On the composition and products of universal mappings

by

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A continuous mapping $f\colon X\to Y$ of a topological space X into a topological space Y is said to be a *universal mapping* if for any continuous mapping $g\colon X\to Y$ there exists a point $x\in X$ such that $f(x)=g(x)$.

Some positive results about the inverse systems and products of the universal mappings are contained in papers [3], [5] and [6]. An example of the inverse system of 1-dimensional continua with universal projections such that the inverse limit does not possess the fixed point property, is given in [4] (cf. [8]).

In this paper we shall show that composition and product of universal mappings may not be universal. A simple relation between universality of the composition of mappings and the product of the same mappings is given in section 3.

§ 1. Universal mappings which raise dimension. Let $I=[-1,1]$ be the closed interval. We start from the following direct consequence of Theorem from [2] (see also Theorem B from [5]).

(1.1) Proposition. *If $f\colon X\to Y$ is a universal mapping, where X, Y are normal spaces with* $\dim X<\dim Y$, *then for every integer n such that* $\dim X<n\leqslant\dim Y$ *and for every universal mapping* $g_n\colon Y\to I^n$ *the mapping* $g_n\circ f\colon X\to I^n$ *is a non-universal mapping which is the composition of two universal mappings.*

Now we introduce an auxiliary notion of S-mapping. Let $S=(X_0,\{X_t\}_{t\in T},A)$, where

X_0 is a topological space,

A and X_t are connected subsets of X_0 for any $t\in T$,

$\bigcup_{t\in T}(X_t\cup A)=X_0$,

$X_t\cap A\neq\emptyset$ for every $t\in T$.

Then we shall say that a continuous mapping $f\colon X_0\to Y$ of X_0 onto a space Y is an S-mapping if the following three conditions hold:

(i) $A=f^{-1}(a)$ *for some point* $a\in Y$,

(ii) $a\in f(X_t)\cap f(X_u)=\{a\}$ *or* $=f(X_t)$ *for any* $t,u\in T$,

(iii) $f|X_t\colon X_t\to f(X_t)$ *is a universal mapping for any* $t\in T$.

The following generalization of Proposition 7 from [3] has almost the same proof.

(1.2) PROPOSITION. *Let $f\colon X_0 \to Y$ be an S-mapping such that for any continuous mapping $g\colon Z \to Y$, where $Z = X_u$ for an arbitrary $u \in T$ or $Z = A$, the set $g(Z) \cap f(X_t) \setminus \{a\}$ is an open subset of the subspace $g(Z)$. Then f is a universal mapping.*

(1.3) COROLLARY. *Let $f\colon X_0 \to Y$ be an S-mapping of X_0 onto a Hausdorff space Y, and let $f(X_t)$ be a closed subset of Y for any $t \in T$. Next, suppose that for any continuous mapping $g\colon Z \to Y$, where Z is an arbitrary space X_t, $t \in T$, or $Z = A$, and for any neighbourhood U of $a \in Y$ we have*

$$g(Z) \cap f(X_u) \setminus U \neq \emptyset$$

for at most finite number of indexes $u \in T$. Then f is a universal mapping.

(1.4) COROLLARY. *Let $f\colon X_0 \to Y$ be an S-mapping of X_0 onto a Hausdorff space Y such that for any connected and locally connected compact subspace W of Y and for any neighbourhood U of $a \in Y$ the set $W \setminus U$ is contained in a finite union of the subspaces $f(X_t)$. Next, suppose that A and X_t, $t \in T$, are locally connected compact Hausdorff spaces. Then f is a universal mapping.*

(1.5) EXAMPLE. Let $g_n\colon X \to K^n$ be a one-to-one continuous mapping of a subspace X of the Cantor discontinuum D onto the nth power of the Knaster's hereditarily indecomposable continuum, where n is a positive integer or $n = \aleph_0$. Then the cone mapping $Cg_n\colon CX \to CK^n$ is, by Corollary (1.4), a one-to-one universal mapping of a 1-dimensional separable metric space onto a $(n+1)$-dimensional continuum, $n = 1, 2, \ldots, \aleph_0$.

(1.7) EXAMPLE. Let $f_n\colon D \to K^n$ be a continuous mapping of D onto K^n (see (1.5)), where n is any positive integer or $n = \aleph_0$. Then the cone mapping $Cf_n\colon CD \to CK^n$ as a continuous extension of Cg_n from (1.5) (or by Corollary (1.4)) is a universal mapping of a 1-dimensional continuum onto a $(n+1)$-dimensional continuum, $n = 1, 2, \ldots, \aleph_0$.

