Mittelgerade besitzt. Sei $A \in a$ $(A \neq O)$. Das Lot zu $a$ durch $A$ schneide $b$ in $C$ (Fig. 6), das Lot zu $b$ durch schneide $a$ in $B$. Dann ist $\vdash = \vdash_{A;O,B;C}$. Da $\vdash$ Euklidisch ist, gilt nach Theorem 2 nicht $\surd (A;\, O, B)$, also gilt gemäß Voraussetzung $\surd (O;\, A, B)$. Nach dem Lemma besitzt damit $\{a, b\}$ eine Mittelgerade.

Sind umgekehrt $O, A, B$ kollineare Punkte und gilt $\surd(A;\, O, B)$ nicht, so ist für beliebiges $C$ $\vdash_{A;O,B;C}$ Euklidisch. Da diese Relation nach Voraussetzung kommensurabel ist, gilt auf Grund des Lemmas offenbar $\surd(O;\, A, B)$. Aus Symmetriegründen gilt auch $\surd(B;\, O, A)$, womit Theorem 4 bewiesen ist.

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# Periodic homeomorphisms on chainable continua

by

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## 1. Preliminaries.

Introduction. In this paper we begin a study of the periodic homeomorphisms on chainable continua. It is well known that an arc admits period two homeomorphisms, but does not admit homeomorphisms of finite order $n$, $n > 2$. We will show that this result does not generalize to chainable continua. We first define *regularly chainable continua* and show that every regularly chainable continuum admits a period two homeomorphism. The arc and pseudo-arc are examples of such continua. We then use these results to construct a chainable continuum which admits period four homeomorphisms.

We note that in [5] F. B. Jones shows, by a proof similar to the one in this paper, that the pseudo-arc admits period two homeomorphisms.

Convention. *All spaces are separable metric.*

Basic definitions. Most of the following definitions are well-known, but are included for completeness.

Definition 1.1. A homeomorphism $h \neq e$ of a continuum $X$ onto itself is called *periodic* provided that there exists an integer $n > 1$ such that $h^n$ is the identity. If $h^n = e$, but $h^k \neq e$ for $0 < k < n$, then $h$ is said to be of *period* $n$ or *order* $n$.

Definition 1.2. A *chain* is a finite collection of open sets $\mathcal{U}$: $U_1, U_2, \ldots, U_n$ such that

(1) $U_i \cap U_j \neq \emptyset$ iff $|i-j| \leqslant 1$,

(2) $\overline{U}_i \cap \overline{U}_j \neq \emptyset$ iff $|i-j| \leqslant 1$ and

(3) $U_i \not\subset U_j$ for any pair $i, j$.

$\mathcal{U}^*$ denotes the union of the elements of $\mathcal{U}$. $\mathcal{U}$ is a *chain from $p$ to $q$* iff $\mathcal{U}$ is a chain, $p \in U_1 - U_2$, and $q \in U_n - U_{n-1}$. If $\mathcal{U}$: $U_1, U_2, \ldots, U_n$ is a chain, then $h(\mathcal{U})$ denotes the chain whose elements are $h(U_1), h(U_2), \ldots, h(U_n)$.

Definition 1.3. A chain $\mathcal{V}$ is a *refinement* of the chain $\mathcal{U}$ provided that each element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$. $\mathcal{V}$ is called a *closed refinement* of $\mathcal{U}$ iff the closure of each element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$.

DEFINITION 1.4. If $\mathfrak{U}$ is a chain of open sets, then *mesh of* $\mathfrak{U}$, denoted by $\mu(\mathfrak{U})$, is the diameter of the largest element of $\mathfrak{U}$.

DEFINITION 1.5. $X$ is a *chainable continuum* iff for every $\varepsilon > 0$, there exists a chain cover $\mathfrak{U}$ of $X$ such that $\mu(\mathfrak{U}) < \varepsilon$. If $X = \bigcap_{i=1}^{\infty} \mathfrak{U}_i^*$ where $\mathfrak{U}_i$ is a chain cover of $X$ of mesh $< 1/i$ and $\mathfrak{U}_{i+1}$ is a closed refinement of $\mathfrak{U}_i$ for all $i$, then $\{\mathfrak{U}_i\}_{i=1}^{\infty}$ is called a *defining sequence of chains for* $X$.

