Universality of the product mappings onto products of $I^n$ and snake-like spaces

by

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In paper [5] we proved, in quite an elementary manner, that the product of the continuous mappings of connected compact spaces onto snake-like spaces (*) is a universal mapping. In this paper we shall show that the product of such a product mapping by a universal mapping of a compact space onto $I^n$ is, by hypothesis, also universal (see Theorem (3.4)). Moreover, the product of a family of universal mappings, defined on the compact spaces, such that one of those mappings is onto $I^n$ and the remaining mappings are onto the snake-like spaces, is, by hypothesis, a universal mapping.

As a corollary we shall obtain also the well-known theorem (see [6]) about the dimension of the product of a paracompact space by a compact space of positive dimension (see Theorem (3.3)) and a related set-combinatorial result (see Theorem (5.1)).

The following assertions on the universal mappings will be used:

**Lemma A** (see [31]). Let $X$ be a normal space. Next, let

$$I^n_k = \{(x_1, \ldots, x_n) \in I^n : x_{k|k} = \text{sgn} k\}$$

for $k = \pm 1, \pm 2, \ldots, \pm n$, where $I = [-1, 1]$ is a closed segment. A continuous mapping $f: X \to I^n$ is universal if and only if the intersection of any sequence of closed sets $P_i$, $i = 1, 2, \ldots, n$, which are partitions between $f^{-1}(I^n_k)$ and $f^{-1}(I^n)_{\bar{k}}$ is a non-empty set.

**Theorem B** (see [3]). Let $X$ be a normal space. Then $\dim X \geq n$ if and only if there exists a universal mapping $f: X \to I^n$.

(*) A compact space $X$ is said to be a snake-like space if for any open covering $P$ of $X$ there exists a finite covering $P' = \{G_1, \ldots, G_k\}$ of $X$ which is refinement of $P$ and is such that $G_i \cap G_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Thus for any open cover $P$ of a snake-like space $X$ there exists a $P$-mapping of $X$ onto $I$. In this paper a compact space means a compact Hausdorff space.
Theorem C (see [5], Theorem 3.1). Let any finite product of the mappings $f_i : X_i \to Y_i$, $i \in I$, be a universal mapping. Then the product mapping

$$f = \prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

is universal if $X_i$ is compact for any $i \in I$.

§ 1. Let $f : I \to I^n$ be a continuous mapping of a space $X$ into the cube $I^n$, where $I = [-1, 1]$ is a closed interval, and let $S^{n-1}$ be the boundary of $I^n$ (i). Then the following easy assertions hold:

(1.1) Proposition. The mapping $f$ is not a universal mapping if and only if there exists a continuous mapping $g : X \to S^{n-1}$ such that $g(x) = f(x)$ for any $x \in f^{-1}(S^{n-1})$.

The above proposition immediately implies the following one:

(1.2) Proposition. Let $A = \overline{A} \subset f^{-1}(S^{n-1})$. We put

$$f_\delta = f|A : A \to S^{n-1}.$$

If for a cohomology theory with arbitrary coefficients (see [2]) we have

$$f_\delta(\sigma^{n-1}) \neq \operatorname{Im} \{i_1^* : H^n(X) \to H^n(A)\},$$

where $i_1 : A \to X$ is the identity imbedding and $\sigma^{n-1} \neq 0$ is an element of $H^n(S^{n-1})$, then $f$ is a universal mapping.

(1.4) Proposition. Let us consider the Čech–Dowker cohomology theory with integer coefficients (see [2] chapter IX). Let $n = 1$ or 2 and let $X$ be a paracompact space. Then the mapping $f$ is universal if and only if condition (1.3) holds for $A = f^{-1}(S^{n-1})$ and a generator $\sigma^{n-1}$ of $H^n(S^{n-1})$.

(1.5) Proposition. Let us consider the Čech–Dowker cohomology theory with integer coefficients and let $X$ be a paracompact space of covering dimension $< n$. Then the mapping $f$ is universal if and only if condition (1.5) holds for $A = f^{-1}(S^{n-1})$ and a generator $\sigma^{n-1}$ of $H^n(S^{n-1})$.

Propositions 1.4 and 1.5 are direct consequences of (1.1) and the respective Dowker theorems (see [1]).

As a corollary to Theorem B and Proposition (1.5) we obtain the following Alexander Theorem

(1.6) Theorem. Let $X$ be a finite-dimensional paracompact space. Then $n = \dim X$ is the greatest integer such that $g_2 : H^n(X) \to H^n(A)$ is not an epimorphism for a certain closed subset $A$ of $X$.