From Proposition (1.1) and Examples (1.5), (1.6) we immediately obtain the following result:

(1.7) THEOREM. *The composition $g \circ f\colon X \to I^n$ of two universal mapping is not any universal mapping in general, even if Y is a continuum and either X is a 1-dimensional continuum or X is a 1-dimensional separable metric space and f is one-to-one.*

Example (1.5) can be generalized.

(1.8) DEFINITION (see [7]). An arcwise connected space X is said to be a *B-space* if it has the following property:

For any one-to-one continuous mapping f of the ray $[0, \infty)$ into X, the closure of the set $P = f([0, \infty))$ is a simple arc.

Let us remark that in a B-space X there exists the unique simple arc $\widetilde{x, y}$ with the end-points x, y for any $x, y \in X$.

The following theorem is proved in [7] (see also [7], Theorem (3.4)).

(1.9) THEOREM. *Let X be a B-space and $Q \subseteq X \times X$ be a non-empty relation such that the following two conditions are satisfied:*

(1.10) *$Q \cap (A \times A)$ is a closed subset of $A \times A$ for any simple arc A in X,*

(1.11) *there exists a function $h\colon Q \times X \to X$ such that $(a', h(a, b, a')) \in Q$ and the simple arc $\widetilde{b, h}(a, b, a')$ is contained in $\bigcup_{p \in \widetilde{a, a'}} \{q \in X\colon (p, q) \in Q\}$ for any $(a, b) \in Q$ and $a' \in X$.*

Then there exists an $x \in X$ such that $(x, x) \in Q$.

Now we shall prove the following theorem, which is a generalization of Example (1.5).

(1.12) THEOREM. *If $f\colon X \to Y$ is a one-to-one continuous mapping of a B-space X onto a Hausdorff B-space Y, then f is a universal mapping.*

Proof. Let $g\colon X \to Y$ be a continuous mapping. Then

$$Q = \{(x, y) \in X \times X\colon g(x) = f(y)\} = (f \times g)^{-1}(\Delta_Y)$$

is a closed subset of $X \times X$, as Y is a Hausdorff space (Δ_Y is the diagonal in $Y \times Y$). Thus condition (1.10) holds.

Now, let us put $h(a, b, a') = f^{-1} \circ g(a')$ for any $(a, b, a') \in Q \times X$. Then $g(a') = f(h(a, b, a'))$, and hence

$$(a', h(a, b, a')) \in Q \quad \text{for any } (a, b, a') \in Q \times X.$$

Next,

$$\bigcup_{p \in \widetilde{a, a'}} \{q \in X\colon (p, q) \in Q\} = f^{-1} \circ g(\widetilde{a, a'})$$

is an arcwise connected subset of X, as X is an arcwise connected space and Y is a B-space. Hence, since (a, b) and $(a', h(a, b, a'))$ belong to Q,

$$\widetilde{b, h}(a, b, a') \subseteq f^{-1} \circ g(\widetilde{a, a'}).$$

Thus condition (1.11) is also satisfied and, by Theorem (1.9), $g(x) = f(x)$ for a point $x \in X$.

The following example has some relation to the product:

(1.13) EXAMPLE. *Let $f_1\colon D \to K$ be a continuous mapping onto, where D and K are as in Example (1.5), and let X be a space obtained from $D \times I$ by identification of points $(x, 1)$, $(y, 1)$ such that $f_1(x) = f_1(y)$, $x, y \in D$. Then X is a 1-dimensional continuum.*

We shall show that the mapping $f\colon X \to K \times I$, given by $f(x, t) = (f_1(x), t)$ for $(x, t) \in X$, is universal.

Indeed, let $p\colon K \times I \to K$ be the projection and let $h\colon X \to K \times I$ be a continuous mapping. Then $p \circ h(x, 1) = f_1(x)$ for a certain $x \in D$

since K has the fixed point property. Hence $h(I_x) \subseteq \{f_1(x)\} \times I$ for such $x \in D$ and $I_x = \{(x, t)\colon t \in I\}$. On the other hand,

$$f|I_x\colon I_x \to \{f_1(x)\} \times I$$

is a universal mapping. Thus $f(x, t) = h(x, t)$ for some $t \in I$. Universality of f is proved.

Now, let $g\colon K \to I$ be a continuous mapping onto, and let $i\colon I \to I$ be the identity mapping. Then, by Theorem (2.7) from [5], the product mapping $g \times i\colon K \times I \to I^2$ is universal yet, by Proposition (1.1), the composition $(g \times i) \circ f\colon X \to I^2$ is not a universal mapping.

