Λ-spaces and fixed point theorems

by

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1. A Lefschetz space is defined in [2] to be a space X such that, for every continuous map $f\colon X\to X$, the Lefschetz number $\Lambda(f) = \sum_{n=0}^{\infty}(-1)^n \operatorname{tr}(f_{*n})$ is well-defined and when $\Lambda(f)\neq 0$, f has a fixed point (f_{*n} is the induced homomorphism of singular homology groups with rational coefficients). In particular, this implies that the homology of X is of finite type.

Since the appearance of Leray–Schauder fixed point results there has been interest in considering more general spaces and compact self-mappings of these spaces. The purpose of this note is to generalize fixed point theorems to "Λ-spaces." A Λ-space is a space X such that, for every compact map $f\colon X\to X$, the Lefschetz number is defined and the Lefschetz theorem holds. We show that spaces which are, in a certain way, dominated by Λ-spaces are again Λ-spaces; and that every polyhedron with the Whitehead topology is a Λ-space. As a corollary of these results we find that (metric) absolute neighborhood retracts are Λ-spaces. This fact has been proved also by A. Granas [3]. A weaker result along these same lines was obtained by F. E. Browder in [2], Theorem 3.

2. For the purpose of this note, the nature of a homology theory under consideration is important only to the extent that the homology groups be vector spaces; that they agree with the usual homology groups with rational coefficients for compact polyhedra; and that they constitute a functor H_* satisfying the homotopy axiom and the dimension axiom for the category of topological spaces under consideration. Thus H_* may be the singular homology, the Cech homology or any other functor satisfying the above requirements. The homomorphism induced by a map $f\colon X\to Y$ will be denoted, as usual, by $f_{*n}\colon H_n(X)\to H_n(Y)$.

Let V be a finite dimensional vector space over a field C (in our case, $C=Q$, the field of rational numbers), let V^* be the dual of V and let

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$\operatorname{End}(V) = \operatorname{Hom}(V, V)$ be the vector space of endomorphisms $V \to V$. Then we have a canonical linear form $\tau\colon V^* \otimes V \to C$ and a canonical isomorphism $\theta\colon V^* \times V \overset{\cong}{\to} \operatorname{End}(V)$.

The trace, tr_V, is the canonical linear form

$$\operatorname{tr}_V = \tau \circ \theta^{-1}\colon \operatorname{End}(V) \to C.$$

When no confusion can arise, we write $\operatorname{tr}_V = \operatorname{tr}$.

PROPOSITION. (see [1], page 112). *If V and W are finite dimensional vector spaces and $g\colon V \to W$, $h\colon W \to V$ are linear maps then* $\operatorname{tr}_V(h \circ g) = \operatorname{tr}_W(g \circ h)$.

Let W be a subspace of a vector space V and let

$$\operatorname{End}(V; W) = \{f \in \operatorname{End}(V)\colon f(V) \subset W\}.$$

Let $i\colon W \sim V$ be the inclusion map and let $i_*\colon \operatorname{Hom}(V, W) \to \operatorname{End}(V; W)$ and $i^*\colon \operatorname{Hom}(V, W) \to \operatorname{End}(W)$ be the maps $f \to i \circ f$ and $f \to f \circ i$, respectively. We have the inclusion $\operatorname{End}(V; W) \subset \operatorname{End}(V)$ and a natural map $\alpha\colon \operatorname{End}(V; W) \to \operatorname{Hom}(V, W)$ such that $i_* \circ \alpha = 1_{\operatorname{End}(V;W)}$. Let $\beta = i^* \circ \alpha\colon \operatorname{End}(V; W) \to \operatorname{End}(W)$.

(2.1) LEMMA. *If V is finite dimensional then*

$$\operatorname{tr}_W \circ \beta = \operatorname{tr}_V | \operatorname{End}(V; W)\colon \operatorname{End}(V; W) \to C.$$

Proof. Let $f \in \operatorname{End}(V; W)$. If $g = \alpha(f)$ then $i \circ g = f$ and $g \circ i = \beta(f)$. By the Proposition quoted above, $\operatorname{tr}_W(g \circ i) = \operatorname{tr}_V(f)$.