Indeed, if $g_2$ is not an epimorphism, then $H^n(X, A) \neq 0$ and consequently $\dim X \geq n$. On the other hand, if $\dim X = n$, then by Theorem B there exists a universal mapping $f : X \to I^n$. Thus, by (1.5), Theorem (1.6) holds.

§ 2. Let us consider a cohomology theory with arbitrary coefficients, defined on the category of all pairs $(X, A)$ of topological spaces (where $A$ is a closed subspace of $X$) and all continuous mappings.

(2.1) Lemma. Let $f : X \to I^n$ be a continuous mapping of a space $X$ such that condition (1.3) holds for a closed subset $A$ of $X$. We put

$$g = f \times i_1 : X \times I \to I^{n+1},$$

where $i_1 : I \to I$ is the identity, and let $g_\delta = g|B : B \to S^n$. Then

$$g_\delta(\sigma^{n-1}) \neq \operatorname{Im} \{i_1^* : H^n(X \times I) \to H^n(B)\},$$

where $i_1 : B \to X \times I$ is the identity embedding and $\sigma\delta$ is the image of $\sigma^{n-1}$ (i)

Hence, by Proposition (1.2), the product mapping $f \times i_1$ is universal.

Proof. We put

$$X^- = \{(x, y) : x \in X, y < 0\} \quad \text{and} \quad X^+ = \{(x, y) : x \in X, y > 0\}.$$

The triads $(B, X^-, X^+)$ is a proper triads such that $B = X^- \cup X^+$. We can identify $\overline{A}$ and $X^- \cap X^+$. Let us consider the cohomological additional exact sequence of the triads $(B, X^-, X^+)$ (see [2] I.15.2c).

$$\cdots \to H^{n-1}(X^-) \to H^n(B) \xrightarrow{i_1^*} H^n(X) \to H^n(X^-) \to \cdots$$

We have, by (1.3), $f_\delta(\sigma^{n-1}) \neq \operatorname{Im} H^n(B)$. Hence

$$\sigma \delta = A \neq f_\delta(\sigma^{n-1}) \neq 0.$$

Now, let us consider the imbedding of the triads $(B, X^-, X^+)$ into the triads $(X \times I, X \times [-1, 0), X \times (0, 1])$ and the induced homomorphism of the cohomological additional exact sequence of the second triads into the sequence of the first one. Then the composition

$$H^n(X \times I) \xrightarrow{f_\delta} H^n(X \times [-1, 0]) \to H^n(X) \to \cdots$$

(1) In this paper $\sigma$ denotes a positive integer. Only in Theorem (4.4) $\sigma$ denotes a non-negative integer.
is a monomorphism onto the „diagonal” of the last group. On the other hand, \( \delta' \in \ker(\varepsilon: H^n(B) \to H^n(X) \times H^n(X)) = \text{Im}(\eta: H^{n-1}(A) \to H^n(B)) \).

Thus \( \delta' \notin \text{Im}(\varepsilon: H^n(X \times I) \to H^n(B)) \), where \( \varepsilon: B \to X \times I \) is the identity embedding. But, by the commutation of the diagram

\[
\begin{array}{ccc}
H^{n-1}(A) & \xrightarrow{\delta'} & H^n(B) \\
\varepsilon | & & | \varepsilon \\
H^{n-1}(S_{n-1}) & \xrightarrow{-\delta'} & H^n(S^n) \\
\end{array}
\]

we have

\( r' = \delta \circ f(e^{n-1}) = g \circ \delta(e^{n-1}) = g \delta(e^n) . \)

The lemma is proved.

The following assertion is easier than Lemma (2.1).

(2.1') Lemma. Under the assumptions of (2.1) the suspension mapping \( Sf: SX \to S\Pi^k \) is universal.

(2.3) Lemma. Let \( k: I \to I \) be a continuous mapping onto such that \( k(-1) = -1 \) and \( k(1) = 1 \). Then under the assumptions of Lemma (2.1) condition (2.2) holds for \( g = f \circ k \) and \( g_0 = g_B : B \to S^n \).

Proof. The mapping \( f \times k_B: B \to S^n \) is homotopically equivalent to the mapping \( f \times k_I: B \to S^n \), whence

\( (f \times k)(B) = (f \times k_I)(B) : H^n(S^n) \to H^n(B) \)

and, by Lemma (2.1), condition (2.2) holds.