We can give a direct proof of the last fact. Let G be a closed-open subset of D such that

$$(g \circ f_1)^{-1}(-1) \subseteq G \quad \text{and} \quad (g \circ f_1)^{-1}(1) \subseteq D \backslash G .$$

Then the mapping $h\colon X \to I^2$ given by the formula

$$h(x, t) = \begin{cases} \big(\min(1, g \circ f_1(x) + 1 - t), -1\big) & \text{if } x \in G, \\ \big(\max(-1, g \circ f_1(x) - 1 + t), -1\big) & \text{if } x \in D \backslash G \end{cases}$$

is continuous. Evidently $h(p) \neq (g \times i) \circ f(p)$ for any $p \in X$.

(1.14) Remark. One can prove, analogously as for f, that the mapping $f \times i$ is universal (see (1.13)).

§ 2. Compositions of the universal mappings of polyhedra. Let us recall (see Proposition (1.1) of [6]) that a continuous mapping $f\colon X \to I^n$ is not a universal mapping if and only if there exists a continuous mapping $g\colon X \to S^{n-1}$ (where, in this case, S^{n-1} is the boundary of I^n) such that $g(x) = f(x)$ for any $x \in f^{-1}(S^{n-1})$.

(2.1) Example. Let C be the complex plane and

$$Q = \{z \in C\colon |z| \leqslant 1\} \quad \text{and} \quad S^1 = \{z \in C\colon |z| = 1\} .$$

Next, let M be a Möbius strip obtained from the subspace

$$P = \{z \in C\colon \tfrac{1}{2} \leqslant |z| \leqslant 1\}$$

by the identification of the points $z_1, z_2 \in P$ such that $|z_1| = |z_2| = \frac{1}{2}$ and $z_1^2 = z_2^2$. Then the mapping $f\colon M \to Q$ given by the formula

$$f(z) = \Big(2 - \frac{1}{|z|}\Big) z \quad \text{for any } z \in M$$

is a well-defined universal mapping and the composition $g \circ f\colon M \to Q$ of the universal mappings $f\colon M \to Q$ and $g\colon Q \to Q$, $g(z) = z^2$ for $z \in Q$, is not a universal mapping.

More generally, let $P_{n,m}$, where n, m are the arbitrary integers, be a space obtained from P by identification of the points $z_1, z_2 \in P$ such that either $|z_1| = |z_2| = \frac{1}{2}$ and $z_1^n = z_2^n$ or $|z_1| = |z_2| = 1$ and $z_1^m = z_2^m$. Next, let $f_{n,m}^k\colon P_{n,m} \to Q$ be a mapping given by formula

$$f_{n,m}^k(z) = \Big(2 - \frac{1}{|z|}\Big)^{km} z^{km} \quad \text{for} \quad z \in P_{n,m} .$$

Obviously, $P_{n,m}$ is a polyhedron and $f_{n,m}^k$ is a well-defined continuous mapping for any integer k. This mapping, under a triangulation of $P_{n,m}$, is even a simplicial mapping.

(2.2) Theorem. *The mapping $f_{n,m}^k\colon P_{n,m} \to Q$ is universal if and only if $n \nmid km$. The composition $g_n \circ f_{n,m}^k\colon P_{n,m} \to Q$ of the mappings $f_{n,m}$ and $g_n\colon Q \to Q$, $g_n(z) = z^n$ for $z \in Q$, is never universal.*

Proof. If $n | km$ then the mappings $h\colon P_{n,m} \to Q$, $h(z) = -(z/|z|)^{km}$ for $z \in P_{n,m}$, is a well-defined continuous mapping such that $h(z) \neq f_{n,m}(z)$ for any $z \in P_{n,m}$, i.e. $f_{n,m}^k$ is not a universal mapping.

In particular, $g_n \circ f_{n,m}^k = f_{n,m}^{n,k}$ is not a universal mapping.

Now let $n \nmid km$. Let S_m^1 denote the subspace of $P_{n,m}$ obtained from S^1 by identification of the points $z_1, z_2 \in S^1$ such that $z_1^m = z_2^m$. Then

$$H_2(P_{n,m}, S_m^1; Z_n) \neq 0 \quad \text{and} \quad (f_{n,m}^k)_*(e) \neq 0 \in H_2(Q, S^1; Z_n)$$

for any generator e of the relative homology group $H_2(P_{n,m}, S_m^1; Z_n)$, where Z_n is the group of the rests modulo n. Hence, if $f\colon P_{n,m} \to Q$ is a continuous mapping such that $f(x) = f_{n,m}^k(x)$ for any $x \in S_m^1 = (f_{n,m}^k)^{-1}(S^1)$, then

$$\delta \circ f_*(e) = (f|S_m^1)_* \circ \delta(e) = (f_{n,m}^k|S_m^1)_* \circ \delta(e) = \delta \circ (f_{n,m}^k)_*(e) \neq 0$$

since $\delta\colon H_2(Q, S^1; Z_n) \to H_1(S^1)$ is an isomorphism. Thus $f(P_{n,m}) \nsubseteq S^1$. This means that $f_{n,m}^k$ is a universal mappings. The theorem is proved.