## 2. Period two homeomorphisms.

In this section we show that the pseudo-arc and certain other chainable continua admit period two homeomorphisms.

DEFINITION 2.1. Let $\mathfrak{U}$: $U_1, U_2, \ldots, U_n$ be a chain. Then the chain $\mathfrak{V}$: $V_1, V_2, \ldots, V_n$, where $V_i = U_{n-i+1}$, is called the *reverse chain of* $\mathfrak{U}$.

DEFINITION 2.2. (As in [1].) Let $\mathfrak{U}$: $U_1, U_2, \ldots, U_n$ be a chain, and let $\mathfrak{V}$: $V_1, V_2, \ldots, V_m$ be a refinement of $\mathfrak{U}$. If the $i$th element of $\mathfrak{V}$ is a subset of the $x_i$th element of $\mathfrak{U}$, then $\mathfrak{V}$ is said to follow the *pattern* $(1, x_1), (2, x_2), \ldots, (m, x_m)$ in $\mathfrak{U}$.

We note that if some element of $\mathfrak{V}$ is a subset of two elements of $\mathfrak{U}$, then at least two patterns exist for $\mathfrak{V}$ in $\mathfrak{U}$. Thus, a pattern may not be unique.

DEFINITION 2.3. Let $\mathfrak{U}$ be a chain and let $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. $\mathfrak{V}$ is said to be *regular* in $\mathfrak{U}$ iff there is a pattern for $\mathfrak{V}$ in $\mathfrak{U}$ which is also a pattern for the reverse chain of $\mathfrak{V}$ in the reverse chain of $\mathfrak{U}$.

DEFINITION 2.4. Let $\{\mathfrak{U}_i\}$ be a sequence of chains such that $\mathfrak{U}_{i+1}$ is regular in $\mathfrak{U}_i$. Then $\{\mathfrak{U}_i\}$ is called a *regular sequence of chains*. If $X$ is a continuum and $\{\mathfrak{U}_i\}$ is a regular sequence of chains covering $X$ such that

(1) $\mu(\mathfrak{U}_i) < 1/i$,

(2) $\mathfrak{U}_{i+1}$ is a closed refinement of $\mathfrak{U}_i$, and

(3) $X = \bigcap_{i=1}^{\infty} \overline{\mathfrak{U}_i^*}$,

then $\{\mathfrak{U}_i\}$ is called a *regular defining sequence of chains for* $X$.

DEFINITION 2.5. A continuum $X$ is called *regularly chainable* iff there exists a regular defining sequence of chains for $X$. $X$ is called *regularly chainable from* $p$ *to* $q$ iff each of the chains in some regular defining sequence runs from $p$ to $q$.

Note. For the definitions of *crooked chains* and the *pseudo-arc*, see [1], [2].

THEOREM 2.1. *Let $X$ be a regularly chainable continuum. Then there exists a period two homeomorphism $h$ of $X$ onto itself, keeping exactly one point of $X$ fixed. If $X$ is regularly chainable from $p$ to $q$, then $h$ may be chosen so that $p$ and $q$ are interchanged.*



Proof. Let $\{\mathfrak{U}_i\}$ be a regular defining sequence of chains for $X$. Let $\{\mathfrak{V}_i\}$ be the sequence of reverse chains of $\{\mathfrak{U}_i\}$; that is, if $V_{i,j} \in \mathfrak{V}_i$ then $V_{i,j} = U_{i,n_i-j+1} \in \mathfrak{U}_i$, where $n_i$ is the number of elements in $\mathfrak{U}_i$. Clearly $\{\mathfrak{V}_i\}$ is also a regular defining sequence of chains for $X$. Let $g_i$: $\mathfrak{U}_i \to \mathfrak{V}_i$ be defined by $g_i(U_{i,j}) = V_{i,j}$. For each $x \in X$, there exists a sequence $\{U_{i,j_x}\}_{i=1}^{\infty}$ such that $x \in U_{i,j_x} \in \mathfrak{U}_i$. Then $\bigcap_{i=1}^{\infty} \overline{V}_{i,j_x}$ is a point. Let

$$h(x) = \bigcap_{i=1}^{\infty} \overline{g_i(U_{i,j_x})} = \bigcap_{i=1}^{\infty} \overline{V}_{i,j_x}.$$