(2.2) LEMMA. *Let X be a vector space and $f \in \operatorname{End}(X)$ be such that $f^n(X)$ is finite dimensional for some integer n. Let V be a finite dimensional subspace of X such that $f^n(X) \subset V$ and $f(V) \subset f^n(X)$. Then if $f'\colon V \to V$ and $f''\colon f^n(X) \to f^n(X)$ are defined by the restriction of f, we have* $\operatorname{tr}(f') = \operatorname{tr}(f'')$.

Proof. Apply (2.1) to V, $W = f^n(X)$, and $f' \in \operatorname{End}(V; W)$.

(2.3) DEFINITION. Let X be a vector space and $f \in \operatorname{End}(X)$. Suppose that there exists an integer n such that $V = f^n(X)$ is finite dimensional. Let $f'\colon f^n(X) \to f^n(X)$ be defined by the restriction of f. Define the trace of f by $\operatorname{tr}(f) = \operatorname{tr}_V(f')$. By (2.2), $\operatorname{tr}(f)$ is well-defined.

By saying "$\operatorname{tr}(f)$ is defined" we shall mean that the assumptions of (2.3) are fulfilled.

(2.4) LEMMA. *Let X, Y be vector spaces and let $g\colon X \to Y$, $h\colon Y \to X$ be linear maps such that $\operatorname{tr}(h \circ g)$ is defined. Then $\operatorname{tr}(g \circ h)$ is defined and* $\operatorname{tr}(h \circ g) = \operatorname{tr}(g \circ h)$.

Proof. Since $\operatorname{tr}(h \circ g)$ is defined, there exists an integer n such that $(h \circ g)^n(X)$ is finite dimensional. Then $(g \circ h)^{n+1}(Y) \subset g\big((h \circ g)^n(X)\big)$ and $(g \circ h)^{n+1}(Y)$ is finite dimensional. Thus $\operatorname{tr}(g \circ h)$ is defined. Let $V = (h \circ g)^{n+1}(X)$ and $W = g\big((h \circ g)^n(X)\big)$; then V and W are finite dimensional subspaces of X and Y, respectively, and $g(V) \subset W$, $h(W) \subset V$. Let $g'\colon V \to W$ and $h'\colon W \to V$ be the maps defined by the restrictions of g and h, respectively.

Let $e = h \circ g \in \operatorname{End}(X)$, $f = g \circ h \in \operatorname{End}(Y)$. Observe that $e(V) \subset V$, $f(W) \subset W$ and that the maps $e'\colon V \to V$, $f'\colon W \to W$ defined by the restrictions of e and f are $e' = h' \circ g'$ and $f' = g' \circ h'$. By definition, $\operatorname{tr}(h \circ g) = \operatorname{tr}(e')$. Moreover, $(g \circ h)^{n+1}(Y) \subset W$ and again by (2.2), $\operatorname{tr}(g \circ h) = \operatorname{tr}(f')$.

Thus $\operatorname{tr}(h \circ g) = \operatorname{tr}(e') = \operatorname{tr}(h' \circ g')$ and $\operatorname{tr}(g \circ h) = \operatorname{tr}(f') = \operatorname{tr}(g' \circ h')$. Since V and W are finite dimensional, in view of the Proposition we have $\operatorname{tr}(h' \circ g') = \operatorname{tr}(g' \circ h')$.

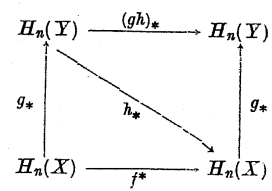
(2.5) DEFINITION. Let X be a topological space and $f\colon X \to X$ a continuous map. Then f is said to be a *Lefschetz map* if $\Lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_{*n})$ is well-defined and $\Lambda(f) \neq 0$ implies that f has a fixed point (in particular this implies that $\operatorname{tr}(f_{*n})$ is defined).