(2.4) Corollary. Let \( h: Y \to I \) be a continuous mapping of a normal space \( Y \) onto \( I \) such that \( h(Y_0) = I \) for a simple arc \( Y_0 \) in \( Y \) with the endpoints \( a_+ \) and \( a_- \). Then, under the assumptions of Lemma (2.1) about the mapping \( f \times k_I \), condition (2.2) holds for \( B = X \times h^{-1}(S^0) \cup A \times Y_0 \), \( g = f \times k \) and \( g_0 = g_B : B \to S^n \), where \( S^0 = \{-1,1\} \).

Proof. Without loss of generality we can assume that \( h^{-1}(-1) = \{a_-\} \) and \( h^{-1}(1) \cap Y_0 = \{a_+\} \).

Thus there exists a retraction \( r: Y \to Y_0 \) such that

\( r(h^{-1}(-1)) = \{a_-\} \) and \( r(h^{-1}(1)) = \{a_+\} \).

We put \( B_0 = X \times \{a_-\} \cup X \times \{a_+\} \cup A \times Y_0 \). The mappings

\[ r_1 = \text{id} \times r: X \times Y \to X \times Y_0 \quad \text{and} \quad r_0 = (\text{id} \times r)B: B \to B_0 \]

are also retractions, where \( \text{id}: X \to X \) is the identity. Thus in the commutative diagram

\[
\begin{array}{ccc}
H^n(X \times Y) & \xrightarrow{\delta} & H^n(B) \\
\varepsilon | & & | \varepsilon \\
H^n(X \times Y_0) & \xrightarrow{-\delta} & H^n(S^n) \\
\end{array}
\]

\( r^* \) and \( r_0^* \) are monomorphisms and, by Lemma 2.3, \( r^* = g_0^*(e^n) \circ \text{Im} i_B \).

Hence \( g_0^*(e^n) \notin \text{Im} i_B \) and condition (2.2) holds.

§ 3. Let us consider a continuous cohomology theory, defined on the category of all pairs of spaces and all continuous mappings, with the coefficients in category \( \mathcal{B}_R \) (for example, let it be the Čech-Dowker theory; see [2], Chapter IX). We shall show that the following theorem holds.

(3.1) Theorem. Let \( h: Y \to I \) be a continuous mapping of a connected compact space \( Y \) onto \( I \) and let \( f: X \to \Pi^k \) be as in Lemma (2.1). Then the product mapping \( f \times h: X \times Y \to \Pi^{k+1} \) is universal. Moreover, condition (3.2) holds for \( B = X \times h^{-1}(S^0) \cup A \times Y_0 \), \( g = f \times h \), \( g_0 = g_B \).

First we shall note the following fact:

(3.2) Lemma. Any closed connected subspace \( Y \) of a Tychonoff cube \( I^k \) is the intersection of an inverse system \( \langle Y_1, \delta_1 \rangle \) of arcwise connected closed subspaces of \( I^k \) such that the projections \( \delta_k \) are identity imbeddings.

Proof. The required inverse system is formed by the spaces \( P \times I^{k,k} \), where \( P \) is a polyhedron and \( L \) is a finite subset of \( X \) such that \( P \subseteq I^k \) and \( Y \subseteq P \times I^{k,k} \).

Proof of Theorem (3.1). We can assume that \( Y \) is a subspace of a Tychonoff cube \( I^k \). Hence \( Y \) is the intersection of an inverse system \( \langle Y_1, \delta_1 \rangle \) of arcwise connected subspaces of \( I^k \). Let \( h: I^k \to I \) be a continuous extension of \( h \) and \( h = h | Y_0 \). We put also

\( g_0 = f \times h_1, B_0 = X \times h^{-1}(S^0) \cup A \times Y_1 \), \( g_0 | B_0 : B_0 \to S^n \).

We have obtained the mapping

\( (g_0): \langle (X \times Y_1, B_0), (I^{k+1}, S^0) \rangle \to (I^{k+1}, S^0) \)

of the inverse system of the pairs \( (X \times Y_1, B_0) \), and the pair \( (X \times Y, B) \) is the intersection of this system. We can identify the pair \( (X \times Y, B) \) with the inverse limit of this inverse system (see [2], Chapter IX). By Corollary (2.4)

\( g_0^*(e^n) \notin \text{Im} i_B : H^n(X \times Y_1) \to H^n(B_0) \).
Hence

\[ q_{i+1}^{n+1} = \delta \ast g_{i}(e^{n}) \in H^{n+1}(X \times Y, B_{i}) \\setminus \{0\} \]

for any index \( i \) and \( (\xi_{i} \times \xi_{i}^{n})(e^{n+1}) = q_{i+1}^{n+1} \) for any \( s < t \). Thus, by the continuity of the cohomology theory in question,

\[ \delta \ast g_{i}(e^{n}) = (\delta q_{i+1}^{n+1})(X \times Y, B_{i}) \setminus \{0\} . \]

Thus \( g_{i}(e^{n}) \notin \text{Im}(\delta) \).

