting manifold is fibered by an n-torus over an $(m+1)$-torus. It can be represented as having two coordinate neighborhoods and structure group $\{\varphi^0, \varphi\}$ where φ^0 is the identity and φ is the homeomorphism $\mathrm{Id}x\psi$: $V_i \times T_n \to p^{-1}(V_i)$. Any automorphism of T_n commutes with φ, and the coordinate neighborhoods are large enough to satisfy the condition above. *Therefore this $(n+m+1)$-manifold admits an expansive homeomorphism.* It has elements in its fundamental group which do not commute, namely: an element generated by a copy of any factor circle in the representation of T_n whose orientation is reversed by ψ and an element generated by one of those copies of I whose ends were identified. Hence, this manifold is not an H-space and therefore, a fortiori, not an abelian group space.

Added in proof: Professor Michael Shub has kindly shown the authors some related work done by him and by Epstein and Shub. David Epstein and Michael Shub have shown that every flat manifold admits an expanding endomorphism. Each such endomorphism is positively expansive.

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