Proof. Let $f\colon P\to P$ be a compact mapping. Let C be a compact subset of P containing $f(P)$. Then there is a finite subpolyhedron P' of P containing C. As before let $f_{P'}\colon P'\to P'$ denote the restriction of f. It is well known that $f_{P'}$ is a Lefschetz map. Thus by Corollary (3.3) (b), f is also a Lefschetz map.

(5.2) COROLLARY. *Every (metric) absolute neighborhood retract X is a Λ-space.*

Proof. For each open cover α of X there is a polyhedron P_α (with the Whitehead topology) and mappings $g_\alpha\colon X\to P_\alpha$ and $h_\alpha\colon P_\alpha\to X$ such that $h_\alpha\circ g_\alpha$ is α-homotopic to 1_X (see [4], p. 138). In particular, $h_\alpha\circ g_\alpha\simeq 1_X$ and $h_\alpha\circ g_\alpha$ and 1_X are α-near. Then by Theorem (4.2), X is a Λ-space.

6. Note that the theorems of § 3 and § 4 also hold for Lefschetz spaces in the sense that "Λ-space" can be replaced by "Lefschetz space" throughout. When this is done the compactness conditions on the mappings can be dropped.

References

[1] N. Bourbaki, *Algèbre*, Chapitre 2 (Algèbre Linéaire), Herman, Paris, 1962.

[2] F. E. Browder, *Fixed point theorems on infinite dimensional manifolds*, Trans. Amer. Math. Soc. 119 (1965), pp. 179-194.

[3] A. Granas, *Generalizing the Hopf—Lefschetz fixed point theorem for non-compact ANR-s*, Proc. Symp. on Infinite Dimensional Topology, Baton Rouge 1967.

[4] S. T. Hu, *Theory of Retracts*, Wayne State Univ. Press, Detroit, 1965.

Reçu par la Rédaction le 1. 9. 1967

On choosing subsets of n-element sets

by

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1. Introduction. Let n be a positive integer. Mostowski ([6]) and others have studied the axioms of choice for finite sets, $[n]$, in which an element is chosen from each set of an arbitrary set of n-element sets. We wish to introduce some new axioms which are concerned with the choice of a subset or of a partition, rather than a single element, from each element of an arbitrary set of n-element sets. We shall discuss the interdependence of these axioms and their relationship to the axioms $[n]$.

2. Notation. We shall operate within a set theory of the Gödel–Bernays type (see the proof of theorem 7); our logical framework will be the first-order predicate calculus with identity. For statements $\alpha_1, \alpha_2, \ldots, \alpha_n$, we write $\alpha_1\to\alpha_2\to\ldots\to\alpha_n$ in lieu of $(\alpha_1\to\alpha_2)\,\&\,(\alpha_2\to\alpha_3)\,\&\ldots$ $\ldots\,\&\,(\alpha_{n-1}\to\alpha_n)$; a similar remark applies to $\alpha_1\leftrightarrow\alpha_2\leftrightarrow\ldots\leftrightarrow\alpha_n$.

By the (nonnegative) *integers* we mean the von Neumann integers—0 (the empty set), $1=\{0\}$, $2=1\cup\{1\}$, $3=2\cup\{2\}$, etc. A set is *finite* iff every nonempty set of subsets of X has a maximal element with respect to set inclusion. If there exists a function which maps the set X one-one onto the positive n, then X is called an *n-element set* and we say that the *number of elements of X is n*; in this case we let $\boldsymbol{n}(X)$ denote the unique integer n for which such a mapping exists.

For each integer n, let I_n be the set of integers $\geqslant n$, let J_n be the relative complement of I_{n+1} in I_1, $I_1\setminus I_{n+1}$, and let $K_n=J_n\setminus\{1\}$. Let Π represent the set of prime numbers and let $\Pi_n=\Pi\cap I_n$.

For any set X let $\mathcal{P}(X)$ designate the power set of X, let $\mathcal{P}^*(X)$ $=\mathcal{P}(X)\setminus 1$, let $\mathcal{P}^{\#}(X)$ be the set of finite subsets of X, and let $\mathcal{P}^{\#*}(X)$ $=\mathcal{P}^{\#}(X)\setminus 1$.

* This research formed part of the author's Ph. D. thesis (Yeshiva University, 1967) under the supervision of Professor Martin Davis of New York University. The work was supported by a National Science Foundation (U.S.A.) Science Faculty Fellowship. We note that Professor Conway's announcement that [3] & [5] & [13] $\not\to$ [15] as well as his other independence results, which were mentioned in the thesis, have recently been retracted.

Furthermore, let $X_{(n)}$ be the set of n-element subsets of elements of X, $n \in I_1$.

Throughout this paper, we shall let $h(n)$ be the greatest integer less than or equal to $n/2$, for $n \in I_1$.

DEFINITION 1. Let $n \in I_2$ and let N be a nonempty subset of J_{n-1}. Let $(n; SN)$ denote the statement: "For every set X of n-element sets there is a function f on X such that for all $A \in X$, $f(A)$ is a subset of A such that $\boldsymbol{n}(f(A)) \in N$." We shall refer to these functions f as *multiple choice functions on* X.

We write $S(n)$ in place of $S(n; J_{n-1})$; thus $S(n)$ simply says that $f(A)$ is a nonempty proper subset of A, for each $A \in X$. If $k \in J_{n-1}$, then if $N_1 = \{k\}$ and if N_2 is the set of integers between k and $n-k$ (inclusive), write $S(n; k)$ for $S(n; N_1)$ and $S^*(n; k)$ for $S(n; N_2)$. Such multiple choice functions f will be specified as k-ary (k^*-ary) choice functions. If for $r \in I_2$, $Z = \{n_1, n_2, \ldots, n_r\}$, then $S(Z; 1)$ will denote $S(n_1; 1)$ & $S(n_2; 1)$ & ... ... & $S(n_r; 1)$. Let $H(n) = S(n; h(n))$.

Clearly, for all $n \in I_2$, $S(n; 1)$ iff $[n]$ and $S^*(n; 1)$ iff $S(n)$. Moreover, for any nonempty subset N of J_{n-1}, $S(n; N)$ iff $S(n; M)$, where $n' \in M$ iff $n' \in N$ or $n-n' \in N$; in particular, for all $k \in J_{n-1}$, $S(n; k)$ iff $S(n; n-k)$, $S^*(n; k)$ iff $S^*(n; n-k)$, and for n odd, $H(n)$ iff $S(n; h(n)+1)$. We shall find it expedient to assume, henceforth, that the number of elements of the "chosen" subset is in $J_{h(n)}$.

DEFINITION 2. Let $n \in I_2$ and let A be an n-element set. By a decomposition D of A is meant a set D satisfying

(i) $\bigcup D = A$,

(ii) $c \in D \to c \neq 0$,

(iii) $(c_1, c_2 \in D \;\&\; c_1 \neq c_2) \to c_1 \cap c_2 = 0$.

