Plane sets with intersections of prescribed power

by

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1. Mazurkiewicz [3] proved that there is a subset of the plane that meets each line in exactly two points. Bagemihl [1] proved a theorem on intersections of abstract sets and used it to extend Mazurkiewicz's theorem. He showed that, if we assign to each line L in the plane a cardinal number n_L such that $2 \leqslant n_L \leqslant \aleph_0$, then there is a set A which meets each line L in exactly n_L points. Sierpiński [4] extended this to the case where $2 \leqslant n_L \leqslant 2^{\aleph_0}$, and this result was also obtained by Bagemihl and Erdös [2], by extending the theorem on intersections of abstract sets (see Theorem A in § 2 of the present paper). We obtain (Theorem 2.1) a more general and simpler form of Theorem A and use it to deduce certain new results on intersections of plane sets.

We shall use the axiom of choice freely but not the continuum hypothesis.

A *congruent copy* of a set A in the real Euclidean space R^2 is the image of A under any isometry of R^2. For each integer $n \geqslant 0$, let $\mathcal{S}(n)$ denote the family of all sets A in R^2 such that there exists a set B in R^2 which meets every congruent copy of A in exactly n points. The results quoted above imply that a line is in $\mathcal{S}(n)$ for all $n \geqslant 2$.

For a given set A in R^2, the problem of determining the values of n for which A is in $\mathcal{S}(n)$ may be difficult. For example, although it is not hard to show that the edges of a triangle form a set that is not (1) in $\mathcal{S}(2)$, it does not seem to be easy to determine whether this set is in $\mathcal{S}(3)$. We reverse the question and consider the following problem: Given a subset N of the positive integers, is there a set A in R^2 such that A is in $\mathcal{S}(n)$ if and only if n is in N? In other words, is

$$\bigcap \{\mathcal{S}(n):\ n \in N\} - \bigcup \{\mathcal{S}(n):\ n \notin N\} = \mathcal{Q}(N)\,,$$

(1) If each congruent copy of a triangle T meets a set B in exactly two points then B is uncountable. Hence B contains a pair of "close" points x, y, say. Let U be the union of all congruent copies of T that meet B in $\{x, y\}$. Then $U - \{x, y\}$ does not meet B but a copy of T is contained in this set, which gives a contradiction.

or (2.2) holds. By the argument used to establish (2.3) we may now show that for all $A \in \mathcal{A}$

$$(2.6) \qquad |A \cap \{a_\xi\colon \xi < m\}| \leqslant n(A)\,.$$

Choose any $A \in \mathcal{A}$. If (2.1) holds for some η with $A_\eta = A$ then clearly

$$(2.7) \qquad |A \cap \{a_\xi\colon 0 \leqslant \xi < m\}| \geqslant n(A)\,.$$

If not, then (2.2) holds for every η with $A_\eta = A$, and we see that

$$\eta \to a_\eta \in A$$

defines an injection from $\{\eta\colon A_\eta = A,\ 0 \leqslant \eta < m\}$ to A, and since $|\{\eta\colon A_\eta = A,\ 0 \leqslant \eta < m\}| = n(A)$, (2.7) again follows. It follows from (2.6), (2.7) that $B = \{a_\xi\colon 0 \leqslant \xi < m\}$ has the required property.

In the sequel the cardinal of the continuum is denoted by c instead of $2^{\aleph_0}$. If A, B are sets in R^2 and B is the image of A under an isometry of R^2, then we say that A and B are *congruent* and write $A \sim B$.

We now give a corollary of Theorem 2.1 that is directly applicable to the problems that we consider in § 3, 4. In reading this corollary it may help to bear in mind the example where the $C_1, \ldots, C_k$ are circles of different radii and $r_1 = \ldots = r_k = 2$.

COROLLARY 2.2. *Let $r_1, \ldots, r_k$, $s_1, \ldots, s_k$ (k finite) be positive integers with $r_i \leqslant s_i$. Let $C_1, \ldots, C_k$ be subsets of R^2 no two of which are congruent, and suppose that for i, j such that $1 \leqslant i \leqslant j \leqslant k$ the intersection of C_i with any congruent copy of C_j (other than C_i itself when $j = i$) has power at most $\aleph_0$. For $i = 1, \ldots, k$, let $|C_i| = c$, and suppose that if T is any set in R^2 of power r_i, then there are at most $\aleph_0$ congruent copies of C_i that contain T. Then there exists a set B in R^2 such that each congruent copy of C_i meets B in exactly s_i points.*

Proof. Let $\mathcal{A}$ denote the family of all congruent copies of the C_i. Then each A in $\mathcal{A}$ is the congruent copy of a unique C_i, so we may define a function n on $\mathcal{A}$ by putting $n(A) = s_i$ if $A \sim C_i$.

We now verify that the hypothesis of Theorem 2.1 is satisfied with this definition of $\mathcal{A}$ and n. Choose any $S \subset R^2$ with

$$|S| < \sum \{n(A)\colon A \in \mathcal{A}\} \leqslant c$$

(the sum is less than c only if $k = 1$ and $A_1 = R^2$); and, if possible, choose $A' \in \mathcal{A}$ with $|S \cap A'| < n(A')$. Put

$$\mathcal{A}^* = \{A\colon |S \cap A| = n(A)\}\,.$$

Since $r_i \leqslant s_i$,

$$\begin{aligned}\mathcal{A}^* &= \{A\colon \text{for some } i,\ A \sim C_i \text{ and } |S \cap A| = s_i\}\\ &\subset \{A\colon \text{for some } i,\ A \sim C_i \text{ and } |S \cap A| \geqslant r_i\}\\ &= \bigcup \{\mathcal{A}_{T,i}\colon i = 1, \ldots, k,\ T \subset S \text{ and } |T| = r_i\}\,,\end{aligned}$$

where $\mathcal{A}_{T,i} = \{A\colon A \sim C_i \text{ and } A \supset T\}$. By hypothesis each $\mathcal{A}_{T,i}$ contains at most $\aleph_0$ sets; and for each $i = 1, \ldots, k$ there are less than c subsets of S with power r_i. Since c is not the sum of $\aleph_0$ cardinals each less than c, we deduce that $|\mathcal{A}^*| < c$. Now A' is not in $\mathcal{A}^*$, because $|S \cap A'| < n(A')$. Hence, for all $A^* \in \mathcal{A}^*$, $|A^* \cap A'| \leqslant \aleph_0$. Since $|A'| = c$, it follows that $|A' - \bigcup \mathcal{A}^*| = c$. Hence, because $|S| < c$,

$$A' - (S \cup \bigcup \{A\colon |S \cap A| = n(A)\}) = A' - (S \cup \bigcup \mathcal{A}^*) \neq \emptyset,$$

as required.

It was stated in the introduction that Theorem 2.1 is a stronger version of a theorem due to Bagemihl and Erdös (Theorem 1 of [2]). This requires a little proof so we proceed to state the Bagemihl–Erdös theorem, with some modifications in the notation, and show that its hypothesis implies that of Theorem 2.1.

THEOREM A. *Let a be an arbitrary, fixed ordinal number, and let $\mathcal{A}$ be a family of sets with $|\mathcal{A}| \leqslant \aleph_a$.*

For every $A \in \mathcal{A}$ and every $\mathcal{A}' \subset \mathcal{A} - \{A\}$ with $|\mathcal{A}'| < \aleph_a$, let $|A - \bigcup \mathcal{A}'| \geqslant \aleph_a$.

For every $A \in \mathcal{A}$, let $l(A)$ be a cardinal number with $1 \leqslant l(A) \leqslant \aleph_a$, such that the following holds: If $S \subset \bigcup \mathcal{A}$ and $|S| < \aleph_a$, then

$$|\{A\colon |S \cap A| \geqslant l(A)\}| < \aleph_a\,.$$

For every $A \in \mathcal{A}$, let $n(A)$ be a cardinal number satisfying $l(A) \leqslant n(A) \leqslant \aleph_a$. There exists a set $B \subset \bigcup \mathcal{A}$ such that $|B \cap A| = n(A)$ for every $A \in \mathcal{A}$.