(2.6) DEFINITION. A topological space X is said to be a *Λ-space* if each compact map $f\colon X \to X$ is a Lefschetz map. (A map f is compact if there is a compact subset of X which contains $f(X)$.)

3. For simplicity, when writing the induced homomorphisms, the dimension subscript will be omitted.

(3.1) THEOREM. *Let X be any space and let $f\colon X \to X$ be a compact map. Suppose that there exists a Λ-space Y and maps $g\colon X \to Y$ and $h\colon Y \to X$ such that h is compact and $h \circ g = f$. Then f is a Lefschetz map.*

Proof. The mapping $g \circ h\colon Y \to Y$ is compact and hence is a Lefschetz map. Considering the induced homomorphisms on the homology groups, we have $f_* = h_* \circ g_*$ at each dimension.



We have $g_*\colon H_n(X) \to H_n(Y)$ and $h_*\colon H_n(Y) \to H_n(X)$ linear maps of vector spaces such that $\operatorname{tr}(g_* h_*)$ is defined (since $\Lambda(gh)$ exists). By Lemma 2.4, $\operatorname{tr}(h_* g_*)$ is defined and $\operatorname{tr}(h_* g_*) = \operatorname{tr}(g_* h_*)$. Then since $f_* = h_* g_*$, $\operatorname{tr}(f_*) = \operatorname{tr}(g_* h_*)$. Hence $\Lambda(f)$ is defined and $\Lambda(f) = \Lambda(gh)$.

Suppose $\Lambda(f) \neq 0$. Then there is a point $y \in Y$ such that $gh(y) = y$. Let $x = h(y) \in X$. Then $f(x) = x$. Q.E.D.

Let $f\colon X\to X$ be a compact map. For any compact set K containing $f(X)$, let $f_K\colon K\to K$ denote the restriction defined by f. Clearly, f has a fixed point iff f_K has a fixed point. The following theorem is along this line and is a useful tool in the study of compact maps. The proof is omitted since it follows very closely the proof of theorem (3.1).

(3.2) THEOREM. *Let $f\colon X\to X$ be a compact mapping.*

(a) *If $\Lambda(f)$ exists, then $\Lambda(f_K)$ exists for all compact K containing $f(X)$ and $\Lambda(f)=\Lambda(f_K)$.*

(b) *If $\Lambda(f_K)$ exists for some compact K containing $f(X)$, then $\Lambda(f)$ exists and $\Lambda(f)=\Lambda(f_K)$.*

In particular, we have

(3.3) COROLLARY. *Let $f\colon X\to X$ be a compact mapping.*

(a) *If f is a Lefschetz map, then f_K is a Lefschetz map for all compact K containing $f(X)$.*

(b) *If f_K is a Lefschetz map for some compact K containing $f(X)$, then f is a Lefschetz map.*

(3.4) COROLLARY. *If X is a topological space and $f\colon X\to X$ can be factored through a Lefschetz space, then f is a Lefschetz map.*

(3.5) THEOREM. *Let X be a topological space such that there is a Λ-space Y and mappings $g\colon X\to Y$ and $h\colon Y\to X$ with $h\circ g=1_X$ ($1_X=$ identity map on X). Then X is a Λ-space.*

Proof. Let $f\colon X\to X$ be a compact mapping. Then $f\circ h\colon Y\to Y$ is a compact mapping and $f=(f\circ h)\circ g$. Then by Theorem (3.1), f is a Lefschetz map. Q.E.D.

(3.6) COROLLARY. *A retract of a Λ-space is again a Λ-space.*

4. An essential fact in the proof of Theorem (3.1) was that $f_*=h_*\circ g_*$. The condition that $f=h\circ g$ is certainly not necessary. By placing a suitable restriction on the space X, Theorem (3.1) can be generalized as follows.

(4.1) THEOREM. *Let X be a regular, T_1 space (i.e., T_3 space) and $f\colon X\to X$ a compact mapping. Suppose that for each open cover α of X, there is a Λ-space Y_α and mappings $g_\alpha\colon X\to Y_\alpha$ and $h_\alpha\colon Y_\alpha\to X$ satisfying*

(a) *h_α is compact,*

(b) *$h_\alpha\circ g_\alpha\simeq f$, and*

(c) *$h_\alpha\circ g_\alpha$ and f are α-near (i.e., for each $x\in X$, there is an element U of α containing both $h_\alpha g_\alpha(x)$ and $f(x)$).*

Then f is a Lefschetz map.