The elements of a decomposition D of A will be called the *cells* of D; in particular, a k-element cell will be referred to as a *k-cell*. If $k_1 < k_2 < \ldots < k_r$, and if $l_1, l_2, \ldots, l_r$ are any positive integers, then a decomposition D of A which consists of l_1 k_1-cells, l_2 k_2-cells, ..., and l_r k_r-cells will be called an *$(l_1, k_1; l_2, k_2; \ldots; l_r, k_r)$-type decomposition.*

Let $n \in I_4$. Let $D_0(n)$ be the statement: "For every set X of n-element sets there is a function f on X such that for all $A \in X$, $f(A)$ is a decomposition of A satisfying ((i), (ii), (iii), above, as well as)

(iv) $\boldsymbol{n}(f(A)) \geqslant 2$,

(v) $c \in f(A) \to \boldsymbol{n}(c) \geqslant 2$."

Let $n \in L_3$. Let $D_1(n)$ be the statement: "For every set X of n-element sets there is a function f on X such that for all $A \in X$, $f(A)$ is a decomposition of A satisfying ((i), (ii), (iii)) (iv) as well as

(v′) *there is some* $c \in f(A)$ *such that* $\boldsymbol{n}(c) \geqslant 2$."

Let $n \in L_3$. Let $D_2(n)$ be the statement: "For every set X of n-element sets there is a function f on X such that for all $A \in X$, $f(A)$ is a decomposition of A satisfying ((i), (ii), (iii)) (iv), (v′), and $\sim$(v)."

Functions f effecting these decompositions on the elements of X will be called *$D_i(n)$-functions on* X, $i \in 3$ and $n \in I_3$.

If in each of the statements of definitions 1 and 2 the sets $A \in X$ are assumed to be pairwise-disjoint, the modified statement is equivalent to the original one. The proofs of these equivalences are similar to the proof of that of the so-called "principle of choice" with the axiom of choice ([7], pp. 93-95). When it suits our convenience to do so, we shall assume pairwise-disjointness.

3. Positive results.

THEOREM 1. $(\forall n \in I_3)\big(S(n) \leftrightarrow D_2(n)\big)$.

Proof. Assume $n \geqslant 3$ and let X be a nonempty set of n-element sets.

$S(n) \to D_2(n)$: Let f be a multiple choice function on X. Then for $A \in X$, if $\boldsymbol{n}(f(A)) = 1$, let $g(A) = \{f(A), A \setminus f(A)\}$; if $\boldsymbol{n}(f(A)) > 1$, let $g(A) = \big\{\{a\}\colon a \in f(A)\big\} \cup \{A \setminus f(A)\}$.

$D_2(n) \to S(n)$: Let G be a $D_2(n)$-function on X. Then for $A \in X$, $G(A)$ contains at least one 1-cell and at least one k-cell for some $k \geqslant 2$. Let A' be the union of all the 1-cells of $G(A)$. Define $F(A)$ to be A' if $\boldsymbol{n}(A') \leqslant h(n)$; otherwise, let $F(A) = A \setminus A'$.

THEOREM 2. (a). $(\forall n \in I_4)(\forall k \in K_{h(n)})\big(S^*(n; k) \to D_0(n) \to D_1(n)\big)$.

(b) $(\forall n \in I_3)(\forall k \in J_{h(n)})\Big(S\big(\tbinom{n}{k}; 1\big) \to S(n; k) \to S^*(n; k) \to S(n) \to D_1(n)\Big)$.

$(\forall n \in I_4)(\forall k \in K_{h(n)})\big(S^*(n; k) \to S^*(n; k-1)\big)$.

(c) $(\forall p \in \Pi_5)\big(D_0(p) \to S(p)\big)$.

(d) $(\forall p \in \Pi_3)\big(D_1(p) \leftrightarrow S(p)\big)$.

(e) $(\forall n \in I_2)(\forall k \in I_1)\big(S(kn; Z) \to S(n)\big)$ *if* Z *is any subset of* $J_{h(kn)}$ *which contains no multiple of* n.

(f) *Suppose that* k *and* l *are nonnegative integers which are not both* 0. *Then*

$(\forall m \in I_1)(\forall n \in I_m)\Big(\big(S(km+ln; 1) \;\&\; S(m+n; m)\big) \to S(m+n; 1)\Big)$.

(g) $(\forall p \in \Pi)\big(\forall k \in h(p)\big)(\forall n \in I_2)\big(S(p; k) \to S(np)\big)$.

(h) $(\forall p \in \Pi)(\forall n \in I_2)\big(S(np-1; 1) \to D_0(np)\big)$.

(i) $(\forall n \geqslant 2)\big(S(2^n-2; 1) \to S(n)\big)$.

Proof. (a) and (b). Let X be a nonempty set of n-element sets.

$S^*(n; k) \to D_0(n)$ for $k \in K_{h(n)}$: Let f by a k-ary choice function on X. The function F defined on X by $F(A) = \{f(A), A \setminus f(A)\}$ is a $D_0(n)$-function on X.

$S(n) \to D_1(n)$: If g is a multiple choice function on X, then $G(A) = \{g(A), A \setminus g(A)\}$ defines a $D_1(n)$-function on X.

The other implications of (a) and (b) follow directly from definitions 1 and 2.

(c) Let X be a nonempty set of p-element sets and let f be a $D_0(p)$-function on X. For each $A \in X$, not all of the cells of $f(A)$ have the same number of elements; let $g(A)$ be the union of all the cells with the minimal number of elements. Let $F(A)$ be $g(A)$ if $\boldsymbol{n}(g(A)) \in J_{h(p)}$ and let $F(A) = A \setminus g(A)$, otherwise.

(d) $S(p) \to D_1(p)$, by (b).

$D_1(p) \to S(p)$: Let X be a nonempty set of p-element sets and let f be a $D_1(p)$-function on X. If for $A \in X$, $f(A)$ satisfies (v) of definition 2, then $p \geqslant 5$, and we can define $F(A)$ as in (c). If $f(A)$ satisfies $\sim$(v), then $f(A)$ is a $D_2(n)$-decomposition of A, and hence, we can define $F(A)$ as in theorem 1.

Examples in which $D_i(n)$ does not imply $S(n)$ for n composite, $i = 0, 1$, will be given in theorem 12.

(e) is a generalization of [6], lemma 13 ([7], p. 99, theorem 2). Let X be a nonempty set of n-element sets. For $A \in X$, let A' be the set of ordered pairs $\langle a, i \rangle$, where $a \in A$ and $i \in J_k$. Let X' be the collection of all A' corresponding to $A \in X$. By the axiom of substitution, X' is a set; each element of X' is a kn-element set. Let f be a multiple choice function on X'. Then for each $A' \in X'$, not all the elements $a \in A$ appear the same number of times as first coordinates of members of $f(A')$. Let $g(A)$ be either the collection of elements of A which appear the minimal number of times, or else the complement of this collection with respect to A.

(f) is an extension of [6], lemma 14. Let X be a nonempty set of $(m+n)$-element sets and let f be any m-ary choice function defined on X. Then for $Y \in X$ the sets $f(Y)$ and $Y \setminus f(Y)$ satisfy the hypothesis of [6], lemma 14; hence there is a function g on X such that $g(Y) \in Y$ for all $Y \in X$.