Proof. We verify that the hypothesis of Theorem 2.1 is satisfied. First note that

$$\sum \{n(A)\colon A \in \mathcal{A}\} \leqslant \aleph_a^2 = \aleph_a$$

since $n(A) \leqslant \aleph_a$ for all $A \in \mathcal{A}$ and $|\mathcal{A}| \leqslant \aleph_a$. Hence, if possible, choose any $S \subset \bigcup \mathcal{A}$ and $A' \in \mathcal{A}$ such that $|S| < \aleph_a$ and $|S \cap A'| < n(A')$. Put

$$\mathcal{A}^* = \{A\colon |S \cap A| = n(A)\}\,.$$

Then

$$\mathcal{A}^* \subset \{A\colon |S \cap A| \geqslant l(A)\}$$

and therefore $|\mathcal{A}^*| < \aleph_a$. Since $A' \notin \mathcal{A}^*$, it follows that

$$|A' - \bigcup \mathcal{A}^*| \geqslant \aleph_a\,.$$

Since $|S| < \aleph_a$, we deduce that

$$|A' - (S \cup \bigcup \mathcal{A}^*)| \geqslant \aleph_a\,,$$

and the proof is complete.

3. Let l_1, l_2, l_3 be non-negative integers. Consider a set $A \subset R^2$ constisting of the union of l_1 circles of radius ϱ_1, l_2 of radius ϱ_2, and l_3 of radius ϱ_3, where $\varrho_1, \varrho_2, \varrho_3$ are distinct positive numbers and the $l_1 + l_2 + l_3$ circles are pairwise disjoint. Applying Corollary 2.2 in the case where $k = 3$, the C_i are circles of radius ϱ_i and $r_1 = r_2 = r_3 = 2$, we see that A is in $\mathcal{S}(n)$ whenever $n = l_1s_1+l_2s_2+l_3s_3$ for some $s_1, s_2, s_3 \geqslant 2$. We shall show in this section that if the circles forming A are chosen in a special way, then A does not belong to $\mathcal{S}(n)$ for any other values of n. If $l_1 = l_2 = l_3 = 0$ then $A = \emptyset$ which belongs to $\mathcal{S}(n)$ if and only if $n = 0$, so we may suppose that $l_1 > 0$.

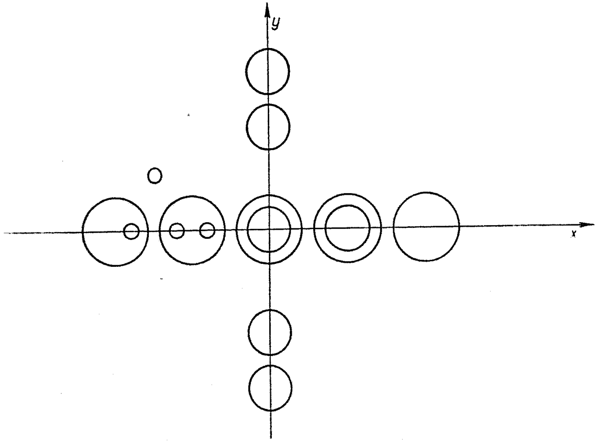

Fig. 1. The set illustrated is in $\mathcal{S}(n)$ if and only if $n = 4s_1+6s_2+5s_3$ for some $s_1, s_2, s_3 \geqslant 2$

The set A that we now define is illustrated in Fig. 1 for the case $l_1 = 4$, $l_2 = 6$, $l_3 = 5$, $\varrho_1 < \varrho_2 < \varrho_3$. The set is defined by reference to Cartesian axes xOy in R^2. The circles with union A are chosen stage by stage and at each stage they may be (and it is assumed that they are) chosen so that they are pairwise disjoint.

First choose l_1-1 circles of radius ϱ_1 with centres on the x-axis; then choose another circle of radius ϱ_1 which does not intersect the x-axis. Next, if $l_2 \geqslant 1$, choose l_2-1 circles of radius ϱ_2 with centres on the y-axis and such that they form a figure symmetric about the x-axis; then choose one more circle of radius ϱ_2 which does not intersect the y-axis and has centre on the x-axis. Finally, if $l_3 \geqslant 1$, choose l_3 circles with radius ϱ_3 and centres evenly distributed on the x-axis and positioned so that they form a figure which is symmetric about the y-axis. Let A be the union of all the circles that have been chosen.

Suppose that B is a set in R^2 such that every congruent copy of A intersects B in just n points.

By the definition of A, the reflection of A in the x-axis is obtained from A by replacing just one of the circles in A with radius ϱ_1 by its reflection in the x-axis. It follows that this circle and its reflection in the x-axis intersect B in the same number of points. Let $d > 0$ be the distance between this circle and its reflection in the x-axis. By considering congruent copies of A and reflecting in the appropriate lines, we see that each pair of circles with radius ϱ_1 and distance d apart intersects B in the same number of points. If C and C' are any two circles of radius ϱ_1, we may clearly form a finite sequence of circles of radius ϱ_1 such that the distance between consecutive circles in the sequence is d, and the first and last circles are C and C'. Hence

$$|C \cap B| = |C' \cap B| = s_1, \text{ say}.$$

Hence every circle of radius ϱ_1 intersects B in just s_1 points.

If $l_2 \geqslant 1$, the reflection of A in the y-axis may be obtained from A by replacing all the circles in A of radius ϱ_1 by their reflections in the y-axis, and replacing just one of the circles in A of radius ϱ_2 by its reflection in the y-axis. Since every circle of radius ϱ_1 intersects B in just s_1 points, we may now use the previous argument to show that every circle of radius ϱ_2 intersects B in the same number of points, s_2 say.

We finally show that, for $l_3 \geqslant 1$, every circle of radius ϱ_3 intersects B in the same number of points. This is obvious now for $l_3 = 1$, so suppose that $l_3 \geqslant 2$. Let $d' > 0$ be the shortest distance between the centres of pairs of circles in A with radius ϱ_3. Let A' be the congruent copy of A obtained by the translation of A that sends the origin $(0, 0)$ to the point $(d', 0)$. Then A' may be obtained from A by this translation of all the circles in A with radius ϱ_1 or ϱ_2, and by translating (in the same direction but by a distance l_3d) just one of the circles in A with radius ϱ_3. Since every circle of radius ϱ_i $(i = 1, 2)$ intersects B in just s_i points, it follows now, as before, that for some s_3 every circle of radius ϱ_3 intersects B in just s_3 points.

We have shown now that $n = l_1s_1+l_2s_2+l_3s_3$. To complete the proof we only have to show that $s_1, s_2, s_3 \geqslant 2$.

The set B is non-empty, because $n \geqslant 1$, so we may choose a point b in B. Suppose that $s_1 = 1$. By considering the circles of radius ϱ_1 that contain b, we see that b is the only point of B that is in the disc with centre b and radius $2\varrho_1$. Any circle of radius ϱ_1 that is in this disc and does not contain b is disjoint from B. This is a contradiction and therefore $s_1 \geqslant 2$. Similarly $s_2, s_3 \geqslant 2$ and the proof is complete.

Remark. The construction of the set B in the above example depends on Theorem 2.1 and therefore on the well ordering theorem. It is interesting to the note that some of the results may be obtained by elementary constructions. For example, for $k = 1, 2, \ldots$, every circle with radius ϱ intersects the set

$$B = \{(x, y)\colon\ y = 0, \pm 2\varrho/k, \pm 4\varrho/k, \ldots\}$$

in just $2k$ points, so we deduce that a circle is in $\bigcap\{\mathcal{S}(2k)\colon k = 1, 2, \ldots\}$. In this connection it is worth recalling Sierpiński's famous unsolved problem of whether there is a Borel set that meets every line in exactly two points. We have shown that there is a closed set that meets every circle of prescribed radius in exactly two points.

By considering the set B defined above it may be shown that a Reuleaux triangle T, say, (and in fact any curve of constant breadth), belongs to $\bigcap\{\mathcal{S}(2k)\colon k = 1, 2, \ldots\}$. It is worth noting that this result cannot be deduced from Corollary 2.2, because there is a congruent copy of T that meets T in c points. Also, because T may be covered by a finite number of copies of itself other than itself, it turns out that a direct argument by transfinite induction of the type used in Theorem 2.1 fails to give this result. I do not know whether T is in $\mathcal{S}(n)$ for any odd $n \geqslant 3$.