That $h$ is a homeomorphism follows from the proof of Theorem 11 of [1]. It is clear from the construction that $h$ is of period two and that $h$ keeps exactly one point fixed. In addition, if $\{\mathfrak{U}_i\}$ is a sequence of chains from $p$ to $q$, then $h$ interchanges $p$ and $q$.

COROLLARY 2.1.1. *Let $X = M \times N$ where $M$ and $N$ are continua and $M$ is regularly chainable. Then $X$ admits a period two homeomorphism.*

Proof. By Theorem 2.1, $M$ admits a period two homeomorphism $h$. Define $g$: $X \twoheadrightarrow X$ by $g(m, n) = (h(m), n)$. Then $g$ is a period two homeomorphism of $X$ onto itself.

EXAMPLE 2.1. The "$\sin(1/x)$ continuum" with limit segment is not regularly chainable and does not admit period two homeomorphisms (since the limit segment must go onto itself).

EXAMPLE 2.2. Let $M_1$ be a pseudo-arc and $M_2$ be an arc. Let $M = M_1 \cup M_2$ at a common endpoint. Then $M$ is not regularly chainable, but *does* admit period two homeomorphisms. (See Lemma 3.1.)

EXAMPLE 2.3. The following is another example of a chainable continuum which is not regularly chainable. Let $\{x_i\}_{i=1}^{\infty}$ be the sequence of points $1/2^i$ converging to 0 on the $x$-axis. For $i$ odd, let $A_i$ be the straight line segment from $x_i$ to $x_{i+1}$. For $i$ even, let $A_i$ be a pseudo-arc of diameter $< 1/2^{i-1}$. Let $A = (\bigcup_{i=1}^{\infty} A_i) \cup \{0\}$. Then $A$ is a chainable continuum which is not regularly chainable. We note that if $h$ is any homeomorphism of $A$ onto itself, $h$ must carry $\{0\} \cup \{x_i\}_{i=1}^{\infty}$ onto itself by the identity.

DEFINITION 2.6. Let $S$ be a rectangle in the plane which is the union of a chain of rectangles $\mathfrak{S}$: $S_1, S_2, \ldots, S_n$, such that $S_i \cap S_{i+1}$ is a common arc. Let $T$ be a polygonal region bounded by a simple closed curve such that

(1) *$T$ is the union of a chain of rectangles $\mathfrak{T}$: $T_1, T_2, \ldots, T_m$, with $T_i \cap T_{i+1} =$ common arc,*

(2) *$\mathfrak{T}$ is a closed refinement of $\mathfrak{S}$,*

(3) *$\mathfrak{T}$ is crooked in $\mathfrak{S}$, and*

(4) *$\mathfrak{T}$ is regular in $\mathfrak{S}$.*

Then $\mathcal{S}$ is called a *standard chain* and $\mathcal{T}$ is said to be a *standard refinement of* $\mathcal{S}$.

LEMMA 2.1. *Let $S$ be a rectangle in the plane such that $S$ is the union of a chain of rectangles, $\mathcal{S}$: $S_1, S_2, \ldots, S_n$. Then there exists a standard refinement $\mathcal{T}$ of $\mathcal{S}$.*

Proof. For $n=5$, we may obtain a regular crooked refinement as in [2]. Clearly this construction may be carried out for any positive integer $n$, in an inductive manner, by induction on $n$.