(g) is a generalization of [6], lemma 15. Let X be a nonempty set of pairwise-disjoint np-element sets. Then $X_{(p)}$ is a set of p-element sets; let f be a k-ary choice function on $X_{(p)}$. For each $A \in X$ and for each $a \in A$, denote by n_a, the number of $y \in X_{(p)}$ for which $a \in f(y)$. Then

$$\sum n_a(a \in A) = k\binom{np}{p}$$

is not divisible by np, whereas $\boldsymbol{n}(\{n_a\colon a \in A\}) = \boldsymbol{n}(A) = np$. Thus not all the n_a are identical. Let $g(A)$ be the set of those $a \in A$ for which n_a is minimal (or else the complement of this set with respect to A).

(h) is a restatement of [6], lemma 16.

(i) There are $2^n - 2$ nonempty proper subsets of an n-element set; either the chosen subset or its complement can be used to satisfy the requirements of $S(n)$.

THEOREM 3. (a) *Let $n \in I_4$, let $k \in J_{h(n)-1}$, and let $N \subseteq J_{h(n)} \setminus J_k$. For each such n, k, and N, let*

$$M_1 = \{m\colon (\exists l, i)(m = l - i \;\&\; l \in N \;\&\; i \in k+1)\},$$

let

$$M_2 = J_{h(n-k)},$$

and let $M = M_1 \cap M_2$. Then $S(n; N) \to S(n-k; M)$.

(b) *Let $n \in I_4$ and let k $(\in I_1)$ be such that $k+1 \leqslant 2k \leqslant n-2$. Then*

$$S^*(n; k+1) \to (\forall m \in J_n \cap I_{n-k}) S(m).$$

(c) *Let $n \in I_4$ and let k and l be integers such that $k+1 \leqslant 2k < n$ and $k+1 \leqslant l \leqslant h(n)$. Let $Z = J_l \cap I_{\max(2, l-k)}$. Then*

$$\big(S(Z; 1) \;\&\; S(n; l)\big) \to S(n-k; 1).$$

Proof. (a) Suppose $S(n; N)$ and let X be a nonempty set of $(n-k)$-element sets. To each $A \in X$, add k new elements—for definiteness, say, the first k positive integers which are not in A. The new sets A^* will each have n elements. We let X^* be the collection of A^* corresponding to $A \in X$. By the axiom of substitution, X^* is a set; let f be a multiple choice function on X^* such that for each $A^* \in X^*$, $\boldsymbol{n}(f(A^*)) \in N$. Moreover, $A \cap f(A^*)$ is a nonempty subset of A for each $A \in X$. Let $g(A) = A \cap f(A^*)$ if $\boldsymbol{n}(A \cap f(A^*)) \leqslant h(n-k)$ and let $g(A) = A \setminus f(A^*)$, otherwise. Then g is a multiple choice function defined on X having the property that $g(A) \in N$ for each $A \in X$.

(b) follows from (a).

(c) If $N = \{l\}$ in (a), then $\boldsymbol{n}(g(A)) \in J = Z \cup \{1\}$ for all $A \in X$. Let X_j be the subset of X consisting of all A for which $\boldsymbol{n}(g(A)) = j$, $j \in J$. Define $G(A) = g(A)$ for $A \in X_1$, and for $z \in Z$, define $G(A) = F_z \cdot g(A)$, $A \in X_z$, where F_z is any (1-ary) choice function on X_z. Then G is the required (1-ary) choice function on X.

THEOREM 4. $(\forall n \in I_3)\Big(\big(D_1(n+1) \;\&\; S(n)\big) \to D_0(n+1) \to S(n)\Big)$.

Proof. Suppose $D_1(n+1) \;\&\; S(n)$. Let Y be a nonempty set of $n+1$-element sets and let f be a $D_1(n+1)$-function on Y. Let Y_1 be the subset of Y consisting of all sets A for which there are no 1-cells in $f(A)$; let Y_2 be the subset of Y consisting of all sets A for which there are at least two 1-cells in $f(A)$; let Y_3 be the subset of Y consisting of all sets A for which there is exactly one 1-cell in $f(A)$. At least one of the sets Y_1, Y_2, Y_3 is nonempty; the nonempty sets Y_i are pairwise-disjoint and their union is Y. For $A \in Y_1$ let $g(A) = f(A)$; for $A \in Y_2$ let

$$g(A) = \{c \in f(A)\colon\ n(c) \geqslant 2\} \cup \bigcup \{c \in f(A)\colon\ n(c) = 1\}\,.$$

Let Y_4 be the set of all $A' = A \backslash \{c_A\}$, where $A \in Y_3$ and c_A is the unique 1-cell in $f(A)$. Then Y_4 is a set of n-element sets; let F be a multiple choice function on Y_4. For $A \in Y_3$ let $g(A) = \{F(A') \cup \{c_A\},\ A' \backslash F(A')\}$. Since in every case each cell of $g(A)$ has at least two elements, g is a $D_0(n+1)$-function on Y.

Suppose $D_0(n+1)$. Let X be a nonempty set of n-element sets. As in the the proof of theorem 3, let X^* be the set of all $A \cup \{a_A\}$, where $A \in X$ and a_A is some definite set which is not in A. Then X^*, as a set of $(n+1)$-element sets, has a $D_0(n+1)$-function f defined on it. Suppose $f(A^*) = \{c_i\colon\ i \in n+1\}$; let i_0 be the index of the cell containing a_A. Then if $g(A) = c_{i_0} \backslash \{a_A\}$ for $A \in X$, g is a multiple choice function on X.

We note that the converse of the second implication is false; $D_0(5)$ is independent of $S(4)$, as will be shown in theorem 12.

THEOREM 5. (a) $(\forall n \in I_2)\big(H(2n) \to H(2n-1)\big)$.

(b) $(\forall n \in I_3)\Big(\big(S(n;\,1)\ \&\ H(n-1)\big) \to H(n)\Big)$.

Proof. (a) follows from theorem 3 (a).

(We note that there are odd integers, $2n+1$, for which $H(2n)$ is independent of $H(2n+1)$; for example, $H(4)$ is independent of $H(5)$ as will be shown in theorem 12.)

(b) If X is a nonempty set of n-element sets, let f be a (1-ary) choice function on X. Let $X' = \{A \backslash f(A)\colon\ A \in X\}$, and let g be an $h(n-1)$-ary choice function on X'. We define an $h(n)$-ary choice function F on X as follows: for $A \in X$, let $F(A) = g\big(A \backslash f(A)\big)$, if n is odd, and let $F(A) = f(A) \cup g\big(A \backslash f(A)\big)$, if n is even.

THEOREM 6. (a) $(\forall n \in I_2)\, S(n;\,1) \to (\forall n \in I_2)(\forall k \in K_{h(n)})\, S(n;\,k)$.

(b) $(\Lambda n \in I_2)\Big((\forall r \in K_n)\, S(r;\,1) \to (\forall m \in K_n)(\forall k \in J_{h(m)})\, S(m;\,k)\Big)$.