4. Let $l_1 \geqslant 0$, $l_2 \geqslant 1$ be integers and let ϱ, λ be positive numbers. Let $A \subset R^2$ be the union of l_1 circles with radius ϱ and l_2 parabolas with latus rectum λ, where the circles and parabolas are pairwise disjoint. If p_1, p_2, p_3 are distinct points in R^2, there are certainly less than $\aleph_0$ parabolas with latus rectum λ that contain $\{p_1, p_2, p_3\}$. Hence, applying Corollary 2.2 when $k = 2$, C_1 is a circle of radius ϱ, C_2 is a parabola of latus rectum λ, and $r_1 = 2$, $r_2 = 3$, we deduce that $A \in \mathcal{S}(n)$ whenever $n = l_1 s_1 + l_2 s_2$ for some $s_1 \geqslant 2$, $s_2 \geqslant 3$. We shall show that if the circles and parabolas are chosen in a special way, then $A \notin \mathcal{S}(n)$ for any other value of n.

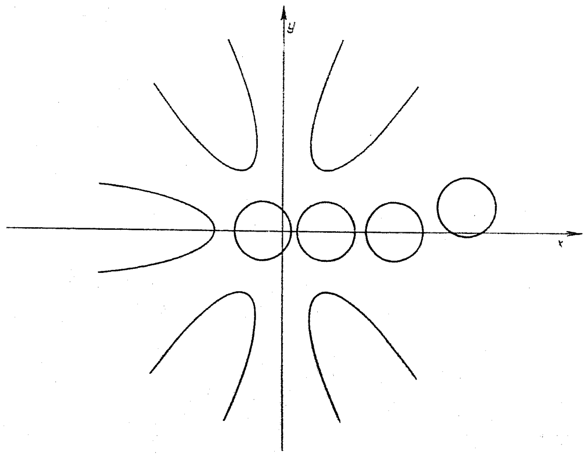

Fig. 2. The set illustrated is in $\mathcal{S}(n)$ if and only if $n = 4s_1+5s_2$ for some $s_1 \geqslant 2$, $s_2 \geqslant 3$

The set that we now define is illustrated in Fig. 2 for the case $l_1 = 4$, $l_2 = 5$. The set is defined by reference to Cartesian axes xOy in R^2.

Let f be the isometry that rotates R^2 about the origin through an angle $2\pi/(l_2+1)$. Let P be the parabola

$$\{(x, y)\colon y^2 = 2\lambda(x - x_0)\}$$

where $x_0 > 0$ is chosen so that P and $f(P)$ are disjoint. Put

$$A_2 = f(P) \cup f^2(P) \cup \ldots \cup f^{l_2}(P)\,.$$

If $l_1 \geqslant 2$, choose $l_1 - 1$ pairwise disjoint circles of radius ϱ that do not intersect A_2, and have centres on the x-axis. If $l_1 \geqslant 1$, choose a circle of radius ϱ, disjoint from the other circles and A_2, and with centre off the x-axis. Let A_1 be the union of all the circles chosen and let $A = A_1 \cup A_2$.

Suppose that B is a set that intersects every congruent copy of A in just n points. Since A_2 is its own reflection in the x-axis, we may argue as in § 3 to show that B intersects every circle of radius ϱ in the same number of points s_1, say. By considering the intersections of B with A, $f(A), \ldots$ $\ldots, f^{l_2}(A)$, we see that the parabolas $P, f(P), \ldots, f^{l_2}(P)$ all intersect B in the same number of points s_2, say. It follows that $n = l_1 s_1 + l_2 s_2$.

To complete the proof we only have to show that $s_1 \geqslant 2$, $s_2 \geqslant 3$.

By the argument used in § 3, we may show that $s_1 \geqslant 2$. It has been shown that every congruent copy of P intersects B in s_2 points. Let b_1, $b_2 \in B$; then there is certainly a congruent copy of P that contains $\{b_1, b_2\}$, so $s_2 \geqslant 2$. In fact, it is easy to see that the union of all congruent copies of P that contain $\{b_1, b_2\}$ contains a half space and therefore contains a congruent copy of P that does not meet $\{b_1, b_2\}$. It follows that $s_2 \geqslant 3$.

5. This section is devoted to a proof of the following

THEOREM 5.1. *Let n be a function that assigns to each pair of real numbers t, t' $(t < t')$ and each isometry f of R^2 a cardinal number $n(t, t', f)$ with $1 \leqslant n(t, t', f) \leqslant c$. Also suppose that n is such that if $n(t, t', f) \geqslant 2$, then whenever r is a reflection we have $n(t, t', r\circ f) \geqslant 2$. Then there exists a family of sets $\{A(t)\colon t \text{ real}\}$ such that for each t, t' $(t < t')$ and each isometry f we have $|A(t) \cap f(A(t'))| = n(t, t', f)$.*

Before giving a proof of this theorem we make a few remarks about the hypothesis concerning $n(t, t', f)$ and $n(t, t', r\circ f)$.

Suppose that A, A' are subsets of R^2, f is an isometry, and $|A \cap f(A')| \geqslant 2$. Then if r is a reflection in a line through any two points of $A \cap f(A')$ we have $|A \cap r(f(A))| \geqslant 2$. Thus we see that the hypothesis may not be omitted. Notice that this hypothesis is satisfied if $n(t, t', f) \geqslant 2$ for all t, t', f, and it is also satisfied if $n(t, t', f) = 1$ for all t, t', f.

Proof of Theorem 5.1. Throughout the proof a letter t denotes a real number and a letter f denotes an isometry of R^2. Form a transfinite sequence $((t_\xi, t'_\xi, f_\xi)\colon 1 \leqslant \xi < c)$ the terms of which are triples (t, t', f) with $t < t'$, and such that each (t, t', f) occurs exactly $n(t, t', f)$ times in the sequence.

For all t and ξ with $0 \leqslant \xi < c$ we define by induction sets $A(t, \xi) \subset R^2$ such that, putting

$$A^*(t, \xi) = \bigcup\{A(t, \xi')\colon \xi' < \xi\},$$
$$A^*(\xi) = \bigcup\{A(t, \xi')\colon \text{all } t, \xi' < \xi\},$$
$$A(\xi) = \bigcup\{A(t, \xi)\colon \text{all } t\},$$

we have the following:

For all $t < t'$, $0 \leqslant \eta \leqslant \zeta < c$ *and all* f

(i) $A(t, \eta) \subset A(t, \zeta)$,

(ii) $|A(\zeta) - A^*(\zeta)| \leqslant 2$,

(iii) $|A(t, \zeta) - A^*(t, \zeta)| \leqslant 1$,

(iv) $|A(t, \zeta) \cap f(A(t', \zeta))| \leqslant n(t, t', f)$,

(v) *if* $|A(t, \eta) \cap f(A(t', \eta))| = n(t, t', f)$ *then*

$$A(t, \zeta) \cap f(A(t', \zeta)) = A(t, \eta) \cap f(A(t', \eta)).$$

It is then shown that the sets $A(t) = \bigcup\{A(t, \xi)\colon \xi < c\}$ have the property required by Theorem 5.1.

Put $A(t, 0) = \emptyset$ for all t. Suppose that $0 < \varkappa < c$, that $A(t, \xi)$ is defined for all t and $\xi < \varkappa$, and that (i)-(v) hold for all f, $t < t'$ and $0 \leqslant \eta \leqslant \zeta < \varkappa$.