§ 3. Product of nniversal mappings. Connection with the composition. We have proved in papers [5] and [6] that the product of universal mappings of the compact spaces onto the snake-like spaces (and, under a condition, onto one n-dimensional cube I^n) is a universal mapping. Now we shall start from some examples of two universal mappings, the product of which is not a universal mapping.

(3.1) Example. Let B be a Boltiansky's 2-dimensional continuum such that $\dim B^2 = 3$ (see [1]). Then, by Theorem of [2], there exists a universal mapping $f\colon B \to I^2$, but $f \times f\colon B^2 \to I^4$ is not a universal mapping.

(3.2) Example. As Knill [9] has shown there exists a 2-dimensional continuum X with the fixed-point property such that $X \times I$ does not possess this property. In the other words, the identities $i_X\colon X\to X$ and $i_I\colon I\to I$ are universal, but $i_X\times i_I$ is not a universal mapping, although

$$\dim(X\times I)=\dim X+\dim I=\dim X+1$$

(cf. Theorem of [2]).

The following theorem gives the relation between the universality of the composition of some mappings and the universality of the product of the same mappings (this is a product criterion of universality of the composition of mappings).

(3.3) Theorem. *Let $f_i\colon X_i\to X_{i+1}$ be a continuous mapping for any $i=1,2,\dots,n$. If the Cartesian product*

$$\prod_{i=1}^{n} f_i\colon \prod_{i=1}^{n} X_i\to\prod_{i=1}^{n} X_{i+1}$$

of mappings $f_1,f_2,\dots,f_n$ is universal, then the composition

$$f=f_n\circ f_{n-1}\circ\dots\circ f_1\colon X_1\to X_{n+1}$$

is also universal.

Proof. Let $g\colon X_1\to X_{n+1}$ be an arbitrary mapping. We shall define a mapping

$$G\colon \prod_{i=1}^{n} X_i\to\prod_{i=1}^{n} X_{i+1}$$

by

$$G(x_1,x_2,\dots,x_n)=\big(x_2,x_3,\dots,x_n,g(x_1)\big)$$

for any $(x_1,x_2,\dots,x_n)\in\prod_{i=1}^{n}X_i$. Then

$$G(x)=\Big(\prod_{i=1}^{n}f_i\Big)(x)$$

for some $x=(x_1,x_2,\dots,x_n)$, and hence

$$x_{i+1}=f_i\circ f_{i-1}\circ\dots\circ f_1(x_1)\quad\text{for}\quad i=1,2,\dots,n-1$$

and

$$g(x_1)=f_n\circ f_{n-1}\circ\dots\circ f_1(x_1)\,,\quad\text{where}\quad x_1\in X_1\,.$$

The theorem is proved.

(3.4) Remark. The converse of Theorem (3.3) is not true. Indeed, let $f\colon X\to I$ be a universal mapping of the continuum X from Example (3.2) into I (in this case that means $f(X)=I$). Then the composition $f\circ i_X=f$ is a universal mapping, but the product mapping $i_X\times f\colon X^2\to X\times I$ is not universal. Theorem (3.3) has a generalization in Theory of Category.

The following assertion is a direct consequence of Proposition (1.1) and Theorem (3.3).

(3.5) Corollary. *If $\dim X<n\leqslant\dim Y$, where X and Y are normal spaces and n is a positive integer, then for any universal mappings $f\colon X\to Y$ and $g\colon Y\to I^n$ the product mapping $f\times g\colon X\times Y\to Y\times I^n$ is not universal.*

We can apply the above Corollary to Examples (1.5) and (1.6).

(3.6) Example. Let the mappings $f\colon X\to K\times I$, $g\colon K\to I$, and $i\colon I\to I$ be given as in Example (1.13). We know (see (1.13)) that the composition $(g\times i)\circ f\colon X\to I^2$ is not a universal mapping. Hence also the product mapping $f\times g\times i\colon X\times K\times I\to K\times I^3$ is not a universal mapping yet the mappings $g\times i$, $f\times i$ are universal (the mapping $f\times g$ is not universal).