(c) *For $j \in I_1$ let q_j be the j-th prime (in order of magnitude) and let Q_j be the set consisting of the first j primes. Then*

$$(\forall j \in I_1)(\forall n \in I_2)\Big(\big(S(Q_j;\,1) \to (\forall m \in K_n)(\forall k \in I_{h(m)})\, S(m;\,k)\big) \leftrightarrow n \in K_{q_{j+1}-1}\Big)\,.$$

(d) $(\forall l \in K_r)\, S(l;\,1)$ *is independent of*

$$(\forall m \in K_n)(\forall k \in J_{h(m)})\, S(m;\,k)\,, \qquad \textit{whenever} \qquad \Pi_{n+1} \backslash \Pi_{r+1} \neq 0\,.$$

Proof. (a), (b): A k-ary choice function is obtained by composing k (1-ary) choice functions.

(c) follows from (b) together with [9], lemma 10.

(d) follows from (b) together with [6], theorem VIII.

4. Negative results. We now generalize theorem III of [6], Mostowski's main result concerning necessary conditions for the implication $S(Z;\,1) \to S(n;\,1)$ (in our notation), in order to obtain the independence of certain of our axioms from $S(Z;\,1)$. Subsequently, we shall be concerned with the independence of $S(Z;\,1)$ from some of our axioms.

For any group $\mathfrak{G}$ let $\mathfrak{G}^\infty$ denote the group whose elements are those infinite sequence $\mathfrak{g} = \langle \mathfrak{g}_1, \mathfrak{g}_2, \ldots \rangle$ whose terms belong to $\mathfrak{G}$ and which are such that almost all of the $\mathfrak{g}_n$ are equal to the unity of $\mathfrak{G}$; multiplication is defined in the obvious way, i.e., by term-wise multiplication in $\mathfrak{G}$.

We shall write "$\mathfrak{g} \in \mathfrak{G}$" in case $\mathfrak{G}$ is the group $\langle X, \cdot \rangle$ and $\mathfrak{g} \in X$.

DEFINITION 3. Let $n \in I_2$ and let $\mathfrak{S}_n$ be the symmetric group on n letters.

If $\Phi \in \mathfrak{S}_n$ and if $D = \{T_1, T_2, \ldots, T_r\}$ is a decomposition of J_n, then

(i) if $T_i = \{t_{i1}, t_{i2}, \ldots, t_{im_i}\}$, $i \in J_r$, let

$$\Phi(T_i) = \{\Phi(t_{i1}), \Phi(t_{i2}), \ldots, \Phi(t_{im_i})\};$$

(ii) let $\Phi(D) = \{\Phi(T_1), \Phi(T_2), \ldots, \Phi(T_r)\}$.

If $\mathfrak{G}$ is a subgroup of $\mathfrak{S}_n$ and if N is a nonempty subset of J_n, then $\mathfrak{G}$ leaves N fixed if $\Phi(N) = N$ for every $\Phi \in \mathfrak{G}$; if D is a decomposition of J_n, then $\mathfrak{G}$ leaves D fixed if $\Phi(D) = D$ for every $\Phi \in \mathfrak{G}$.

$\mathfrak{G}$ satisfies condition (C_1) if it leaves no nonempty proper subset of J_n fixed.

Let L be a nonempty proper subset of $J_{h(n)}$. $\mathfrak{G}$ satisfies condition (C_2) with respect to L if it leaves fixed no subset of J_n the number of whose elements is in L.

$\mathfrak{G}$ satisfies condition (C_3) if it leaves fixed no decomposition D of J_n which satisfies conditions (iv) and (v) of definition 2.

$\mathfrak{G}$ satisfies condition (C_4) if it leaves fixed no decomposition D of J_n which satisfies (iv) and (v′) of definition 2.

For $i \in J_4$, $Z\big(\in \mathfrak{F}^{\#}(I_2)\big)$ and n $(\in I_2)$ satisfy condition (K_i) if for every subgroup $\mathfrak{G}$ of $\mathfrak{S}_n$ which satisfies condition (C_i), there is a group $\mathfrak{H} \subset \mathfrak{G}^\infty$ and a finite number r of (not necessarily different) proper subgroups $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_r$ of $\mathfrak{H}$ such that

$$\sum \operatorname{Ind} \mathfrak{H}/\mathfrak{K}_i (i \in J_r) \in Z\,.$$

THEOREM 7. (a) (K_1) *is necessary for the implication* $S(Z;1)\to S(n)$.

(b) (K_2) *is necessary for the implication* $S(Z;1)\to S(n;L)$.

(c) (K_3) *is necessary for the implication* $S(Z;1)\to D_0(n)$.

(d) (K_4) *is necessary for the implication* $S(Z;1)\to D_1(n)$.

Proof. The model for set theory which Mostowski uses in [6], in order to indicate the necessity of his condition (K) for the implication $S(Z;1)\to S(n;1)$, will suffice for our purpose as well in each of our cases. We shall consider only case (a); the other cases follows along the same lines with only minor modifications.

Let σ be the set theory of [5]; this version of set theory permits the existence of urelements (objects, other than the empty set, which are in the domain but not the range of the ϵ-relation) and it does not include the axiom of choice among its axioms. Let σ^* be σ together with the axiom of choice and an axiom asserting the existence of a denumerable number of urelements. We shall assume that σ^* is consistent; this is equivalent to the assumption that σ (or Gödel's system A, B, C of [3]) is consistent. [1] (See [4], pp. 478-479.) Thus we can look for a model within σ^*.

Suppose Z and n fail to satisfy condition (K_1). Then there is a subgroup $\mathfrak{G}$ of $\mathfrak{S}_n$ which leaves no proper, nonempty subset of J_n fixed and which is such that for any subgroup $\mathfrak{H}$ of $\mathfrak{G}^\omega$ and any proper subgroups $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_r$ of $\mathfrak{H}$,

$$\sum \operatorname{Ind} \mathfrak{H}/\mathfrak{K}_i \ (i \in J_r) \notin Z .$$

Let $N_k = J_{kn}\setminus J_{(k-1)n}$, $k \in I_1$. For any $\Psi \in \mathfrak{G}$, let

$$\Psi\big((k-1)n+i\big) = (k-1)n+\Psi(i), \qquad i \in J_n .$$

We define a set $\mathfrak{K}_\xi$ for each ordinal number ξ [2] and a meaning of $\Psi(x)$ for each $\Psi = \langle \Psi_1, \Psi_2, \ldots\rangle \in \mathfrak{G}^\omega$ and each $x \in \mathfrak{K}_\xi$ by transfinite induction.

Let $\mathfrak{K}_0 = \bigcup N_k \ (k \in I_1)$; for $x \in \mathfrak{K}_0$ let $\Psi(x) = \Psi_k(x)$ when $x \in N_k$, $k \in I_1$.