Proof. Given an open cover α of X, let Y_α, g_α, and h_α satisfy the conditions of the theorem. Then $g_\alpha\circ h_\alpha\colon Y_\alpha\to Y_\alpha$ is a compact mapping and hence is a Lefschetz map. Since $h_\alpha\circ g_\alpha\sim f$, for the induced homomorphisms on the homology groups we have $f_*=h_{\alpha*}\circ g_{\alpha*}$ at each dimension.

Following exactly the proof of Theorem (3.1), we find that $\Lambda(f)$ is defined and that $\Lambda(f)=\Lambda(g_\alpha\circ h_\alpha)$. This is true for every open cover α of X.

Suppose $\Lambda(f)\neq 0$. Then for each open cover α of X, there is a point $y_\alpha\in Y_\alpha$ such that $g_\alpha\circ h_\alpha(y_\alpha)=y_\alpha$. Let $x_\alpha=h_\alpha(y_\alpha)\in X$. Choose some compact set K containing $f(X)$. Then $f(x_\alpha)\in K$ for each α. Now $C=\mathrm{Cov}(X)$, the set of all open covers of X, is directed by the refinement relation: if $\alpha,\alpha'\in C$, $\alpha<\alpha'$ means that α' is a refinement of α; and $\alpha\mapsto f(x_\alpha)$ defines a net $\varphi\colon C\to K$ in K. Since K is compact, there is a directed set D and a cofinal map $\lambda\colon D\to C$ such that the subnet $\varphi\circ\lambda\colon D\to K$ converges to a point $x_0 \parallel K$. Consider the net $\psi\colon D\to X$ defined by $\beta\mapsto x_{\lambda(\beta)}$. It suffices to show that ψ converges to x_0. For then by the continuity of f we have $f(x_0)=x_0$.

First note that since $x_\alpha=h_\alpha\circ g_\alpha(x_\alpha)$, x_α and $f(x_\alpha)$ are both contained in an element of α; thus we can choose a map $U\colon C\to\bigcup_{\alpha\in C}\alpha$, $\alpha\mapsto U_\alpha$, such that $U_\alpha\in\alpha$ and U_α contains both x_α and $f(x_\alpha)$.

Let V be any open neighborhood of x_0. Then there is an open neighborhood W of x_0 such that $\overline{W}\subset V$. Let $\alpha_0=\{V, X-\overline{W}\}\in C$. Since λ is cofinal, there exists an element $\beta_0\in D$ such that for $\beta>\beta_0$ we have $\lambda(\beta)>\alpha_0$; and since $\varphi\circ\lambda$ converges to x_0, there exists an element $\beta_1\in D$ such that for $\beta>\beta_1$ we have $f(x_{\lambda(\beta)})\in W$ and also $f(x_{\lambda(\beta)}), x_{\lambda(\beta)}\in U_{\lambda(\beta)}$. Thus for $\beta>\beta_0,\beta_1$ we have $U_{\lambda(\beta)}\subset V$ and $x_{\lambda(\beta)}\in V$. This means that ψ converges to x_0.

(4.2) THEOREM. *Let X be a regular, T_1 space. Suppose that for each open cover α of X there is a Λ-space Y_α and mappings $g_\alpha\colon X\to Y_\alpha$ and $h_\alpha\colon Y_\alpha\to X$ satisfying*

(a) *$h_\alpha\circ g_\alpha\simeq 1_X$ and*

(b) *$h_\alpha\circ g_\alpha$ and 1_X are α-near.*

Then X is a Λ-space.