(3.7) Example. The product of universal mappings of 2-dimensional polyhedra is not necessarily a universal mapping. For example, if $n\nmid km$, then $f^k_{n,m}\colon P_{n,m}\to Q$ and $g_n\colon Q\to Q$ are universal (see Example (2.1) and Theorem (2.2)) but

$$f^k_{n,m}\times g_n\colon P_{n,m}\times Q\to Q\times Q$$

is not a universal mapping.

Let us put

$$A=P_{n,m}\times S^1_m\cup S^1_m\times P_{n',m'}\subseteq P_{n,m}\times P_{n',m'}\,.$$

Then

(3.8) *We have*

$$H_3(A;R_1)=\{(x,x')\in R_1\oplus R_1\colon nx=n'x'=mx+m'x'=0\bmod 1\}$$

where $R_1=R/Z$ denotes the group of the numbers modulo 1, and

(3.9) $H_3(A;R_1)=0$ *if n, mn' or n', $m'n$ is a pair of relatively prime integers.*

(3.10) Example. Let n, m be a pair of relatively prime integers and $n'=m$ and $m'=n$. Then by (3.9) we have $H_3(A;R_1)=0$.

Evidently $P_{n,m}$ and $P_{m,n}$ are homeomorphic polyhedra. The product mapping $f^1_{n,m}\times f^1_{m,n}\colon P_{n,m}\times P_{m,n}\to Q\times Q$ of the universal mappings $f^1_{n,m}$ and $f^1_{m,n}$ (see Theorem (2.2)) of the homeomorphic polyhedra is not universal.

Indeed,

$$A=(f^1_{n,m}\times f^1_{m,n})^{-1}(S^3)$$

and

$$f^1_{n,m}\times f^1_{m,n}|A\colon A\to S^3$$

is, by the Hopf theorem, a homotopically trivial mapping. Hence the assertion is a consequence of the Borsuk's homotopy extension theorem.

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Euklidische und Minkowskische Orthogonalitätsrelationen

von

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In affinen Ebenen lassen sich Orthogonalitätsrelationen danach unterscheiden, ob sie *selbstorthogonale* (*singulär, isotrop*) Richtungen enthalten oder nicht, anders ausgedrückt, ob die ihnen entsprechenden Involutionen auf der unendlich fernen Geraden von hyperbolischen oder elliptischen Typ sind. Die ersteren seien *Minkowskische* (pseudoeuklidische) die letzeren *Euklidische* Orthogonalitätsrelationen genannt.

Die allgemeinen Eigenschaften dieser Relationen werden gewöhnlich erst nach Einführung von Koordinaten durch die Untersuchung quadratischer Formen, oder mit Hilfsmitteln der Theorie der projektiven Abbildungen untersucht. Es ist vom Standpunkt der Grundlagen der Geometrie, wie auch für die Geometrie selbst von Interesse, diese Eigenschaften innerhalb der Theorie der affinen Ebenen selbst herzuleiten. Bemerkenswert ist, daß die hier genannten Theoreme und Corollare auf synthetischen, d. h. rein geometrischen Wege sehr viel schneller und eleganter als mit analytischen Hilfsmitteln herleitbar sind.

Es wird u.a. ein Kriterium für die Existenz singulärer Richtungen angegeben. Ferner werden Bedingungen für die Kommensurabilität von Orthogonalitätsrelationen genannt. Die Untersuchungen sind von einer Anordnung unabhängig.

Wir betrachten hier i. a. *Translationsebenen*, also affine Ebenen, mit dem sogenannten kleinen Satz von Desargues als zusätzlichem Axiom. (1) Um Sonderfälle auszuschliessen, wollen wir außerdem annehmen, daß die Diagonalen eines Parallelogramms einander schneiden (Fano-Axiom).

Punkte werden mit $A, B, \ldots$, Geraden mit $a, b, \ldots$ bezeichnet. AB bezeichnet die Verbindungsgerade der Punkte A, B; $a*b$ den Schnittpunkt zweier nicht paralleler Geraden a, b, $\|$ bezeichnet die Parallelitätsrelation und $\varrho, \sigma, \ldots$ die Aequivalenzklassen dieser Relation, *Richtungen* genant. $\vdash$ bezeichnet Orthogonalitätsrelationen.

(1) Die folgenden Theoreme gelten meist auch ohne diese Voraussetzung; es ist m. W. unbekannt, ob die Existenz von O-Relationen in affinen Ebenen nicht schon den kleinen Desargues impliziert.