Assume that $\xi > 0$ and that for all $\eta < \xi$, sets $\mathfrak{K}_\eta$ and the meaning of $\Phi(x)$ for $x \in \bigcup \mathfrak{K}_{\eta'}$ $(\eta' \in \eta+1)$ have been defined. Let

$$\mathcal{M}_\xi = \mathfrak{P}\Big(\bigcup \mathfrak{K}_\eta(\eta \in \xi)\Big) .$$

For $x \in \mathcal{M}_\xi \setminus \bigcup \mathfrak{K}_\eta$ $(\eta \in \xi)$, let $\Phi(x) = \{\Phi(y)\colon y \in x\}$. Let $\mathfrak{K}_\xi$ be the set of $x \in \mathcal{M}_\xi$ which satisfy the invariance condition:

[1] The consistency of σ is tacitly assumed in all of our metamathematical statements.

[2] By an *ordinal number* is meant a transitive set which is ordered by the ϵ-relation. (Regularity is included among the axioms of σ.)

(1) *there is an integer* l *such that if* $\Psi = \langle \Psi_1, \Psi_2, \ldots\rangle \in \mathfrak{G}^\omega$ *and if* $\Psi_1 = \Psi_2 = \ldots = \Psi_l = 1$, *then* $\Psi(x) = x$.

The urelements of the model will be the elements of $\mathfrak{K}_0$ and the sets of the model will be the elements of the $\mathfrak{K}_\xi$, $\xi > 0$. (The reader is referred to sections 13-16 of [6] for a verification that certain of the axioms of σ are true in the model as well as for properties of the model.) We proceed to show that $S(n)$ is false in the model; our argument follows that of section 17 of [6].

Let $x = \{N_i\colon i \in I_1\}$; since $\Psi(x) = x$ for every $\Psi \in \mathfrak{G}^\omega$, x is a set of the model. Suppose that y contains exactly one nonempty proper subset R_k of each N_k. We claim that y must fail to satisfy the invariance condition (1), and thus must be disqualified for admission as a set of the model.

Using the axiom of choice (in σ^*), for each $k \in I_1$ we select a permutation $\Psi_k \in \mathfrak{G}$ such that $\Psi_k(R_k) \neq R_k$. For any l let Ψ_{l+1} be the sequence whose $(l+1)$st term is Ψ_{l+1} and all of whose other terms are the identity (of $\mathfrak{S}_n$). Thus $\Phi_1 = \Phi_2 = \ldots = \Phi_l = 1$; yet $\Phi(y) \neq y$.

The argument of [6], section 18 serves to establish the validity of $S(Z;1)$ in the model.

THEOREM 8. *Let* $p \in \Pi$, *let* $r \in I_1$, *and let* Z *be any finite set of positive integers which contains no multiple of* p. *Then* $S(p^r)$ *is independent of* $S(Z;1)$.

Proof. Take $\mathfrak{G}$ to be the cyclic group on p^r letters; $\mathfrak{G}$ leaves no proper subgroup of J_n fixed. Moreover, each group $\mathfrak{H} \subset \mathfrak{G}^\omega$ has the property that for any proper subgroup $\mathfrak{K}$ of $\mathfrak{H}$, $\operatorname{Ind}(\mathfrak{H}/\mathfrak{K})$ is a multiple of p. Hence, for any number of proper subgroups $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_m$ of $\mathfrak{H}$,

$$\sum \operatorname{Ind}(\mathfrak{H}/\mathfrak{K}_i)\,(i \in J_m) \notin Z .$$

Thus Z and n fail to satisfy condition (K_1).

COROLLARY. *Let* p, Z *and* r *be as in theorem* 8.

(a) *For each* l *and* k *satisfying*

$$1 \leqslant l < k \leqslant h(p^r+l) < p^r ,$$

$S^*(p^r+l;k)$ *is independent of* $S(Z;1)$.

(b) *For all* $n \in I_4$, *let* $\mathrm{pp}(n)$ *be the greatest prime power less than and relatively prime to* n. *Then for* $k \in J_{h(n)}\setminus J_{n-\mathrm{pp}(n)}$, $S^*(n;k)$ *is independent of* $S(n;1)$.

(c) *For* $p \in \Pi_5$, $D_0(p)$ *is independent of* $S(Z;1)$.

(d) *For* $p \in \Pi_3$, $D_1(p)$ *is independent of* $S(Z;1)$.

(e) *For* $r \in I_2$, $S(p^r)$ *is independent of* $D_0(p^r)$.

Proof. (a) follows from theorems 3(b) and 8.

(b) reduces to (a), with $\mathrm{pp}(n)$ playing the role of p^r.

(c) follows from theorems 2(c) and 8.

(d) follows from theorems 2(d) and 8.

(e) follows from theorems 2(h) and 8. Let $\mathcal{M}$ be a model for σ & & $S(p^r-1;1)$ & $\sim S(p^r)$. Then $D_0(p^r)$ is true in $\mathcal{M}$.

Other independence results can be obtained by taking theorem 8 and its corollaries together with previous implicational results.

The above argument cannot be extended to show the independence of $D_0(p^r)$ from $S(Z;1)$ when $r\in I_2$. For, let D be the decomposition of J_{p^r} in which each cell contains a residue class, mod p. Then for each permutation Φ of the cyclic group on p^r letters, $\Phi(D)=D$.

DEFINITION 4. Z $\left(\in \mathfrak{I}^{\#}(I_1)\right)$ and n $(\in I_1)$ *satisfy condition* (M) *if for any decomposition of* n *into a sum of (not necessarily distinct) primes*

$$n=p_1+p_2+\ldots+p_s\,,$$

there are nonnegative integers $r_1,r_2,\ldots,r_s$ *such that*

$$r_1p_1+r_2p_2+\ldots+r_sp_s\in Z\,.$$

Mostowski, [6], theorem IV, shows that condition (M) is a consequence of condition (K) (see definition 4 and theorem 7, above), and thus

(2) (M) *is a necessary condition for the implication* $S(Z;1)\to S(n;1)$.

He also shows (M) to be sufficient for this implication in certain specific cases; in particular ([6], theorem IX),

(3) (M) *is a sufficient condition for the implication* $S(Z;1)\to S(n;1)$ *for arbitrary* Z *and for* $n\in\Pi\cup J_{19}\setminus\{15\}$. [3]

THEOREM 9. (a) *Let* $n\in I_2$, *let* $N\in\mathfrak{I}^*(J_{h(n)})$, *let* m *be the number of subsets* y *of an* n*-element set such that* $\boldsymbol{n}(y)\in N$, *and let* s $(\in I_2)$ *be such that* $\{m\}$ *and* s *fail to satisfy condition* (M). *Then* $S(s;1)$ *is independent of* $S(n;N)$.

(b) *Let* $n_0\in I_4$, *let* $n_1,n_2\in I_3$, *and for* $i\in 3$ *let* $\mathfrak{D}_i$ *be the set of all types of* $D_i(n_i)$*-decompositions. For any* $x_i\in\mathfrak{I}^*(\mathfrak{D}_i)$ *let* $m(x_i)$ *be the number of* $D_i(n_i)$*-decompositions whose type is in* x_i. *Let* $s(x_i)$ $(\in I_2)$ *be such that for some* $x_i\in\mathfrak{I}^*(\mathfrak{D}_i)$, $\{m(x_i)\}$ *and* $s(x_i)$ *fail to satisfy condition* (M). *Then* $S\big(s(x_i);1\big)$ *is independent of* $D_i(n_i)$; *moreover,* $S\big(s(x_2);1\big)$ *is independent of* $S(n_2)$.

