Universality of the product mappings onto products of I^n and snake-like spaces

by

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In paper [5] we proved, in quite an elementary manner, that the product of the continuous mappings of connected compact spaces onto snake-like spaces (1) is a universal mapping. In this paper we shall show that the product of such a product mapping by a universal mapping of a compact space onto I^n is, by hypothesis, also universal (see Theorem (3.4)). Moreover, the product of a family of universal mappings, defined on the compact spaces, such that one of those mappings is onto I^n and the remaining mappings are onto the snake-like spaces is, by hypothesis, a universal mapping.

As a corollary we shall obtain also the well-known theorem (see [6]) about the dimension of the product of a paracompact space by a compact space of positive dimension (see Theorem (3.3)) and a related set-combinatorial result (see Theorem (5.1)).

The following assertions on the universal mappings will be used:

LEMMA A (see [3]). *Let X be a normal space. Next, let*

$$I_k^n = \{(x_1, \ldots, x_n) \in I^n \colon x_{|k|} = \operatorname{sgn} k\}$$

for $k = \pm 1, \pm 2, \ldots, \pm n$, where $I = [-1, 1]$ is a closed segment. A continuous mapping $f\colon X \to I^n$ is universal if and only if the intersection of any sequence of closed sets F_i, $i = 1, 2, \ldots, n$, which are partitions between $f^{-1}(I_{-k}^n)$ and $f^{-1}(I_k^n)$ is a non-empty set.

THEOREM B (see [3]). *Let X be a normal space. Then $\dim X \geqslant n$ if and only if there exists a universal mapping $f\colon X \to I^n$.*

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(1) A compact space X is said to be a *snake-like space* if for any open covering P of X there exists a finite covering $P' = \{G_1, \ldots, G_n\}$ of X which is refinement of P and is such that $G_i \cap G_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. Thus for any open cover P of a snake-like space X there exists a P-mapping of X onto I. In this paper a *compact space* means a *compact Hausdorff space*.

THEOREM C (see [5], Theorem 3.1). *Let any finite product of the mappings* $f_t\colon X_t \to Y_t$, $t \in T$, *be a universal mapping. Then the product mapping*

$$f = \prod_{t\in T} f_t\colon \prod_{t\in T} X_t \to \prod_{t\in T} Y_t$$

is universal if X_t *is compact for any* $t \in T$.

§ 1. Let $f\colon X \to I^n$ be a continuous mapping of a space X into the cube I^n, where $I = [-1, 1]$ is a closed interval, and let S^{n-1} be the boundary of I^n (²). Then the following easy assertions hold:

(1.1) PROPOSITION. *The mapping* f *is not a universal mapping if and only if there exists a continuous mapping* $g\colon X \to S^{n-1}$ *such that* $g(x) = f(x)$ *for any* $x \in f^{-1}(S^{n-1})$.

The above proposition immediately implies the following one:

(1.2) PROPOSITION. *Let* $A = \bar{A} \subseteq f^{-1}(S^{n-1})$. *We put*

$$f_0 = f|A\colon A \to S^{n-1}.$$

If for a cohomology theory with arbitrary coefficients (see [2]) *we have*

$$(1.3)\qquad f_0^*(e^{n-1}) \notin \operatorname{Im}\big(i^*\colon H^{n-1}(X) \to H^{n-1}(A)\big),$$

where $i\colon A \to X$ *is the identity imbedding and* $e^{n-1} \neq 0$ *is an element of* $H^{n-1}(S^{n-1})$, *then* f *is a universal mapping.*

(1.4) PROPOSITION. *Let us consider the Čech–Dowker cohomology theory with integer coefficients* (see [2] chapter IX). *Let* $n = 1$ *or* 2 *and let* X *be a paracompact space. Then the mapping* f *is universal if and only if condition* (1.3) *holds for* $A = f^{-1}(S^{n-1})$ *and a generator* e^{n-1} *of* $H^{n-1}(S^{n-1})$.