We now show that for all $t < t'$, f

(iv*) $|A^*(t, \varkappa) \cap f(A^*(t', \varkappa))| \leqslant n(t, t', f)$,

(v*) *if* $|A(t, \eta) \cap f(A(t', \eta))| = n(t, t', f)$ *for some* $\eta < \varkappa$ *then*

$$A^*(t, \varkappa) \cap f(A^*(t', \varkappa)) = A(t, \eta) \cap f(A(t', \eta)).$$

Choose any $t < t'$, f. If $n(t, t', f)$ is finite then (iv*) is an immediate consequence of (iv). If $n(t, t', f)$ is infinite and (iv*) does not hold, let $\varkappa'$ be the smallest ordinal such that

$$|A^*(t, \varkappa') \cap f(A^*(t', \varkappa'))| > n(t, t', f), \tag{5.1}$$

then $\varkappa' \leqslant \varkappa$. Because (iii) holds for all t and $\zeta < \varkappa$, we deduce that for some $\varkappa'' < \varkappa'$

$$|A(t, \varkappa'') \cap f(A(t', \varkappa''))| = n(t, t', \varkappa). \tag{5.2}$$

Since (v) holds for $\eta \leqslant \zeta < \varkappa$, (5.2) implies that

$$A(t, \zeta) \cap f(A(t', \zeta)) = A(t, \varkappa'') \cap f(A(t', \varkappa''))$$

for all ζ with $\varkappa'' \leqslant \zeta < \varkappa$. Hence

$$A^*(t, \varkappa') \cap f(A^*(t', \varkappa')) = A(t, \varkappa'') \cap f(A(t', \varkappa''))$$

which contradicts (5.1), (5.2). Hence (iv*) holds. If

$$|A(t, \eta) \cap f(A(t', \eta))| = n(t, t', f)$$

for some $\eta < \varkappa$, then, by the argument just used,

$$A^*(t, \varkappa) \cap f(A^*(t', \varkappa)) = A(t, \eta) \cap f(A(t', \eta)).$$

Hence (v*) is also established.

We next define the sets $A(t, \varkappa)$ for all t and show that (i)-(v) hold for all $t < t', f$, $0 \leqslant \eta \leqslant \zeta \leqslant \varkappa$. This constitutes a major part of our proof. It is clearly sufficient to consider $0 \leqslant \eta \leqslant \zeta = \varkappa$. In view of (iv*) we have either

(iv*a) $|A^*(t_\varkappa, \varkappa) \cap f_\varkappa(A^*(t'_\varkappa, \varkappa))| = n(t_\varkappa, t'_k, f'_\varkappa)$

or

(iv*b) $|A^*(t_\varkappa, \varkappa) \cap f_\varkappa(A^*(t'_\varkappa, \varkappa))| < n(t_\varkappa, t'_\varkappa, f'_\varkappa)$.

If (iv*a) holds, put $A(t, \varkappa) = A^*(t, \varkappa)$ for all t. Then for $0 \leqslant \eta \leqslant \zeta = \varkappa$ (i)-(iii) are trivial, and (iv), (v) reduce to (iv*), (v*) respectively. Hence we may suppose that (iv*b) holds.

Define sets $C_{\varkappa 1}, \ldots, C_{\varkappa 4}$ in R^2 as follows:

$$C_{\varkappa 1} = \{p\colon |p - a_1| = |a_2 - a_3| \text{ for some } a_1, a_2, a_3 \in A^*(\varkappa)\},$$
$$C_{\varkappa 2} = \{p\colon |p - f_\varkappa(a')| = |a_1 - a_2| \text{ for some } a' \in A^*(t'_\varkappa, \varkappa) \text{ and } a_1, a_2 \in A^*(\varkappa)\},$$
$$C_{\varkappa 3} = \{p\colon |p - a| = |p - f_\varkappa(a')| \text{ for some } a \in A^*(t_\varkappa, \varkappa) \text{ and } a' \in A^*(t'_\varkappa, \varkappa) \text{ with } a \neq f_\varkappa(a')\},$$
$$C_{\varkappa 4} = \{p\colon p, a_1, a_2 \text{ are collinear for some } a_1, a_2 \in A^*(t_\varkappa, \varkappa),\ a_1 \neq a_2\}.$$

From (ii) and the fact that $A(0) = \emptyset$ we deduce that $|A^*(\varkappa)| \leqslant 2|\varkappa| - 2 \leqslant 2|\varkappa| < c$. Hence $C_{\varkappa 1}$ is the union of at most $2^3|\varkappa|^3 < c$ circles; $C_{\varkappa 2}$ is the union of less than c circles; and $C_{\varkappa 3}$, $C_{\varkappa 4}$ are each the union of less than c lines. Hence

$$C_\varkappa = C_{\varkappa 1} \cup C_{\varkappa 2} \cup C_{\varkappa 3} \cup C_{\varkappa 4} \neq R^2$$

and we may choose a point $q_\varkappa \in R^2 - C_\varkappa$.

Now define

$$A(t, \varkappa) = A^*(t, \varkappa) \quad \text{if} \quad t \neq t_\varkappa \text{ and } t \neq t'_\varkappa,$$
$$A(t_\varkappa, \varkappa) = A^*(t_\varkappa, \varkappa) \cup \{q_\varkappa\},$$
$$A(t'_\varkappa, \varkappa) = A^*(t'_\varkappa, \varkappa) \cup \{f_\varkappa^{-1}(q_\varkappa)\}.$$

Choose any $t < t'$, f, $\eta \leqslant \varkappa$. Then, with $\zeta = \varkappa$, (i) and (iii) are obvious and (ii) follows because

$$A(\varkappa) - A^*(\varkappa) \subset \{q_\varkappa, f_\varkappa^{-1}(q_\varkappa)\}.$$

To prove (iv), (v) we first establish the following:

(a) *If* $q_\varkappa \in f(A^*(t', \varkappa))$ *then*

$$A^*(t_\varkappa, \varkappa) \cap f(A^*(t', \varkappa)) = \emptyset .$$

(b) *If* $f(q_\varkappa) \in A^*(t, \varkappa)$ *then*

$$A^*(t, \varkappa) \cap f(A^*(t_\varkappa, \varkappa)) = \emptyset .$$

(c) *If* $f(f_\varkappa^{-1}(q_\varkappa)) \in A^*(t, \varkappa)$ *then*

$$A^*(t, \varkappa) \cap f(A^*(t'_\varkappa, \varkappa)) = \emptyset .$$

(d) *If* $f_\varkappa^{-1}(q_\varkappa) \in f(A^*(t', \varkappa))$ *then*

$$A^*(t'_\varkappa, \varkappa) \cap f(A^*(t', \varkappa)) = \emptyset .$$

(e) *If* $q_\varkappa \in f(A^*(t'_\varkappa, \varkappa))$ *then*

$$f(f_\varkappa^{-1}(q_k)) \notin A^*(t'_\varkappa, \varkappa) .$$

(f) *If* $f(f_\varkappa^{-1}(q_\varkappa)) \in A^*(t_\varkappa, \varkappa)$ *then*

$$q_\varkappa \notin f(A^*(t_\varkappa, \varkappa)) .$$

To prove (a), suppose the contrary. Then for some $a'_1, a'_2 \in A^*(t', \varkappa)$ and $a \in A^*(t_\varkappa, \varkappa)$ we have $q_\varkappa = f(a'_1)$, $a = f(a'_2)$. Then $|q_\varkappa - a| = |f(a'_1) - f(a'_2)| = |a'_1 - a'_2|$ which is impossible because $q_\varkappa \notin C_{\varkappa 1}$, so (a) holds. Similarly, if (b) is false, then for some $a_1, a_2 \in A^*(t, \varkappa)$ and $a \in A^*(t_\varkappa, \varkappa)$ we have $f(q_\varkappa) = a_1$, $a_2 = f(a)$. Hence $|q_\varkappa - a| = |a_1 - a_2|$ which is impossible, because $q_\varkappa \notin C_{\varkappa 1}$, so (b) holds. If (c) is false, then for some $a_1, a_2 \in A^*(t, \varkappa)$ and $a' \in A^*(t'_\varkappa, \varkappa)$ we have $f(f_\varkappa^{-1}(q_\varkappa)) = a_1$, $a_2 = f(a')$. Hence $|a_1 - a_2| = |f(f_\varkappa^{-1}(q_\varkappa) - f(a))| = |q_\varkappa - f_\varkappa(a')|$ which is impossible, because $q_\varkappa \notin C_{\varkappa 2}$, so (c) holds. If (d) is false, then for some $a'_1, a'_2 \in A^*(t', \varkappa)$ and $a' \in A^*(t'_\varkappa, \varkappa)$ we have $f_\varkappa^{-1}(q_\varkappa) = f(a'_1)$ and $a' = f(a'_2)$. Hence $|q_\varkappa - f_\varkappa(a')| = |a'_1 - a'_2|$ which is impossible, because $q_\varkappa \notin C_{\varkappa 2}$, so (d) holds. To prove (e) and (f) it is sufficient to show that it is impossible that $q_\varkappa \in f(A^*(t'_\varkappa, \varkappa))$ and $f(f_\varkappa^{-1}(q_\varkappa)) \in A^*(t_\varkappa, \varkappa)$. If $q_\varkappa = f(a')$ and $f(f_\varkappa^{-1}(q_\varkappa)) = a$ for some $a' \in A^*(t'_\varkappa, \varkappa)$, $a \in A^*(t_\varkappa, \varkappa)$, then $|q_\varkappa - a| = |q_\varkappa - f_\varkappa(a')|$ and, by (a) with $t' = t'_\varkappa$, $a \neq f(a')$. This is impossible, because $q_\varkappa \notin C_{\varkappa 3}$, so (e) and (f) hold.