[3] For other cases in which (M) is sufficient for this implication, see [9], Section 4 and [10].

Proof. (a) With each set X of n-element sets A we associate the set X', where $A'\in X'$ iff there is an $A\in X$ such that A' is the set of all subsets A'' of A such that $\boldsymbol{n}(A'')\in N$. Since $\{m\}$ and s fail to satisfy condition (M), we can take a model $\mathcal{M}$ for σ & $S(m;1)$ & $\sim S(s;1)$. Now for each $A\in X$, the selection of a unit subset of A' yields a subset A^* of A with $\boldsymbol{n}(A^*)\in N$. Thus $S(n;N)$ is true in $\mathcal{M}$.

(b) Utilizes the implications $S\big(m(x_i);1\big)\to D_i(n_i)$, $i\in 2$, and follows along the same lines as (a). We note that for each $n\in I_2$ and for each $i\in 3$, there are only a finite number of types of $D_i(n)$-decompositions. Moreover, theorem 1 enables us to apply information about $D_2(n)$ to the study of $S(n)$.

COROLLARY 1. *Assume the same notation as in theorem* 9. *Let* $p\in\Pi$, *let* $r\in I_1$, *and let* l *and* k *satisfy* $1\leqslant l<k\leqslant h(p^r+l)<p^r$. *Then*

(a) *if* p *does not divide* m, *each of the following axioms is independent of* $S(n;N)$:

(i) $S(p^r)$,

(ii) $S^*(p^r+l;k)$,

(iii) $D_0(p)$, *if* $p\in\Pi_5$,

(iv) $D_1(p)$, *if* $p\in\Pi_3$.

Also, if $m\in I_4$, *if* $\mathrm{pp}(m)$ *is as in corollary* (b) *of theorem* 8, *and if* $t\in J_{h(m)}\setminus J_{m-\mathrm{pp}(m)}$, *then* $S^*(m;t)$ *is independent of* $S(n;N)$.

(b) *For* $i\in 3$ *if* p *does not divide* $m(x_i)$ *for some* $x_i\in\mathfrak{I}^*(\mathfrak{D}_i)$, *then each of the axioms* (i), (ii), (iii), *and* (iv) *of* (a) *is independent of* $D_i(n_i)$. *Moreover, if in addition* $m(x_i)\in I_4$ *and if* t *is an integer satisfying* $m(x_i)-\mathrm{pp}\big(m(x_i)\big)<t\leqslant h\big(m(x_i)\big)$, *then* $S^*\big(m(x_i);t\big)$ *is independent of* $D_i(n_i)$. *In particular, if* p *does not divide* $m(x_2)$ *for some* $x_2\in\mathfrak{I}^*(\mathfrak{D}_2)$, *then each of the axioms* (i), (ii), (iii), *and* (iv) *of* (a) *is independent of* $S(n_2)$; *further, if* $m(x_2)\geqslant 4$ *and if* t *is as above (with* $i=2$), *then* $S^*\big(m(x_2);t\big)$ *is independent of* $S(n_2)$.

Proof. This follows from theorem 9 together with theorem 1 and theorem 8 and its corollary. In the proof of theorem 9 it is shown that $S(m;1)\to S(n;N)$, m,n,N as indicated.

COROLLARY 2. *For each prime* p, $S(p;1)$ *is independent of* H_{2p}.

Proof. $p\nmid\binom{2p}{p}$; the result follows from (a) of the theorem.

For the purpose of proving our next theorem, we now define the multiple choice axioms of M. N. Bleicher ([1] and [2]).

DEFINITION 4. For $m\in I_2$, $FS_m^{\#}$ will denote the following statement. "For any nonempty set X of nonempty finite sets there is a function f defined on X such that for each $A\in X$, $f(A)\in\mathfrak{I}^*(A)$ and $\big(\boldsymbol{n}(f(A)),m\big)=1$."

THEOREM 10. *For m ($\neq 4$) composite, $S(m; 1)$ is independent of $S(m)$.*

Proof. Theorem 5 of [11] asserts that

(4) *for composites $m > 6$, $S(m; 1)$ is independent of $FS^{\#}_m$.*

But for these values of m,

$$FS^{\#}_m \to (\forall A, X \in \mathfrak{U})\big((A \in X \to n(A) = m) \to FS_m(X)\big) \to S(m) .$$

(Here $\mathfrak{U}$ is the universal class.) Thus the implication $S(m) \to S(m; 1)$ would yield a contradiction of (4).

For $m = 6$: By condition (M), there is a model $\mathcal{M}$ for σ & $S(2; 1)$ &) & $\sim S(3; 1)$. Thus $S(6; 1)$ is false in $\mathcal{M}$, since $S(6; 1)$ iff $S(2; 1)$ & $S(3; 1$ ([7], p. 101, theorem 4). Let $p = 2$, $k = 1$, and $n = 3$ in theorem 2(g); then $S(6)$ is true in $\mathcal{M}$.

THEOREM 11. (a) *For all n which are not divisible by 6 and for all subsets $N \subseteq J_{h(n)}$, $S(n; N)$ is independent of σ. In particular, for such n, $S(n)$ is independent of σ.*

(b) *For $n \in I_2$ and for $k \in J_{h(n)}$, $S(n; k)$ is independent of σ.*

(c) *For $n \in I_2$ and for $N \in \mathfrak{I}^*(K_{h(n)})$, $S(n; N)$ is independent of σ.*

Proof. Let $n \geqslant 3$ be an odd integer and let $\mathfrak{G}$ be the cyclic group on n letters. Then for every group $\mathfrak{K} \subset \mathfrak{G}^{\omega}$ and for every proper subgroup $\mathfrak{K}$ of $\mathfrak{H}$, $\mathrm{Ind}\,\mathfrak{H}/\mathfrak{K} \geqslant 3$. Let $Z = \{2\}$ and apply theorem 7(a) and (b). Then for any (proper or improper) nonempty subset N of $J_{h(n)}$, $S(n; N)$ is independent of $S(2; 1)$, i.e., $\big(\sigma \,\&\, S(2; 1)\big) \not\to S(n; N)$. It follows that $S(n; N)$ is independent of σ.

Let n be a positive integer which is not divisible by 3; we repeat the above argument with $Z = \{3\}$. Again, $\mathrm{Ind}\,\mathfrak{H}/\mathfrak{K} \notin Z_3$, and we conclude that $S(n; N)$ is independent of σ.

The above two paragraphs together yield (a).

For each $m \in I_2$, $S(m; 1)$ is relatively consistent with σ, since, in fact, stronger forms of the axiom of choice have this property. Let $n \geqslant 2$ and let p be any prime which is not a factor of n. Then $\big(\sigma \,\&\, S(p; 1)\big) \not\to S(n; 1)$, by (2). Hence $S(n; 1)$ is independent of σ.