(1.5) PROPOSITION. *Let us consider the Čech–Dowker cohomology theory with integer coefficients and let* X *be a paracompact space of covering dimension* $\leqslant n$. *Then the mapping* f *is universal if and only if condition* (1.3) *holds for* $A = f^{-1}(S^{n-1})$ *and a generator* e^{n-1} *of* $H^{n-1}(S^{n-1})$.

Propositions 1.4 and 1.5 are direct consequences of (1.1) and the respective Dowker theorems (see [1]).

As a corollary to Theorem B and Proposition (1.5) we obtain the following Alexandrov Theorem

(1.6) THEOREM. *Let* X *be a finite-dimensional paracompact space. Then* $n = \dim X$ *is the greatest integer such that* $i_A^*\colon H^{n-1}(X) \to H^{n-1}(A)$ *is not an epimorphism for a certain closed subset* A *of* X.

Indeed, if i_A^* is not an epimorphism, then $H^n(X, A) \neq 0$ and consequently $\dim X \geqslant n$. On the other hand, if $\dim X = n$, then by Theorem B

(²) In this paper n denotes a positive integer. Only in Theorem (4.4) n denotes a non-negative integer.

there exists a universal mapping $f\colon X \to I^n$. Thus, by (1.5), Theorem (1.6) holds.

§ 2. Let us consider a cohomology theory with arbitrary coefficients, defined on the category of all pairs (X, A) of topological spaces (where A is a closed subspace of X) and all continuous mappings.

(2.1) LEMMA. *Let* $f\colon X \to I^n$ *be a continuous mapping of a space* X *such that condition* (1.3) *holds for a closed subset* A *of* X. *We put*

$$B = X \times \{-1\} \cup X \times \{1\} \cup A \times I$$

and

$$g = f \times i_I\colon X \times I \to I^{n+1},$$

where $i_I\colon I \to I$ *is the identity, and let* $g_0 = g|B\colon B \to S^n$. *Then*

$$(2.2)\qquad g_0^*(e^n) \notin \operatorname{Im}\big(i_B^*\colon H^n(X \times I) \to H^n(B)\big),$$

where $i_B\colon B \to X \times I$ *is the identity embedding and* e^n *is the image of* e^{n-1}.(³) *Hence, by Proposition* (1.2), *the product mapping* $f \times i_I$ *is universal.*

Proof. We put

$$X^- = \{(x, y) \in B\colon y \leqslant 0\} \quad \text{and} \quad X^+ = \{(x, y) \in B\colon y \geqslant 0\}.$$

The triada (B, X^-, X^+) is a proper triada such that $B = X^- \cup X^+$. We can identify A and $X^- \cap X^+$. Let us consider the cohomological additional exact sequence of the triada (B, X^-, X^+) (see [2] I.15.2c)

$$\ldots \xrightarrow{\varphi} H^{n-1}(X^-) \dotplus H^{n-1}(X^+) \xrightarrow{\psi} H^{n-1}(A) \xrightarrow{\Delta} H^n(B) \xrightarrow{\varphi} H^n(X^-) \dotplus H^n(X^+) \xrightarrow{\psi} \ldots$$

We have, by (1.3), $f_0^*(e^{n-1}) \notin \operatorname{Im}\psi$. Hence

$$e^I \overset{\text{df}}{=} \Delta \circ f_0^*(e^{n-1}) \neq 0.$$

Now, let us consider the imbedding of the triada (B, X^-, X^+) into the triada $(X \times I, X \times [-1, 0], X \times [0, 1])$ and the induced homomorphism of the cohomological additional exact sequence of the second triada into the sequence of the first one. Then the composition

$$\begin{array}{ccc} H^n(X \times I) \xrightarrow{\varphi} & H^n(X \times [-1, 0]) \dotplus H^n(X \times [0, 1]) \\ & \downarrow \\ & H^n(X^-) \dotplus H^n(X^+) \end{array}$$