We now return to the proof of (iv), (v) for $\eta \leqslant \zeta = \varkappa$, i.e.

(iv′) $|A(t, \varkappa) \cap f(A(t', \varkappa))| \leqslant n(t, t', f)$,

(v′) *if* $|A(t, \eta) \cap f(A(t', \eta))| = n(t, t', f)$ *for some* $\eta \leqslant \varkappa$ *then*

$$A(t, \varkappa) \cap f(A(t', \varkappa)) = A(t, \eta) \cap f(A(t', \eta)) .$$

We consider various cases.

Case 1. $\{t, t'\} \cap \{t_\varkappa, t'_\varkappa\} = \emptyset$. Then

$$A(t, \varkappa) \cap f(A(t', \varkappa)) = A^*(t, \varkappa) \cap f(A^*(t', \varkappa)) ,$$

so (iv′), (v′) reduce to (iv*), (v*), respectively.

Case 2. $\{t, t'\} \cap \{t_\varkappa, t'_\varkappa\} = t_\varkappa$ or $t'_\varkappa$. Suppose first that $t = t_\varkappa$, $t' \neq t'_\varkappa$. Then

$$A(t, \varkappa) \cap f(A(t', \varkappa)) = [A^*(t, \varkappa) \cup \{q_\varkappa\}] \cap f(A^*(t', \varkappa)) .$$

If $q_\varkappa \notin f(A^*(t', \varkappa))$, then (iv′), (v′) follow from (iv*), (v*), respectively. If $q_\varkappa \in f(A^*(t', \varkappa))$ then, by (a),

$$A(t, \varkappa) \cap f(A(t', \varkappa)) = \{q_\varkappa\} ,$$

so (iv′) follows, because $n(t, t', f) \geqslant 1$. Also (v′) holds because for $\eta < \varkappa$

$$A(t, \eta) \cap f(A(t', \eta)) \subset A^*(t, \varkappa) \cap f(A^*(t', \varkappa)) = \emptyset .$$

The other possibilities that occur in Case 2 are as follows:

$$t' = t_\varkappa, \quad t \neq t'_\varkappa;$$
$$t' = t'_\varkappa, \quad t \neq t_\varkappa;$$
$$t = t'_\varkappa, \quad t' \neq t_\varkappa.$$

These may be dealt with by similar arguments but using (b), (c), (d) respectively in place of (a).

Case 3. $\{t, t'\} \cap \{t_\varkappa, t'_\varkappa\} = \{t_\varkappa, t'_\varkappa\}$. Then $t = t_\varkappa$ and $t' = t'_\varkappa$, so

$$A(t, \varkappa) \cap f(A(t', \varkappa)) = [A^*(t_\varkappa, \varkappa) \cap f(A^*(t'_\varkappa, \varkappa))] \cup [\{q_\varkappa\} \cap f(A^*(t'_\varkappa, \varkappa))] \cup$$
$$\cup [A^*(t_\varkappa, \varkappa) \cap \{f(f_\varkappa^{-1}(q_\varkappa))\}] \cup [\{q_\varkappa\} \cap \{f(f_\varkappa^{-1}(q_\varkappa))\}]$$
$$= I_1 \cup I_2 \cup I_3 \cup I_4 = I , \quad \text{say}.$$

It is convenient to subdivide this case.

Case 3.1. $I_2 = I_3 = I_4 = \emptyset$. Then $I = I_1$ and (iv′), (v′) follow from (iv*), (v*), respectively.

Case 3.2. $I_2 \neq \emptyset$, i.e. $q_\varkappa \in f(A^*(t'_\varkappa, \varkappa))$. Then by (a) $I_1 = \emptyset$, and by (e) $I_3 = \emptyset$. Hence $I = \{q_\varkappa\} \cup I_4 = \{q_\varkappa\}$, and (iv′), (v′) follow as in Case 2.

Case 3.3. $I_3 \neq \emptyset$, i.e. $f(f_\varkappa^{-1}(q_\varkappa)) \in A^*(t_\varkappa, \varkappa)$. Then by (c) $I_1 = \emptyset$, and by (f) $I_2 = \emptyset$. Hence $I = f(f_\varkappa^{-1}(q_\varkappa))$, and (iv′), (v′) follow as in Case 2.

Case 3.4. $I_4 \neq \emptyset$, i.e. $q_\varkappa = f(f_\varkappa^{-1}(q_\varkappa))$. If $I_1 = \emptyset$, then $I = \{q_\varkappa\}$ and (iv′) (v′) follow as in Case 2.

Suppose that $|I_1| = 1$. Then there exist $a \in A^*(t_\varkappa, \varkappa)$ and $a' \in A^*(t'_\varkappa, \varkappa)$ such that $a = f(a')$ and $I = \{a, q_\varkappa\}$. Then

$$|a - q_\varkappa| = |f(a') - f(f_\varkappa^{-1}(q_\varkappa))| = |f_\varkappa(a') - q_\varkappa| .$$

Hence, since $q_\varkappa \notin C_{\varkappa 3}$, $a = f_\varkappa(a')$. Thus

$$a = f(f_\varkappa^{-1}(a)) , \qquad q_\varkappa = f(f_\varkappa^{-1}(q_\varkappa)) \cdot$$

Also $q_\varkappa \neq a$, because $q_\varkappa \notin C_{\varkappa 1}$. Thus the isometry $f \circ f_\varkappa^{-1}$ fixes the two points a, $q_\varkappa$, so it follows that $f = f_\varkappa$ or $r \circ f_\varkappa$ where r is the reflection in the line through a and $q_\varkappa$. Hence $a \in A^*(t_\varkappa, \varkappa) \cap f_\varkappa(A^*(t'_\varkappa, \varkappa))$, so since (iv*b) holds, $n(t_\varkappa, t'_\varkappa, f_\varkappa) \geqslant 2$. Hence by the hypothesis of our theorem and the fact that $f = f_\varkappa$ or $r \circ f_\varkappa$ we deduce that $n(t, t', f) = n(t_\varkappa, t'_\varkappa, f) \geqslant 2$. The result (iv′) is now immediate, because $I = \{a, q_\varkappa\}$. Also (v′) follows, because

$$A(t, \eta) \cap f(A(t', \eta)) \subset I_1 \quad \text{and} \quad |I_1| = 1 .$$

Now suppose that $|I_1| \geqslant 2$. Then for $i = 1, 2$ there exist $a_i \in A^*(t_\varkappa, \varkappa)$, $a'_i \in A^*(t'_\varkappa, \varkappa)$ such that $a_i = f(a'_i)$, $a_1 \neq a_2$. Just as in the preceeding praragraph, we may show that $a_i = f(f_\varkappa^{-1}(a_i))$ for $i = 1, 2$. Thus the isometry $f \circ f_\varkappa^{-1}$ fixes the three points $a_1, a_2, q_\varkappa$. Since $q_\varkappa \notin C_{\varkappa 4}$, these three points are not collinear and therefore $f = f_\varkappa$. The results (iv′), (v′) now follow, because $|I_1| < n(t_\varkappa, t'_\varkappa, f_\varkappa)$, $I = I_1 \cup \{q_\varkappa\}$, and $A(t_\varkappa, \eta) \cap f_\varkappa(A(t'_\varkappa, \eta)) \subset I_1$.

The definition of the sets $A(t, \xi)$ for all t and all ξ with $0 \leqslant \xi < c$ is complete, and (i)–(v) hold for $t < t'$, $0 \leqslant \eta \leqslant \zeta < c$, and all f. We now show that the sets

$$A(t) = A^*(t, c) = \bigcup \{A(t, \xi): \ \xi < c\}$$

have the required property.