If N is any nonempty subset of $K_{h(n)}$, let $M_1 = \{m\colon m \in N \vee m+1 \in N\}$, and let $M = M_1$ if n is odd and $M = M_1 \setminus \{h(n)\}$ if n is even. Suppose that $S(n; N)$ is a theorem of σ. Then by theorem 3(a), so is $S(n-1; M)$. By (a), above, this would mean that 6 divides both n and $n-1$. This contradiction together with the results of the preceding paragraph establish both (b) and (c).

5. The interdependence of the axioms for $n \in K_6$.

THEOREM 12. *The following "matrix" indicates the interdependence among the various axioms for $2 \leqslant n \leqslant 6$. In listing these axioms we take*

the following into consideration:

(a) $S(2) \leftrightarrow S(2; 1) \leftrightarrow S(4; 1)$.

(b) $S(3) \leftrightarrow S(3; 1) \leftrightarrow D_1(3) \leftrightarrow D_0(4)$.

(c) $S(5) \leftrightarrow D_1(5) \leftrightarrow D_0(6)$.

(d) $H(5) \leftrightarrow D_0(5)$.

Proof. (a). $S(2; 1) \leftrightarrow S(4; 1)$ is demonstrated in [7], pp. 98-99.

(b). $S(3; 1) \to D_0(4)$, by theorem 2(h).

$D_0(4) \to S(3; 1)$: Let X be a nonempty set of 3-element sets, and let X' be the set of all $A' = A \cup \{a_A\}$, $A \in X$, where a_A is the first positive integer not in A. Let f be a $D_0(4)$-function on X'; then for each $A' \in X'$, $f(A')$ consists of two 2-cells. For each $A \in X$ define $g(A)$ to be the unique element of A which shares a cell with a_A.

(c). $S(5) \leftrightarrow D_1(5)$ follows from theorem 2(b) and (d). $D_0(6) \to S(5)$ by theorem 4; the converse implication was demonstrated by J. H. Conway (to be published).

(d). $H(5) \to D_0(5)$ by theorem 2(a); the converse implication follows from the observation that every $D_0(5)$-function effects only $(1, 2; 3, 1)$-type decompositions.

In the "matrix" an even number indicates that the left-hand axiom implies the upper axiom; (4) and odd number indicates that the upper axiom is independent of the left-hand one. The particular integer employed is the key to the explanation, below. (A blank space indicates an open question.)

0. $\Gamma \to \Gamma$.
1. by (2).
2. by (3).
3. by (a), (b), above, and theorem 2(b) and 8.
4. by (c), above, and theorem 2(a).
5. by (a), (b), above, and corollary (a) of theorem 8.
6. by (a), above, and theorem 2(b).
7. by (a), (b), (d), above, and corollary (c) of theorem 8.
8. by (a), (b), above, and theorem 2(g).
9. by (a), (b), above, and theorem 9(a). We let n, N, and m be as in theorem 9.

If $n = 4$ and $N = \{1, 2\}$, then $m = 10$; if $n = 4$ and $N = \{2\}$, then $m = 6$; if $n = 5$ and $N = \{1, 2\}$, then $m = 15$; if $n = 6$ and $N = \{1, 2, 3\}$, then $m = 41$; if $n = 6$ and $N = \{2\}$, then $m = 15$; if $n = 6$ and $N = \{2, 3\}$, then $m = 35$; if $n = 6$ and $N = \{3\}$, then $m = 20$.

(4) Several of the implications in our matrix were also obtained independently by J. H. Conway.

THE INTERDEPENDENCE OF THE AXIOMS FOR $n \in K_6$

	$S(2)$	$S(3)$	$S(4)$	$H(4)$	$D_1(4)$	$S(5)$	$S(5;1)$	$H(5)$	$S(6)$	$S(6;1)$	$S(6;2)$	$H(6)$	$S^*(6;2)$	$D_1(6)$
$S(2)$	0	1	6	5	6	3	1	7	8	1	17	5	5	22
$S(3)$	1	0	3	3	22	3	1	5	8	1	17	5	5	22
$S(4)$		9	0	17	6	17	17	17		9	17	17	17	24
$H(4)$		10	6	0	4	17	9	17	22		17	17	17	22
$D_1(4)$	11	11	17	17	0	17	11	17		11	17	17	17	24
$S(5)$	9	13	13	17		0		17		9	17	13		24
$S(5;1)$	1	1	3	5		6	0	5		1	17	5		22
$H(5)$		11	10	17	22	4		0		11	17			24
$S(6)$	9	9	13	17	20	17	9	17	0	15	17	17	17	6
$S(6;1)$	2	2	22	14	22	3	1	7	6	0	17	5	5	6
$S(6;2)$	9	16	13	17	22	10	16	17	6	9	0	17	6	6
$H(6)$	18	9	10	17	22	10	18	12	6	9	17	0	6	4
$S^*(6;2)$	9	9	13	17	22	10	18	17	6	9	17	17	0	4
$D_1(6)$	11	11	17	17		17	17	17		11	17	17	17	0

10. by (b), above, and theorem 3.

11. by (a), (b), (d), above, and theorem 9(b). For $i \in 2$ we let n_i, $\mathfrak{D}_i$, x_i, and $m(x_i)$ be as in theorem 9: If $n_1 = 4$ then $m(\mathfrak{D}_1) = 13$; if $n_0 = 5$, then $m(\mathfrak{D}_0) = 10$; if $n_1 = 6$, then $m(\mathfrak{D}_1) = 201$; if $n_1 = 6$ and $x_1 = (3,1;1,3)$, then $m(x_1) = 20$.

12. by theorem 5(a).

13. by corollary 1 of theorem 9. The data of 9. applies here, as well; also, if $n_2 = 5$, then $m(\mathfrak{D}_2) = 40$.

14. by the proof of theorem 9(a).

15. by theorem 10.

16. $S(6;2) \to S(3)$: Let X be a nonempty set of 3-element sets, and for each $A \in X$, let A' be the set of all nonempty, proper subsets of A. Let X' be the set of A' corresponding to $A \in X$. Each $A' \in X'$ has 6 elements; we choose two of these. If (exactly) one of these is a unit subset of A, we let $f(A)$ be this subset; if both of the choices are unit subsets of A, we let $f(A)$ be the third unit subset of A; if both of the choices are 2-element subsets of A, we let $f(A)$ be their intersection.

$S(6;2) \to S(5;1)$: Let Y be a nonempty set of 5-element sets and let Y' be the set obtained by adding one new element to each element of Y. Choose a 2-element subset B'' of each $B' \in Y'$ (corresponding to $B \in Y$). Then if $B \cap B''$ contains one element, let $f(B)$ be this intersection; if $B \cap B''$ contains 2 elements, use $S(3)$ to choose an element from $B \setminus B''$.

17. $((\Theta$ independent of $\Xi)$ & $(\Xi \to \Gamma)) \to \Theta$ independent of Γ; $((\Theta$ independent of $\Xi)$ & $(\Delta \to \Theta)) \to \Delta$ independent of Ξ.

18. $H(6) \to S(2)$, by theorem 2(e).

$H(6) \to S(5;1)$: Let Y, Y', B, B' be as in the second paragraph of 16. Choose a 3-element subset B''' of each $B' \in Y'$. Then either $B \cap B'''$ or else $B \setminus B'''$ contains 2 elements; use $H(6) \to S(2)$.