(³) We identify S^{n-1} and $\{(x_1, \ldots, x_{n+1}) \in S^n\colon x_{n+1} = 0\}$. Let $S_-^n = \{(x_1, \ldots, x_{n+1}) \in S^{n+1}\colon x_{n+1} \leqslant 0\}$ and $S_+^n = \{(x_1, \ldots, x_{n+1}) \in S^{n+1}\colon x_{n+1} \geqslant 0\}$. Then (S^n, S_-^n, S_+^n) is a proper triada (see [2] I.14.1) and $e^n = \Delta(e^{n-1})$ (see [2], I, 15,2c).

is a monomorphism onto the „diagonal" of the last group. On the other hand,

$$e^I \in \operatorname{Ker}\big(\varphi\colon H^n(B)\to H^n(X^-)+H^n(X^+)\big) = \operatorname{Im}\big(\Delta\colon H^{n-1}(A)\to H^n(B)\big) .$$

Thus $e^I \notin \operatorname{Im}\big(i_B^*\colon H^n(X\times I)\to H^n(B)\big)$, where $i_B\colon B\to X\times I$ is the identity embedding. But, by the commutation of the diagram

$$\begin{array}{ccc} H^{n-1}(A) & \xrightarrow{\ \Delta\ } & H^n(B) \\ {\scriptstyle f_0^*}\uparrow & & \uparrow{\scriptstyle g_0^*} \\ H^{n-1}(S^{n-1}) & \xrightarrow{\ \Delta\ } & H^n(S^n) \end{array}$$

we have

$$e^I = \Delta\circ f_0^*(e^{n-1}) = g_0^*\circ\Delta(e^{n-1}) = g_0^*(e^n) .$$

The lemma is proved.

The following assertion is easier than Lemma (2.1).

(2.1′) LEMMA. *Under the assumptions of* (2.1) *the suspension mapping* $Sf\colon SX\to SI^n \underset{\text{top}}{=} I^{n+1}$ *is universal.*

(2.3) LEMMA. *Let* $h\colon I\to I$ *be a continuous mapping onto such that* $h(-1)=-1$ *and* $h(1)=1$. *Then under the assumptions of Lemma* (2.1) *condition* (2.2) *holds for* $g=f\times h$ *and* $g_0=g|B\colon B\to S^n$.

Proof. The mapping $f\times h|B\colon B\to S^n$ is homotopically equivalent to the mapping $f\times i_I\,|B\colon B\to S^n$, whence

$$(f\times h|B)^* = (f\times i_I|B)^*\colon H^n(S^n)\to H^n(B)$$

and, by Lemma (2.1), condition (2.2) holds.

(2.4) COROLLARY. *Let* $h\colon Y\to I$ *be a continuous mapping of a normal space* Y *onto* I *such that* $h(Y_0)=I$ *for a simple arc* Y_0 *in* Y *with the end-points* a_{-1} *and* a_1. *Then, under the assumption of Lemma* (2.1) *about the mapping* f, *condition* (2.2) *holds for* $B=X\times h^{-1}(S^0)\cup A\times Y$, $g=f\times h$ *and* $g_0=g|B\colon B\to S^n$, *where* $S^0=\{-1,1\}$.

Proof. Without loss of generality we can assume that

$$h^{-1}(-1)\cap Y_0=\{a_{-1}\} \quad\text{and}\quad h^{-1}(1)\cap Y_0=\{a_1\} .$$

Thus there exists a retraction $r\colon Y\to Y_0$ such that

$$r\big(h^{-1}(-1)\big)=\{a_{-1}\} \text{ and } \quad r\big(h^{-1}(1)\big)=\{a_1\} .$$