Choose any $t < t', f$. By the argument used to establish (iv*) we may show that

$$|A^*(t, c) \cap f(A^*(t', c))| \leqslant n(t, t', f) . \tag{5.3}$$

If

$$|A(t, \varkappa) \cap f(A(t', \varkappa))| = n(t, t', f)$$

for any $\varkappa < c$, then by the argument that establishes (v*) we can prove that

$$A^*(t, c) \cap f(A^*(t', c)) = A(t, \varkappa) \cap f(A(t', \varkappa)) .$$

so

$$|A^*(t, c) \cap f(A^*(t, c))| = n(t, t', f) .$$

Hence, by (iv), we may suppose that for all $\varkappa < c$

$$|A(t, \varkappa) \cap f(A(t', \varkappa))| < n(t, t', f) .$$

Then for all $\varkappa$ such that $(t_\varkappa, t'_\varkappa, f_\varkappa) = (t, t', f)$ (iv*b) holds and therefore $q_\varkappa$ is defined. Now for all ξ for which q_ξ is defined, by definition, $q_\xi \in A(t, \xi)$ for some t and $q_\xi \notin A^*(\xi)$, because $q_\xi \notin C_{\xi 1}$. Hence no two of the q_ξ are equal. Hence the function defined by

$$\varkappa \to q_\varkappa$$

is an injection from $\{k\colon (t_\varkappa, t'_\varkappa, f_\varkappa) = (t, t', f), \varkappa < c\}$ into $A^*(t, c) \cap f(A^*(t', c))$. Since

$$|\{\varkappa\colon t_\varkappa, t'_\varkappa, f_\varkappa) = (t, t', f), \varkappa < c\}| = n(t, t', f) ,$$

it follows that

$$|A^*(t, c) \cap f(A^*(t', c))| \geqslant n(t, t', f) .$$

Using also (5.3) we deduce

$$|A^*(t, c) \cap f(A^*(t', c))| = n(t, t', f) .$$

The proof of Theorem 5.1 is now complete.

It was stated in the introduction that Theorem 5.1 implies that for each cardinal number n with $1 \leqslant n \leqslant c$, there is a family $\mathcal{A}$ consisting of c subsets of R^2 such that each intersects each congruent copy of another in exactly n points. This is trivial for $n = c$, because we may, for example, take $\mathcal{A}$ to be a family of sets with bounded complements. Therefore suppose that $1 \leqslant n < c$ and let $\mathcal{A} = \{A(t)\colon t \text{ real}\}$ be the family determined by Theorem 5.1 with $n(t, t', f) = n$ for all $t < t', f$. To show that this family has power c it is sufficient to show that the $A(t)$ are all distinct. This is a consequence of a result that we prove in the next section (Theorem 6.1).

6. The principal results of this section are Theorems 6.11, 6.12. We begin with Theorem 6.1 which gives the (negative for $n \neq 0, c$) answer to the following question:

For $0 \leqslant n \leqslant c$, is there a subset of R^2 that intersects each congruent copy of itself in exactly n points?

The section continues with various modifications of this question that eventually lead to Theorems 6.11 and 6.12.

We have already observed in § 1 that the answer to the above question is affirmative for $n = 0, c$.

THEOREM 6.1. *If n is a cardinal number such that $1 \leqslant n < c$, then there is no subset of R^2 that intersects each congruent copy of itself in exactly n points.*

Proof. Suppose that A intersects each congruent of itself in exactly n points. Then $|A| = n$, because A is a congruent copy of itself. Since there are at most n^2 $(< c)$ different distances between pairs of points of A, there is a translation f such that $A \cap f(A) = \emptyset$.

In view of the proof of Theorem 6.1 it is natural to consider the set set $\mathcal{F}_1$ consisting of all isometries of R^2 other than the identity, and pose the following question: *Is there a set $A \subset R^2$ such that for all $f \in \mathcal{F}_1$ $|A \cap f(A)| = n$?* It is shown that the answer is affirmative for all infinite $n < c$ and negative for all finite $n \geqslant 1$.

THEOREM 6.2. *If n is a cardinal number with $\aleph_0 \leqslant n \leqslant c$, then there is a set A in R^2 such that for all $f \in \mathcal{F}_1$ $|A \cap f(A)| = n$.*

Proof. Let S, T be sets of real numbers such that every nondegenerate interval of real numbers intersects S and T in sets of power n, and for all $s, s' \in S$ and $t, t' \in T$ $s \neq -s'$, $s \neq \pm t$, $t \neq -t'$. Put

$$E_s = \{(x, y)\colon\ y = se^x\} \quad \text{for} \quad s \in S\,,$$

and

$$E_t = \{(x, y)\colon\ y = te^{-x}\} \quad \text{for} \quad t \in T\,,$$

and let

$$\mathcal{S} = \{E_s\colon\ s \in S\}\,, \qquad \mathcal{T} = \{E_t\colon\ t \in T\}\,.$$

It is easy to show that if $E, E' \in \mathcal{S} \cup \mathcal{T}$ and f is an isometry of R^2, then $|E \cap f(E')| < \aleph_0$ unless $E = E'$ and f is the identity. Using this fact it is easily shown that $A = \bigcup \mathcal{S} \cup \bigcup \mathcal{T}$ has the required property.

THEOREM 6.3. *Let $\mathcal{F}$ be the set of all isometries that rotate R^2 through the angle π. If n is a positive integer, then there is no set $A \subset R^2$ such that for all $f \in \mathcal{F}$ $|A \cap f(A)| = n$.*

Proof. Suppose on the contrary that $A \subset R^2$ is such that $|A \cap f(A)| = n$ for all $f \in \mathcal{F}$.

Suppose that n is even. Certainly A is not empty, so we may choose $a \in A$. Let f be the isometry that rotates R^2 about a through the angle π. Since $n \geqslant 2$, there is a point $a' \in A \cap f(A)$, $a' \neq a$. Then $f(a') \in A$ because $f(f(A)) = A$. Thus all points $(\neq a)$ in $A \cap f(A)$ may be paired so that a is the mid point of each pair. Since $a \in A \cap f(A)$ and n is even, this is a contradiction.

When n is odd a contradiction may be obtained similarly by considering the rotation through angle π about a point $b \notin A$.

In view of Theorem 6.3 it is now natural to consider the set $\mathcal{F}_\pi$ of all $f \in \mathcal{F}_1$ that do not rotate the plane through the angle π, and pose the following question:

If n is a positive integer, is there a set $A \subset R^2$ such that for all $f \in \mathcal{F}_\pi$ $|A \cap f(A)| = n$?

We show that the answer is negative for $n = 1, 2$ and affirmative for all $n \geqslant 3$. By further reducing the family of isometries considered we also obtain a positive result for $n = 2$. To show that these reductions are necessary, we prove two theorems each of which also shows that the answer to the last question is negative for $n = 2$. We first dispose of the case $n = 1$ by observing that if L is a line that contains two points of a set A, then the intersection of A and its reflection in L contains these two points.

THEOREM 6.4. *Let $0 < \theta < 2\pi$, and let $\mathcal{F}$ be the set of all isometries that rotate the plane through the angle θ together with all the reflections. Then there is no set A in R^2 such that for all $f \in \mathcal{F}$ $|A \cap f(A)| = 2$.*

Proof. Suppose that $A \subset R^2$ is such that for all $f \in \mathcal{F}$ $|A \cap f(A)| = 2$. Choose $a \in A$ and let $f \in \mathcal{F}$ be the rotation through angle θ about the point a. Let $a' \in A \cap f(A)$, $a' \neq a$. Then the points a, a', $f^{-1}(a')$ are in A and are the vertices of an isosceles triangle. We obtain a contradiction by considering the reflection in the line that bisects the angle at the vertex a of this triangle.