As corollaries to assertions (1.4), (1.5), (1.6), (3.1) we obtain:

(3.3) THEOREM. If \( X \) is a paracompact space and \( Y \) is a compact space of \( \dim Y \geq 1 \), then \( \dim X \times Y \geq \dim X + 1 \).

Remark. We can use Theorem 1.6 since \( X \times Y \) is a paracompact space (see [7]).

(3.4) THEOREM. If \( f \colon X \rightarrow \Gamma \) is a universal mapping of a paracompact space \( X \), where

(3.5) either \( n < 2 \) or \( \dim X < n \),

and \( g \colon Y \rightarrow \Gamma \) is a continuous mapping of a connected compact space \( Y \) onto \( I \), then the product mapping \( f \times g \colon X \times Y \rightarrow \Gamma \) is universal.

Now we shall prove the main theorem of this paper:

(3.6) THEOREM. Given a universal mapping \( f \colon X \rightarrow \Gamma \) of a compact space \( X \) onto \( \Gamma \) such that condition (3.5) holds, and a family \( g_{i} \colon Y_{i} \rightarrow S_{i} \), \( i \in T \), of connected compact spaces \( Y_{i} \), onto \( S_{i} \). Then the product mapping

\[ f \times \prod_{i \in T} g_{i} \colon X \times \prod_{i \in T} Y_{i} \rightarrow \Gamma \times \prod_{i \in T} S_{i} \]

is universal.

Proof. In the case of a one-element set \( T = \{ t \} \) and \( S_{t} = I \), Theorem 3.6 is a consequence of Theorem 3.4). Hence, by Lemma 1 of [4], Theorem (3.1) holds for \( T = \{ t \} \) and any snake-like space \( S_{t} \). Hence, step by step, we infer that the theorem holds for any finite set \( T \). Thus it follows from Theorem (3.1) of [5] (or Theorem C of this paper) that the theorem holds for any \( T \).

§ 4. We shall give some remarks.

Firstly, let us remark that if the product of a family of mappings is a universal mapping, then any mapping of the family is universal. Thus if \( f \) is in Theorem (3.6) we put \( X = I \), \( n = 1 \) and an identity instead of \( f \), then we obtain the following result (see Theorem (3.2) of [5]).

(4.1) THEOREM. If \( g_{i} \colon Y_{i} \rightarrow S_{i} \) is a continuous mapping of a connected compact space \( Y_{i} \) onto a snake-like space for any \( t \in T \), then the product mapping

\[ \prod_{i \in T} g_{i} \colon \prod_{i \in T} Y_{i} \rightarrow \prod_{i \in T} S_{i} \]

is universal.

Now we shall prove

(4.2) THEOREM. If \( f \colon X \rightarrow \Gamma \) is a universal mapping of a compact space \( X \) onto \( \Gamma \)-space \( \Gamma \), then \( f(C) = \Gamma \) for some connected subset \( C \) of \( X \).

Proof. Let \( C_{x} \) be the component of \( x \) for any point \( x \in X \). If \( f(C_{x}) \neq \Gamma \) for any \( x \in X \), then there exists a family \( (U_{x} \times \Gamma) \) of the closed-open sets \( U_{x} \) such that \( C_{x} \subseteq U_{x} \) and \( f(U_{x}) \neq \Gamma \) for any \( x \in X \). This family is a cover and it contains a finite cover \( V_{0}, V_{1}, ..., V_{n} \) of \( X \). Let \( y_{i} \in Y \), \( f(V_{i}) \) and let the mapping \( g \colon X \rightarrow \Gamma \) be given as follows:

\[ g(x) = y_{i} \]

for \( x \in V_{i} \setminus \bigcup_{j \neq i} V_{j} \), \( i = 1, 2, ..., n \).

Then \( g \) is a continuous mapping and \( g(x) \neq f(x) \) for any \( x \in X \) in contradiction to the universality of the mapping \( f \). Thus \( f(C_{x}) = \Gamma \) for some \( x \in X \).

Now we can formulate the following generalization of Theorems (3.6) and (4.1).