We put $B_0 = X\times\{a_{-1}\}\cup X\times\{a_1\}\cup A\times Y_0$. The mappings

$$r_1 = i_X\times r\colon X\times Y\to X\times Y_0 \quad\text{and}\quad r_0=(i_X\times r)|B\colon B\to B_0$$

are also retractions, where $i_X\colon X\to X$ is the identity. Thus in the commutative diagram

$$\begin{array}{ccccc} H^n(X\times Y) & \xrightarrow{\ i_B^*\ } & H^n(B) & \xleftarrow{\ g_0^*\ } & \\ {\scriptstyle r_1^*}\downarrow & & \downarrow{\scriptstyle r_0^*} & & H^n(S^n) \\ H^n(X\times Y_0) & \xrightarrow[\ i_{B_0}^*\ ]{} & H^n(B_0) & \xleftarrow{\ r_0^*\circ g_0^*\ } & \end{array}$$

r_1^* and r_0^* are monomorphisms and, by Lemma 2.3, $r_0^*\circ g_0^*(e^n)\notin \operatorname{Im} i_{B_0}^*$. Hence $g_0^*(e^n)\notin\operatorname{Im} i_B^*$ and condition (2.2) holds.

§ 3. Let us consider a continuous cohomology theory, defined on the category of all pairs of spaces and all continuous mappings, with the coefficients in category $\mathfrak{G}_R$ (for example, let it be the Čech–Dowker theory; see [2], Chapter IX). We shall show that the following theorem holds.

(3.1) THEOREM. *Let* $h\colon Y\to I$ *be a continuous mapping of a connected compact space* Y *onto* I *and let* $f\colon X\to I^n$ *be as in Lemma* (2.1). *Then the product mapping* $f\times h\colon X\times Y\to I^{n+1}$ *is universal. Moreover, condition* (2.2) *holds for* $B=X\times h^{-1}(S^0)\cup A\times Y$, $g=f\times h$, $g_0=g|B$.

First we shall note the following fact:

(3.2) LEMMA. *Any closed connected subspace* Y *of a Tychonoff cube* I^K *is the intersection of an inverse system* $\langle Y_t, i_s^t\rangle$ *of arcwise connected closed subspaces of* I^K *such that the projections* i_s^t *are identity imbeddings.*

Proof. The required inverse system is formed by the spaces $P\times I^{K\setminus L}$, where P is a polyhedron and L is a finite subset of K such that $P\subseteq I^L$ and $Y\subseteq P\times I^{K\setminus L}$.

Proof of Theorem (3.1). We can assume that Y is a subspace of a Tychonoff cube I^K. Hence Y is the intersection of an inverse system $\langle Y_t, i_s^t\rangle$ of arcwise connected subspaces of I^K. Let $h'\colon I^K\to I$ be a continuous extension of h and $h_t = h'|Y_t$. We put also

$$g_t = f\times h_t,\ B_t = X\times h_t^{-1}(S^0)\cup A\times Y_t \quad\text{and}\quad g_{0,t}=g_t|B_t\colon B_t\to S^n .$$

We have obtained the mapping

$$(g_t)_t\colon \langle (X\times Y_t, B_t), i_X\times i_s^t\rangle \to (I^{n+1}, S^n)$$

of the inverse system of the pairs $(X\times Y_t, B_t)$, and the pair $(X\times Y, B)$ is the intersection of this system. We can identify the pair $(X\times Y, B)$ with the inverse limit of this inverse system (see [2], X, 2.5). By Corollary (2.4)

$$g_{0,t}^*(e^n)\notin \operatorname{Im} i_{B_t}\colon H^n(X\times Y_t)\to H^n(B_t) .$$

Hence

$$e_t^{n+1} \stackrel{\mathrm{df}}{=} \delta \circ g_{0,t}^*(e^n) \in H^{n+1}(X \times Y_t, B_t) \setminus \{0\}$$

for any index t and $(i_X \times i_s^t)^*(e_s^{n+1}) = e_t^{n+1}$ for any $s < t$. Thus, by the continuity of the cohomology theory in question,

$$\delta \circ g_0^*(e^n) = (e_t^{n+1})_t \in H^{n+1}(X \times Y, B) \setminus \{0\}.$$

Thus $g_0^*(e^n) \notin \mathrm{Im}\, i_B^*$.