THEOREM 6.5. *Let $\mathcal{F}$ be the set of all isometries that rotate the plane through the angle $\pi/3$ or $2\pi/3$. Then there is no set $A \subset R^2$ such that for all $f \in \mathcal{F}$ $|A \cap f(A)| = 2$.*

Proof. Suppose that $A \subset R^2$ is such that for all $f \in \mathcal{F}$ $|A \cap f(A)| = 2$. Choose $a \in A$ and let $f \in \mathcal{F}$ be the rotation through $\pi/3$ about a. Then there is a point $a' \in A \cap f(A)$, $a' \neq a$. Then a, a', $f^{-1}(a')$ are all in A and are the vertices of an equilateral triangle. If $g \in \mathcal{F}$ is the rotation through angle $2\pi/3$ about the centre of this triangle, then $\{a, a', f^{-1}(a')\} \subset A \cap g(A)$ and we have a contradiction.

In view of Theorems 6.4 and 6.5, to obtain some positive result for $n = 2$ it is reasonable (2) to introduce the set $\mathcal{F}^*$ of all isometries in $\mathcal{F}_\pi$ that are not reflections and do not rotate the plane through the angles $\pm\pi/3$. With this definition of $\mathcal{F}^*$ we have Theorem 6.12.

The proofs of Theorems 6.11 and 6.12 occupy the remainder of this paper. We begin with three elementary lemmas; the first is well known

(2) The restriction of the set of isometries is to some extent arbitrary. In view of Theorem 6.4, it is natural to choose between the reflections and rotations, and since the rotations form the larger set (in terms of parameters) we remove the reflections. Then in view of Theorem 6.5 we remove the rotations by $\pm\pi/3$. This choice of the restriction is further justified since it leads to a positive result.

and no proof is given. These are followed by Lemmas 6.9 and 6.10 that are the main parts of the proofs of Theorems 6.11 and 6.12, respectively.

LEMMA 6.6. *An orientation preserving isometry of R^2 is a translation or a rotation. An orientation reversing isometry is a reflection in a line followed by a translation parallel to that line.*

For our purpose a *conic* is a set of the form $\{(x, y): (x, y) \in R^2, P(x, y) = 0\}$, where P is a polynomial of degree at most 2 and is not identically zero. Thus we allow all the degenerate forms of a conic except R^2.

LEMMA 6.7. *If $l \geq 0$ and f is an isometry that is not a translation by distance l, then $E = \{p: |p-f(p)| = l\}$ is a conic.*

Proof. If f is a translation, then $E = \emptyset$. If f is a rotation, then E is a singleton if $l = 0$ and a circle otherwise. If f is a reflection in a line L followed by a translation by distance d parallel to L, then $E = \emptyset$ if $d > l$, $E = L$ if $d = l$, and E is a pair of lines parallel to L if $d < l$.

LEMMA 6.8. *Let k be an isometry that is not a rotation through angle $\pm\pi/3$ and let p' be any point in R^2. Then $E = \{p: |p-k(p)| = |p-p'|\}$ is a conic.*

Proof. If k is a translation, then E is a circle with centre p' and degenerates to $\{p'\}$ if k is the identity.

Let k be a rotation through angle θ about a point r, where $0 < \theta < 2\pi$, $\theta \neq \pi/3, 5\pi/3$. Then $|p-k(p)| = 2|p-r|\sin\frac{1}{2}\theta$, and therefore $E = \{p: 2|p-r|\sin\frac{1}{2}\theta = |p-p'|\}$. Hence E is the circle of Apollonius if $r \neq p'$, and $E = \{p'\}$ if $r = p'$ because $2\sin\frac{1}{2}\theta \neq 1$.

Let k be the reflection in a line L followed by a translation parallel to L by a distance d. Let p' be distance $3y_0/4$ from L. Then we may choose Cartesian axes so that $L = \{(x, y): y = y_0/4\}$, $p' = (0, y_0)$. Then $k(x, y) = (x \pm d, y_0/2 - y)$, and E is the set of all points (x, y) such that

$$\{x-(x\pm d)\}^2 + \{y-(y_0/2-y)\}^2 = x^2 + (y-y_0)^2,$$

i.e.

$$3y^2 - x^2 = 3y_0^2/4 - d^2.$$

Hence E is a conic.

We use the following notation: If f is an isometry of R^2 and $p_1, \ldots, p_s$ are points in R^2, then $f(p_1, \ldots, p_s) = (f(p_1), \ldots, f(p_s))$. The set of fixed points of f is denoted by $F(f)$.

Note. It follows from Lemma 6.7 that $F(f)$ is a conic unless f is the identity. In fact, it is easy to deduce from Lemma 6.6 that if f is not the identity, then $F(f)$ is empty, a singleton, or a line, and is a line if and only if f is the reflection in that line.

LEMMA 6.9. *Let n be an integer $n \geq 3$, $h \in \mathcal{F}_\pi$, and $B \subset R^2$. Suppose that $|B| < c$, $|B \cap h(B)| < n$, and for all $f \in \mathcal{F}_\pi$ $|B \cap f(B)| \leq n$. Then there exists $B' \supset B$ such that*

$$|B'-B| \leq 2, \quad |B' \cap h(B')| > |B \cap h(B)|,$$

and for all $f \in \mathcal{F}_\pi$

$$|B' \cap f(B')| \leq n.$$

Proof. Let

$$\mathcal{G} = \{g: g \in \mathcal{F}_\pi, |B \cap g(B)| \geq 2\}.$$

Then $|\mathcal{G}| < c$. Let

$$C_1 = \{p: |p-b_1| = |b_2-b_3| \text{ for some } b_1, b_2, b_3 \in B\}.$$

Then C_1 is the union of a family of less than c circles and $B \subset C_1$.

Suppose first that h is the reflection in a line L, say. By the above note, for each $g \in \mathcal{G}-\{h\}$ $L \cap F(g)$ is empty or a singleton. Since $|\mathcal{G}| < c$, it follows that we can choose a point $q \in L - (C_1 \cup \bigcup\{F(g): g \in \mathcal{G}-\{h\}\})$.

We show that $B' = B \cup \{q\}$ has the required property.

Clearly $B' \supset B$ and $|B'-B| \leq 2$. Also, since $h(q) = q$,

$$B' \cap h(B') = [B \cap h(B)] \cup \{q\}.$$

Hence

$$|B' \cap h(B')| = |B \cap h(B)|+1 \leq n.$$

It only remains to be shown that for all $f \in \mathcal{F}_\pi - \{h\}$ $|B' \cap f(B')| \leq n$. Suppose the contrary; then there are subsets S, S' of B' each consisting of $n+1$ points and $f \in \mathcal{F}_\pi - \{h\}$ such that $S' = f(S)$. Then $q \in S \cup S'$, because $|B \cap f(B)| \leq n$. It follows that there are subsets T, T' of S, S' respectively each consisting of 3 points such that $q \in T \cup T'$ and $T' = f(T)$. Hence, for some $b_1, \ldots, b_5 \in B$, either

$$f(b_1, b_2, b_3) = (q, b_4, b_5),$$

or

$$f(q, b_1, b_2) = (b_3, b_4, b_5),$$

or

$$f(q, b_1, b_2) = (b_3, q, b_4),$$

or

$$f(q, b_1, b_2) = (q, b_3, b_4).$$

The first, second, and third equations cannot hold because, since $q \notin C_1$, we have

$$|b_1-b_2| \neq |q-b_4|, \quad |q-b_1| \neq |b_3-b_4| \quad \text{and} \quad |q-b_2| \neq |b_3-b_4|.$$

The last equation cannot hold because, if $f(b_1, b_2) = (b_3, b_4)$, then $f \in \mathcal{G}-\{h\}$ and therefore $q \notin F(f)$ by definition of q. We have a contradiction so the result is established when h is a reflection.

Suppose that h is not a reflection. Put

$$C_2 = \{p\colon |p-h(b_1)| = |b_2-b_3| \text{ for some } b_1, b_2, b_3 \in B\},$$
$$C_3 = \{p\colon |p-b_1| = |p-b_2| \text{ for some } b_1, b_2 \in B,\ b_1 \neq b_2\},$$
$$C_4 = \bigcup\{F(g)\colon g \in \mathcal{G}\},$$
$$C_5 = \bigcup\{F(h\circ g)\colon g \in \mathcal{G}-\{h^{-1}\}\},$$
$$C_6 = \bigcup\{h(F(g))\colon g \in \mathcal{G}\},$$
$$C_7 = F(h\circ h) \cup h^{-1}(B) \cup h(h(B)).$$

Also, if h is not a translation by a distance $|b_1-b_2|$ for any $b_1, b_2 \in B$, put

$$C_8 = \{p\colon |p-h(p)| = |b_1-b_2| \text{ for some } b_1, b_2 \in B\};$$

and, if h is such a translation, then define C_8 as follows: Let h^{-1} be a translation by the vector t, $h^{-1}(p) = p+t$. For each (ordered) pair of points b, b' in B such that $|b-b'| = |t|$ define $L_{b,b'}$, to be the line through b parallel to $t+(b-b')$ if $t+(b-b') \neq 0$ and perpendicular to t if $t+(b-b') = 0$. Let C_8 be the union of all such lines $L_{b,b'}$.