$S^*(6;2) \to S(5;1)$ now follows in this manner from the implications $S(6;2) \to S(5;1)$ and $H(6) \to S(5;1)$.

20. $S(6) \to D_1(4)$: Let X be a nonempty set of 4-element sets. For $A \in X$ let A^* be the set of 2-element subsets of A and let X^* be the set of A^* corresponding to $A \in X$. Let f be a multiple choice function on X^*; such a function exists for, by construction, X^* is a set of 6-element sets. For $A \in X$ we define $g(A)$ as follows: If $\boldsymbol{n}(f(A^*)) = 1$, let $g(A) = \{f(A^*), A \setminus f(A^*)\}$. If $\boldsymbol{n}(f(A^*)) = 2$ and if the two elements of $f(A^*)$ are disjoint, let $g(A) = f(A^*)$. In all other cases the elements of A do not all appear the same number of times as elements of elements of $f(A^*)$. Let $g(A) = \{A_1, A_2\}$, where A_1 is the set of elements of A which appear the maximum number of times in this role and where $A_2 = A \setminus A_1$.

22. by the transitivity of implication.

24. these are the results of J. H. Conway (to be published).

6. A final example. An unsolved problem, which was posed by Mostowski ([6], p. 168) is the status of the implication

$$S(\{3,5,13\};1) \to S(15;1)\,. \tag{5}$$

It is to be noted that $\{3,5,13\}$ and 15 satisfy (M) (of definition 4); thus the question is of particular significance in view of Mostowski's conjecture about the sufficiency of condition (M) for an implication $S(Z;1) \to S(n;1)$. In this regard we reduce (5) to the problem of determining the status of the implication in the hypothesis of

THEOREM 13. $(S(\{3,5,13\};1) \to D_1(8)) \to (S(\{3,5,13\}) \to S(15;1))$.

Proof. We suppose that both $S(\{3,5,13\};1)$ and the implication $S(\{3,5,13\}) \to D_1(8)$ are valid, and we wish to show that $S(15;1)$ must also be valid.

We first note that by (3), it follows that $S(\{3,13\};1) \to S(9;1)$.

Let X be a nonempty set of 15-element sets. By theorem 2(g), there exists a function f_1 on X such that for each $A \in X$, $f_1(A)$ is a subset of A such that $\boldsymbol{n}(f_1(A)) = z$, where $1 \leqslant z \leqslant 7$. Let f_k be a (1-ary) choice function on $X_{(k)}$ for $k = 3, 5, 9$, and 13.

Let $A \in X$. If $\boldsymbol{n}(f_1(A)) = 1$, let $g(A) = f_1(A)$; if $\boldsymbol{n}(f_1(A)) = 2$, let $g(A) = f_{13}(A \setminus f_1(A))$; if $\boldsymbol{n}(f_1(A)) = 3$, let $g(A) = f_3 f_1(A)$; if $\boldsymbol{n}(f_1(A)) = 5$, let $g(A) = f_5 f_1(A)$; if $\boldsymbol{n}(f_1(A)) = 6$, let $g(A) = f_9(A \setminus f_1(A))$.

Suppose $n(f_1(A)) = 4$. By theorem 2(h), there is a function F_4 defined on $X_{(4)}$ such that for $B \in X_{(4)}$, $F_4(B) = \{B_1, B_2\}$, where B_1 and B_2 are disjoint 2-element subsets of B. Now let Y be the set of all sets of the form $(A\backslash f_1(A)) \cup B_i(A)$, $i = 1, 2$, where $A \in X$, $n(f_1(A)) = 4$, and $F_4 f_1(A) = \{B_1(A), B_2(A)\}$. Then $Y \subset X_{(9)}$. To each $A \in X$ with $n(f_1(A)) = 4$, there corresponds one or two distinct elements of A of the form $f_9((A\backslash f_1(A)) \cup B_i(A))$. In case there is just one such element a, let $g(A) = \{a\}$; in case a_1 and $a_2 (\neq a_1)$ are both of this form, let $g(A) = f_{13}(A\backslash\{a_1, a_2\})$.

Suppose $n(f_1(A)) = 7$. By our hypothesis, there is a $D_1(8)$-function F_8 defined on $X_{(8)}$. Consider $F_8(A\backslash f_1(A))$. If any element of $F_8(A\backslash f_1(A))$ is a unit set, the union of all such unit sets is a subset C of $A\backslash f_1(A)$ with $n(C) \leqslant 6$, by (v′) of definition 2; in this case the argument reduces to one of the previous considerations. Otherwise, if any element x of $F_8(A\backslash f_1(A))$ is such that $n(x) = 2$, we take the union of each such set with $f_1(A)$, thereby obtaining a subset of $X_{(9)}$. If we procede as in the preceding paragraph, we obtain a subset C' of A for which $n(C')$ is at most 4; again we have reduced the case to one of prior considerations. Finally, if $F_8(A\backslash f_1(A)) = \{D_1, D_2\}$, where D_1 and D_2 are disjoint sets with $n(D_1) = n(D_2) = 4$, we can obtain another decomposition of $A\backslash f_1(A)$ into $\{E_1, E_2, E_3, E_4\}$ in which the E_i are pairwise-disjoint sets such that each $n(E_i) = 2$, and each E_i is an element of $F_4(D_j)$, $j = 1, 2$.

Remark. We have chosen to work with $D_1(8)$ because theorem 8 and its corollary yield the independence from $S(\{3, 5, 13\}; 1)$ of each of the following axioms: $S(7)$, $S(8)$, $D_0(7)$, and $D_1(7)$.

References

[1] M. N. Bleicher, *Some theorems on vector spaces and the axiom of choice*, Fund. Math. 54 (1964), pp. 95-107.

[2] — *Multiple choice axioms and the axiom of choice for finite sets*, ibid. 57 (1965), pp. 247-252.

[3] K. Gödel, *The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory*, 6th ed., Annals of Math. studies 3, Princeton Univ. Press, Princeton, 1964.

[4] A. Levy, *Axioms of multiple choice*, Fund. Math. 50 (1962), pp. 475-483.

[5] A. Mostowski, *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, ibid. 32 (1939), pp. 201-252.

[6] — *Axiom of choice for finite sets*, ibid. 33 (1945), pp. 137-168.

[7] W. Sierpiński, *Cardinal and ordinal numbers*, 1st ed., Monografie Matematyczne, 34, Warszawa, 1958.

[8] M. M. Zuckerman, *Finite versions of the axiom of choice*, Ph. D. Thesis, Dept. of Math., Yeshiva University, New York (1967).

[9] — *Number theoretic results relating to the axioms of choice for finite sets* (to appear in Illinois J. Math.).

[10] — *Some theorems on the axioms of choice for finite sets* (to appear in Z. Math. Logik Grundlagen Math.).

[11] — *Multiple choice axioms* (to appear in (Axiomatic Set Theory) Proc. Sympos. Pure Math. 13 (1969)).

Reçu par la Rédaction le 6. 9. 1967