As corollaries to assertions (1.4), (1.5), (1.6), (3.1) we obtain:

(3.3) THEOREM. *If X is a paracompact space and Y is a compact space of* $\dim Y \geqslant 1$, *then* $\dim X \times Y \geqslant \dim X + 1$.

Remark. We can use Theorem 1.6 since $X \times Y$ is a paracompact space (see [7]).

(3.4) THEOREM. *If $f\colon X \to I^n$ is a universal mapping of a paracompact space X, where*

(3.5) *either* $n \leqslant 2$ *or* $\dim X \leqslant n$,

and $g\colon Y \to I$ is a continuous mapping of a connected compact space Y onto I, then the product mapping $f \times g\colon X \times Y \to I^{n+1}$ is universal.

Now we shall prove the main theorem of this paper:

(3.6) THEOREM. Given a universal mapping $f\colon X \to I^n$ *of a compact space X onto I^n such that condition* (3.5) *holds, and a family $g_t\colon Y_t \to S_t$, $t \in T$, of connected compact spaces Y_t onto snake-like spaces S_t. Then the product mapping*

$$f \times \prod_{t \in T} g_t\colon X \times \prod_{t \in T} Y_t \to I^n \times \prod_{t \in T} S_t$$

is universal.

Proof. In the case of a one-element set $T = \{t\}$ and $S_t = I$, Theorem (3.6) is a consequence of Theorem (3.4). Hence, by Lemma 1 of [4], Theorem (3.1) holds for $T = \{t\}$ and any snake-like space S_t. Hence, step by step, we infer that the theorem holds for any finite set T. (4) Thus it follows from Theorem (3.1) of [5] (or Theorem C of this paper) that the theorem holds for any T.

§ 4. We shall give some remarks.

Firstly, let us remark that if the product of a family of mappings is a universal mapping, then any mapping of the family is universal. Thus if in Theorem (3.6) we put $X = I$, $n = 1$ and an identity instead of f, then we obtain the following result (see Theorem (3.2) of [5]).

(4.1) THEOREM. *If $g_t\colon Y_t \to S_t$ is a continuous mapping of a connected compact space Y_t onto a snake-like space for any $t \in T$, then the product mapping*

$$\prod_{t \in T} g_t\colon \prod_{t \in T} Y_t \to \prod_{t \in T} S_t$$

is universal.

(4) In the case of $n \leqslant 2$ we use (1,2), (1.4) and (2.1).

Now we shall prove

(4.2) THEOREM. *If $f\colon X \to Y$ is a universal mapping of a compact space X onto T_1-space Y, then $f(C) = Y$ for some connected subset C of X.*

Proof. Let C_x be the component of x for any point $x \in X$. If $f(C_x) \neq Y$ for any $x \in X$, then there exists a family $(U_x)_{x \in X}$ of the closed-open sets U_x such that $C_x \subseteq U_x$ and $f(U_x) \neq Y$ for any $x \in X$. This family is a cover and it contains a finite cover $V_1, V_2, \ldots, V_n$ of X. Let $y_i \in Y \setminus f(V_i)$ and let the mapping $g\colon X \to Y$ be given as follows:

$$g(x) = y_i \quad \text{for} \quad x \in V_i \setminus \bigcup_{j<i} V_j,\ i = 1, 2, \ldots, n.$$

Then g is a continuous mapping and $g(x) \neq f(x)$ for any $x \in X$ in contradiction to the universality of the mapping f. Thus $f(C_x) = Y$ for some $x \in X$.

Now we can formulate the following generalization of Theorems (3.6) and (4.1).

(4.3) THEOREM. *Let $f\colon X \to I^n$ and $g_t\colon Y_t \to S_t$ be universal mappings, where X, Y_t, S_t are compact spaces such that condition* (3.4) *holds, and S_t is a snake-like space for any $t \in T$. Then the product mapping*

$$f \times \prod_{t \in T} g_t\colon X \times \prod_{t \in T} Y_t \to I^n \times \prod_{t \in T} S_t$$

is universal for any $n = 1, 2, \ldots$

This theorem immediately follows from Theorem (3.4), (4.1) and (4.2).