(4.3) THEOREM. Let \( f \colon X \rightarrow \Gamma \) and \( g_{i} \colon Y_{i} \rightarrow S_{i} \) be universal mappings, where \( X, Y_{i}, S_{i} \) are compact spaces such that condition (3.4) holds, and \( S_{i} \) is a snake-like space for any \( t \in T \). Then the product mapping

\[ f \times \prod_{i \in T} g_{i} \colon X \times \prod_{i \in T} Y_{i} \rightarrow \Gamma \times \prod_{i \in T} S_{i} \]

is universal for any \( n = 1, 2, ... \).

This theorem immediately follows from Theorem (3.4), (4.1) and (4.2).

§ 5. In this section we shall give a set-combinatorial result.

(5.1) THEOREM. Let \( (A_{i}, B_{i}), i = 1, 2, ..., n \) be a sequence of pairs of closed subsets \( A_{i}, B_{i} \) of a paracompact space \( X \) such that \( A_{i} \cap B_{i} = \emptyset \) for \( i = 1, 2, ..., n \) and the intersection of any sequence of the partitions \( \mathcal{P}_{i} \) between \( A_{i} \) and \( B_{i} \), \( i = 1, 2, ..., n \), is a non-empty set. Next, let \( A_{0} = A_{0} \cup A_{1} \), \( A_{0} \), be non-empty disjoint closed subsets of a connected compact space \( X \).

We put \( B_{i} = A_{i} \times Y \) for \( i = 1, 2, ..., n \) and \( B_{(n+1)} = X \times A_{(n+1)} \) and \( B_{(n+1)} = X \times A_{(n+1)} \). Then, under the assumption that condition (3.5) holds and if either \( X \) is a compact space or \( A_{(n+1)} \) and \( A_{(n+1)} \) are functionally closed subsets of \( X \), the intersection of any sequence of the partitions \( \mathcal{P}_{i} \) between \( B_{i} \) and \( B_{i} \), \( i = 1, 2, ..., n+1 \), is a non-empty set.

Proof. Let \( B_{i} \) be a partition between \( B_{i} \) and \( B_{i} \) for \( i = 1, 2, ..., n+1 \).
and let $p: X \times Y \to X$ and $q: X \times Y \to Y$ be projections. For the partitions $E_i$ there exist pairs of open sets $G_{-i}, G_i$ such that

$$B_{-i} \subseteq G_{-i}, \quad B_i \subseteq G_i$$

and

$$G_{-i} \cap G_i = \emptyset, \quad G_{-i} \cup G_i = X \times Y \setminus E_i$$

for $i = 1, 2, ..., n+1$. Then, as $X$ is a connected compact space, the set $p(B_i)$ is closed and the sets

$$H_{-i} := p(G_{-i} \setminus p(B_i)) \quad \text{and} \quad H_i := p(G_i \setminus p(B_i))$$

are disjoint open subsets of $X$, for $i = 1, 2, ..., n$. Then there exist continuous mappings $f_i: X \to I$ such that $f_i(a_j) = \text{sgn} j$ and $f_i(x) > 0$ for any $x \in H_i \cup p(B_i)$, $j = \pm i, i = 1, 2, ..., n$. The diagonal product

$$f_x = \bigcap_{i=1}^{n} f_i: X \to I^n$$

is, by Lemma of [3] (or Lemma A of this paper), a universal mapping.

Next, by the alternative assumption of the theorem about $X, A, (A_{(n+1)}, A_{<n+1})$, there exists a continuous mapping $f_{n+1}: Y \to I$ such that $f_{n+1}(a_j) = \text{sgn} j$ and $f_{n+1}(j) < 1$ for any $x \in q(B_{n+1} \cup G_i)$, $j = \pm (n+1)$. The product mapping $f_x \times f_{n+1}: X \times Y \to I^{n+1}$ is, by Theorem (3.4), a universal mapping.

We shall show that $E_i$ is a partition between $f^{-1}(I_{i}^{+1})$ and $f^{-1}(I_{i}^{-1})$ for $i = 1, 2, ..., n+1$.

Indeed, if $1 < i < n$ and $j = \pm i$, then

$$f^{-1}(I_{i}^{+1}) = p^{-1}(f^{-1}(\text{sgn} j)) \subseteq p^{-1}(H_i) \subseteq G_i.$$  

Similarly, if $j = \pm (n+1)$, then

$$f^{-1}(I_{i}^{-1}) = q^{-1}(f^{-1}(\text{sgn} j)) \subseteq G_i,$$

by definition of $f_{n+1}$.

Thus it follows from Lemma A, that $\bigcap_{i=1}^{n+1} E_i \neq \emptyset$. The theorem is proved.

References