THEOREM 2.2. *The pseudo-arc is regularly chainable.*

Proof. Let $p$ and $q$ be two points in $E^2$ and let $\mathcal{U}_1$: $U_{1,1}, U_{1,2}, \ldots, U_{1,n_1}$ be a standard chain from $p$ to $q$. Let $\mathcal{U}_2$: $U_{2,1}, U_{2,2}, \ldots, U_{2,n_2}$ be a chain in $E^2$ from $p$ to $q$ such that $\mathcal{U}_2$ is a standard refinement of $\mathcal{U}_1$. We also require that $\mu(\mathcal{U}_2) < \frac{1}{2}$. Let $h_1$: $E^2 \twoheadrightarrow E^2$ be a homeomorphism which carries the chain $\mathcal{U}_2$ to a standard chain $\mathcal{C}_2$ in $E^2$ and which is the identity outside some disk neighborhood of $\overline{\mathcal{U}_2^*}$. This can be done by the Schoenflies Theorem. Then $h_1^{-1}$ is uniformly continuous, since it is supported on a compact set. Therefore there is a $\delta_1 > 0$ such that if $A$ is a set of diameter $< \delta_1$ then $h_1^{-1}(A)$ is of diameter $< \frac{1}{3}$. Let $\mathcal{D}_3$ be a standard refinement of $\mathcal{C}_2$ from $h_1(p)$ to $h_1(q)$ such that $\mu(\mathcal{D}_3) < \delta_1$. Let $\mathcal{U}_3 = h_1^{-1}(\mathcal{D}_3)$. Then $\mathcal{U}_3$ is a chain of mesh $< \frac{1}{3}$ from $p$ to $q$, and is a regular, crooked, closed refinement of $\mathcal{U}_2$.

We proceed in this manner inductively. Assume that we have $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$, with $\mathcal{U}_{i+1}$ being a regular, crooked, closed refinement of $\mathcal{U}_i$, $i = 1, 2, \ldots, n-1$, and such that $\mu(\mathcal{U}_i) < 1/i$. Let $h_{n-1}$ be a homeomorphism of $E^2$ onto $E^2$ which is the identity outside some disk neighborhood of $\overline{\mathcal{U}_n^*}$, and which carries $\mathcal{U}_n$ to a standard chain $\mathcal{C}_n$ in $E^2$. This can be done by the Schoenflies Theorem. Then $h_{n-1}^{-1}$ is uniformly continuous, since it is supported on a compact set. Therefore there exists $\delta_{n-1} > 0$ such that if $A$ is of diameter $< \delta_{n-1}$ then $h_{n-1}^{-1}(A)$ is of diameter $< 1/(n+1)$. Let $\mathcal{D}_{n+1}$ be a standard refinement of $\mathcal{C}_n$ from $h^{n-1}(p)$ to $h^{n-1}(q)$ such that $\mu(\mathcal{D}_{n+1}) < \delta_{n-1}$. Let $\mathcal{U}_{n+1} = h_{n-1}^{-1}(\mathcal{D}_{n+1})$. Then $\mathcal{U}_{n+1}$ is a regular, crooked, closed refinement of $\mathcal{U}_n$ and $\mu(\mathcal{U}_{n+1}) < 1/(n+1)$. We may make the links overlap slightly, so that we have a tower of chains in the usual sense.

Let $M = \bigcap_{i=1}^{\infty} \overline{\mathcal{U}_i^*}$. Then $M$ is a pseudo-arc by [1], and the sequence $\{\mathcal{U}_i\}$ is a regular defining sequence for $M$. It follows that the pseudo-arc is regularly chainable.

COROLLARY 2.1.1. *The pseudo-arc admits uncountably many period two homeomorphisms.*

Proof. Let $M$ be a pseudo-arc and let $h$ be a period two homeomorphism of $M$ onto itself. Let $\varphi$ be any homeomorphism of $M$ onto itself such that $\varphi \neq h$ and $\varphi \neq e$. Then $\varphi^{-1}h\varphi$ is also of period two. Since $M$ is homogeneous [1], there are uncountably many such homeomorphisms $\varphi$. Thus there are uncountably many period two homeomorphisms of $M$ onto itself.