§ 5. In this section we shall give a set-combinatorial result.

(5.1) THEOREM. *Let (A_{-i}, A_i), $i = 1, 2, \ldots, n$, be a sequence of pairs of closed subsets A_{-i}, A_i of a paracompact space X such that $A_{-i} \cap A_i = \emptyset$ for $i = 1, 2, \ldots, n$ and the intersection of any sequence of the partitions F_i between A_{-i} and A_i, $i = 1, 2, \ldots, n$, is a non-empty set. Next, let $A_{-(n+1)}$, $A_{(n+1)}$ be non-empty disjoint closed subsets of a connected compact space Y. We put $B_i = A_i \times Y$ for $i = \pm 1, \pm 2, \ldots, \pm n$, and $B_{-(n+1)} = X \times A_{-(n+1)}$ and $B_{n+1} = X \times A_{n+1}$. Then, under the assumption that condition* (3.5) *holds and if either X is a compact space or A_{n+1} and $A_{-(n+1)}$ are functionally closed subsets of Y, the intersection of any sequence of the partitions E_i between B_{-i} and B_i, $i = 1, 2, \ldots, n+1$, is a non-empty set.*

Proof. Let E_i be a partition between B_{-i} and B_i for $i = 1, 2, \ldots, n+1$

and let $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ be projections. For the partitions E_i there exist pairs of open sets G_{-i}, G_i such that

$$B_{-i}\subseteq G_{-i}\,,\qquad B_i\subseteq G_i$$

and

$$G_{-i}\cap G_i=\emptyset\,,\qquad G_{-i}\cup G_i=X\times Y\setminus E_i$$

for $i=1,2,\ldots,n+1$. Then, as Y is a connected compact space, the set $p(E_i)$ is closed and the sets

$$H_{-i}=p(G_{-i})\setminus p(E_i)\qquad\text{and}\qquad H_i=p(G_i)\setminus p(E_i)$$

are disjoint open subsets of X, for $i=1,2,\ldots,n$. Then there exist continuous mappings $f_i\colon X\to I$ such that $f_i(A_j)=\operatorname{sgn}j$ and $j\cdot f_i(x)\geqslant 0$ for any $x\in H_j\cup p(E_i)$, $j=\pm i$, $i=1,2,\ldots,n$. The diagonal product

$$f_0=\mathop{\Delta}_{i=1}^{n}f_i\colon X\to I^n$$

is, by Lemma of [3] (or Lemma A of this paper), a universal mapping.

Next, by the alternative assumption of the theorem about X, $A_{-(n+1)}$, A_{n+1}, there exists a continuous mapping $f_{n+1}\colon Y\to I$ such that $f_{n+1}(A_j)=\operatorname{sgn}j$ and $(-j)f_{n+1}(x)<1$ for any $x\in q(E_{n+1}\cup G_j)$, $j=\pm(n+1)$. The product mapping $f_0\times f_{n+1}\colon X\times Y\to I^{n+1}$ is, by Theorem (3.4), a universal mapping.

We shall show that E_i is a partition between $f^{-1}(I^{n+1}_{-i})$ and $f^{-1}(I^{n+1}_i)$ for $i=1,2,\ldots,n+1$.

Indeed, if $1\leqslant i\leqslant n$ and $j=\pm i$, then

$$f^{-1}(I_j^{n+1})=p^{-1}\big(f_i^{-1}(\operatorname{sgn}j)\big)\subseteq p^{-1}(H_j)\subseteq G_j\,.$$

Similarly, if $j=\pm(n+1)$, then

$$f^{-1}(I_j^{n+1})=q^{-1}\big(f_{n+1}^{-1}(\operatorname{sgn}j)\big)\subseteq G_j\,,$$

by definition of f_{n+1}.

Thus it follows from Lemma A, that $\bigcap\limits_{i=1}^{n+1}E_i\neq\emptyset$. The theorem is proved.

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