Proof. Let $f\colon X\to X$ be a compact mapping. Take an open cover α of X. Then $\beta=f^{-1}(\alpha)\in\mathrm{Cov}(X)$ and we have the corresponding Λ-space Y_β and mappings g_β, h_β. Then $f\circ h_\beta\colon Y_\beta\to X$ is compact, $f\circ h_\beta\circ g_\beta\simeq f$, and $f\circ h_\beta\circ g_\beta$ and f are α-near. Thus by Theorem (4.1), f is a Lefschetz map. Q.E.D.

5. The two topologies usually considered on a polyhedron are the metric topology and the Whitehead topology ([4], p. 99). Unless the polyhedron is locally finite these topologies do not coincide.

(5.1) THEOREM. *Every polyhedron P with the Whitehead topology is a Λ-space.*

Proof. Let $f\colon P\to P$ be a compact mapping. Let C be a compact subset of P containing $f(P)$. Then there is a finite subpolyhedron P' of P containing C. As before let $f_{P'}\colon P'\to P'$ denote the restriction of f. It is well known that $f_{P'}$ is a Lefschetz map. Thus by Corollary (3.3) (b), f is also a Lefschetz map.

(5.2) COROLLARY. *Every (metric) absolute neighborhood retract X is a Λ-space.*

Proof. For each open cover α of X there is a polyhedron P_α (with the Whitehead topology) and mappings $g_\alpha\colon X\to P_\alpha$ and $h_\alpha\colon P_\alpha\to X$ such that $h_\alpha\circ g_\alpha$ is α-homotopic to 1_X (see [4], p. 138). In particular, $h_\alpha\circ g_\alpha\simeq 1_X$ and $h_\alpha\circ g_\alpha$ and 1_X are α-near. Then by Theorem (4.2), X is a Λ-space.

6. Note that the theorems of § 3 and § 4 also hold for Lefschetz spaces in the sense that "Λ-space" can be replaced by "Lefschetz space" throughout. When this is done the compactness conditions on the mappings can be dropped.

References

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On choosing subsets of n-element sets

by

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1. Introduction. Let n be a positive integer. Mostowski ([6]) and others have studied the axioms of choice for finite sets, $[n]$, in which an element is chosen from each set of an arbitrary set of n-element sets. We wish to introduce some new axioms which are concerned with the choice of a subset or of a partition, rather than a single element, from each element of an arbitrary set of n-element sets. We shall discuss the interdependence of these axioms and their relationship to the axioms $[n]$.

2. Notation. We shall operate within a set theory of the Gödel–Bernays type (see the proof of theorem 7); our logical framework will be the first-order predicate calculus with identity. For statements $a_1, a_2, \ldots, a_n$, we write $a_1\to a_2\to\ldots\to a_n$ in lieu of $(a_1\to a_2)\,\&\,(a_2\to a_3)\,\&\ldots$ $\ldots\,\&\,(a_{n-1}\to a_n)$; a similar remark applies to $a_1\leftrightarrow a_2\leftrightarrow\ldots\leftrightarrow a_n$.

By the (nonnegative) *integers* we mean the von Neumann integers—0 (the empty set), $1=\{0\}$, $2=1\cup\{1\}$, $3=2\cup\{2\}$, etc. A set is *finite* iff every nonempty set of subsets of X has a maximal element with respect to set inclusion. If there exists a function which maps the set X one-one onto the positive n, then X is called an *n-element set* and we say that the *number of elements of X is n*; in this case we let $\boldsymbol{n}(X)$ denote the unique integer n for which such a mapping exists.

For each integer n, let I_n be the set of integers $\geqslant n$, let J_n be the relative complement of I_{n+1} in I_1, $I_1\backslash I_{n+1}$, and let $K_n=J_n\backslash\{1\}$. Let Π represent the set of prime numbers and let $\Pi_n=\Pi\cap I_n$.

For any set X let $\mathfrak{P}(X)$ designate the power set of X, let $\mathfrak{P}^*(X)$ $=\mathfrak{P}(X)\backslash 1$, let $\mathfrak{P}^{\#}(X)$ be the set of finite subsets of X, and let $\mathfrak{P}^{\#*}(X)$ $=\mathfrak{P}^{\#}(X)\backslash 1$.

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