QUESTION 1. *Let $g, h$ be two period two homeomorphisms of the pseudo-arc. Does there exist a homeomorphism $\varphi$ of the pseudo-arc onto itself such that $g = \varphi^{-1}h\varphi$?*

QUESTION 2. *Same as Question* (1) *for any regularly chainable continuum.*

QUESTION 3. *Do there exist period two homeomorphisms of the pseudo-arc onto itself keeping more than one point fixed?*

QUESTION 4. *Does example 2.3 admit period two homeomorphisms?*

## 3. An example.

In this section we construct an example of a chainable continuum which admits a period four homeomorphism.

LEMMA 3.1. *Let $M$ be the union of two chainable continua $M_1$ and $M_2$ intersecting at a common endpoint $p$, where $M_1$ is a pseudo-arc. Then there exists a homeomorphism $h$: $M \twoheadrightarrow M$ such that $h$ is of order two and $h|M_2$ is the identity.*

Proof. Let $g$ be any period two homeomorphism of $M_1$ onto itself. Since $M_1$ is chainable, there is a point $x_0 \in M_1$ such that $g(x_0) = x_0$. See [3]. Since $M_1$ is homogeneous [1], there exists a homeomorphism $\varphi$: $M_1 \twoheadrightarrow M_1$ such that $\varphi(p) = x_0$. Then $\varphi^{-1}g\varphi$ is a period two homeomorphism of $M_1$ onto itself keeping $p$ fixed. Define $h$: $M \twoheadrightarrow M$ by $h|M_1 = \varphi^{-1}g\varphi$ and $h|M_2$ is the identity. Then $h$ is the desired homeomorphism.

THEOREM 3.1. *There exists a chainable continuum which admits a period four homeomorphism.*

Proof. Let $M_1$ be a pseudo-arc in $E^2$, with an endpoint $p$ on the $y$-axis, but otherwise lying in the left-hand half plane. Let $f$: $E^2 \twoheadrightarrow E^2$ be a reflection through the $y$-axis, and let $M_2 = f(M_1)$. Let $M = M_1 \cup M_2$. Clearly $M$ is chainable. Let $g = f|M$. By Lemma 3.1, there is a homeomorphism $h$: $M \twoheadrightarrow M$ such that $h$ is of period two and $h|M_2$ is the identity. Then it is easy to see that $gh$ is of period four on $M$.

QUESTION. *Does the pseudo-arc admit period $n$ homeomorphisms for $n > 2$?*

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# Expansive homeomorphisms on homogeneous spaces

by

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**1.** In [2] Murray Eisenberg has shown how an expansive homeomorphism may be constructed from a positively expansive map. In this note we show that a positively expansive map on a compact connected manifold must be a covering map of the manifold on itself, and that a manifold admitting such a map cannot be simply connected. Furthermore, if the manifold is triangulable, its Euler characteristic must be zero. Next, positively expansive maps of various manifolds are exhibited; and using Eisenberg's technique, we infer:

A. In every finite dimension greater than one there is a compact connected space, fibered over a manifold by the Cantor set, which admits an expansive homeomorphism and which is not an abelian group space. In every finite dimension greater than 2 there are countably many different such spaces.

Finally we prove:

B. In every finite dimension greater than 3 there is a compact, connected manifold, not an abelian group space which admits an expansive homeomorphism.

In previous examples of expansive homeomorphisms on compact, perfect, homogeneous spaces, the space has been a group space nad the homeomorphism conjugate (in the homeomorphism group) to an automorphism of the topological group carried by the space ([2], [3], [7], [9]). T. S. Wu [10] has shown that compact connected finite dimensional topological groups which admit expansive automorphisms are abelian. It follows that the expansive homeomorphisms constructed here cannot be conjugate to an automorphism of a topological group.

**2.** Let $X$ be a metric $(d)$ space. A map $f$ of $X$ onto itself is called *expansive* provided that there exists a positive constant $c$ such that to each pair $(x, y)$ of distinct points of $X$ there corresponds an integer $n$ with $d[f^n(x), f^n(y)] > c$. The number $c$ is called the *expansive constant*. The distance $d[f^n(x), f^n(y)]$ is to be interpreted as the usual distance between sets. If, to each pair $(x, y)$ of distinct points of $X$, there corresponds