Then C_2 is the union of less than c circles, and C_3 is the union of less than c lines. Also, since $F(f)$ is a conic if f is not the identity, the sets C_4, C_5, C_6 are each the union of less than c conics. Since h is not a reflection and not a rotation through angle π, we can see from Lemma 6.6 that $h\circ h$ is not the identity, and therefore C_7 is also the union of less than c conics. If h is not a translation by distance $|b_1-b_2|$ for any $b_1, b_2 \in B$, then, by Lemma 6.7, C_8 is the union of less than c conics. If h is such a translation then C_8 is the union of less than c lines. Hence $\bigcup\{C_i\colon i = 1, \ldots, 8\} \neq R^2$ and we may choose $q \in R^2 - \bigcup\{C_i\colon i = 1, \ldots, 8\}$.

We show that $B' = B \cup \{q, h^{-1}(q)\}$ has the required property.

Clearly $B' \supset B$ and $|B'-B| \leqslant 2$. Also

$$B' \cap h(B') = [B \cup \{q, h^{-1}(q)\}] \cap [h(B) \cup \{h(q), q\}] = [B \cap h(B)] \cup \{q\},$$

since $h(q) \notin B$ because $q \notin C_7 \supset h^{-1}(B)$, $q \notin B$ because $q \notin C_1 \supset B$, $q \notin h(B)$ because $q \notin C_2 \supset h(B)$, $h^{-1}(q) \notin h(B)$ because $q \notin C_7 \supset h(h(B))$, and $h^{-1}(q) \neq h(q)$ because $q \notin C_7 \supset F(h\circ h)$. Hence $|B' \cap h(B')| = |B \cap h(B)|+1 \leqslant n$. Hence $|B' \cap h^{-1}(B')| \leqslant n$.

Now we only have to show that for all $f \in \mathcal{F}_\pi - \{h, h^{-1}\}$ $|B' \cap f(B')| \leqslant n$. Suppose the contrary. Then by the argument used before we can show that there are subsets T, T' of B each consisting of 4 points such that $T \cup T'$ contains q or $h^{-1}(q)$ (or both) and T' is the image of T under some isometry in $\mathcal{F}_\pi - \{h, h^{-1}\}$. Since the inverse of any isometry in $\mathcal{F}_\pi - \{h, h^{-1}\}$ is itself in $\mathcal{F}_\pi - \{h, h^{-1}\}$, it follows that T is also the image of T' under some isometry in $\mathcal{F}_\pi - \{h, h^{-1}\}$. Hence, for some $b_1, \ldots, b_7 \in B$ and $f \in \mathcal{F}_\pi - \{h, h^{-1}\}$, at least one of the following equations holds (each side of each equation represents a quadruple of distinct points in R^2):

(1) $f(q, b_1, b_2, b_3) = (b_4, b_5, b_5, b_7)$,

(2) " $= (q, b_4, b_5, b_6)$,

(3) " $= (b_4, q, b_5, b_6)$,

(4) $f(h^{-1}(q), b_1, b_2, b_3) = (b_4, b_5, b_6, b_7)$,

(5) " $= (h^{-1}(q), b_4, b_5, b_6)$,

(6) " $= (b_4, h^{-1}(q), b_5, b_6)$,

(7) $f(q, b_1, b_2, b_3) = (h^{-1}(q), b_4, b_5, b_6)$,

(8) " $= (b_4, h^{-1}(q), b_5, b_6)$,

(9) $f(q, h^{-1}(q), b_1, b_2) = (b_3, b_4, b_5, b_6)$,

(10) " $= (q, b_3, b_4, b_5)$,

(11) " $= (b_3, q, b_4, b_5)$,

(12) " $= (b_3, b_4, q, b_5)$,

(13) " $= (h^{-1}(q), b_3, b_4, b_5)$,

(14) " $= (b_3, h^{-1}(q), b_4, b_5)$,

(15) " $= (b_3, b_4, h^{-1}(q), b_5)$,

(16) " $= (q, h^{-1}(q), b_3, b_4)$,

(17) " $= (q, b_3, h^{-1}(q), b_4)$,

(18) " $= (h^{-1}(q), q, b_3, b_4)$,

(19) " $= (b_3, q, h^{-1}(q), b_4)$,

(20) " $= (h^{-1}(q), b_3, q, b_4)$,

(21) " $= (b_3, h^{-1}(q), q, b_4)$,

(22) " $= (b_3, b_4, q, h^{-1}(q))$.

Because $q \notin C_1$ we see that (1), (3), (8), (9), (11), (12), (14), (15), (19), (20), (21) do not hold. Because $q \notin C_2$ we see that (4), (6), (10), (13), (17) do not hold. Because $q \notin C_4$ (2) and (16) do not hold; because $q \notin C_5$ (7) and (18) do not hold; and because $q \notin C_6$ (5) does not hold. Then only (22) remains, and therefore (22) holds.

If h is not a translation by a distance $|b-b'|$ for some $b, b' \in B$, then, recalling the definition of C_8 in this case and the fact that $q \notin C_8$, we obtain a contradiction from (22). Hence h is such a translation. Let $h^{-1}(p)$

$=p+t$, say. From (22) we have $|q-b_1|=|q-b_3|$, and since $q \notin C_3$, it follows that $b_1=b_3$. Hence f interchanges q, b_1. We can easily prove from Lemma 6.6 that the only isometries that interchange a pair of points are the reflections and rotations by π. Since $f \in \mathcal{F}_\pi$, it follows that f is a reflection in a line L, say. Hence, by (22), $b_4=b_2$. Thus q and $h^{-1}(q)$ are reflections of b_1 and b_2 respectively in L, and it follows that q is on the line L' through b_1 perpendicular to L. However, L' is parallel to $t+(b_1-b_2)$ if $t+(b_1-b_2)\neq 0$ and perpendicular to t if $t+(b_1-b_2)=0$. Recalling the definition of C_8 in this case we see that this is contrary to the definition of $q \notin C_8$. This contradiction completes the proof.

LEMMA 6.10. *Let $h \in \mathcal{F}^*$ and $B \subset R^2$. Suppose that $|B|<c$, $|B \cap h(B)| < 2$, and for all $f \in \mathcal{F}^*$ $|B \cap f(B)| \leqslant 2$. Then there exists $B' \supset B$ such that $|B'-B| \leqslant 2$, $|B' \cap h(B')| > |B \cap h(B)|$, and for all $f \in \mathcal{F}^*$ $|B' \cap f(B')| \leqslant 2$*

Proof. Put $\mathcal{G}=\{f\colon f \in \mathcal{F}^*,\ |B \cap f(B)|=2\}$. Define $C_1, \ldots, C_6$ as in the proof of Lemma 6.9. Put

$$C_7=\{p\colon |p-h(b_1)|=|p-h(b_2)| \text{ for some } b_1, b_2 \in B,\ b_1\neq b_2\},$$

$$C_8=\{p\colon |p-h^{-1}(p)|=|p-b| \text{ for some } b \in B\},$$

$$C_9=\{p\colon |p-h^{-1}(p)|=|p-h(b)| \text{ for some } b \in B.$$

Then C_8 is the union of less than c lines, and by Lemma 6.8 C_8, C_9 are each the union of less than c conics. Hence $\bigcup\{C_i\colon i=1,\ldots,9\}\neq R^2$, and we may choose $q \in R^2-\bigcup\{C_i\colon i=1,\ldots,9\}$.

We show that $B'=B\cup\{q, h^{-1}(q)\}$ has the required property. Clearly $B'\supset B$ and $|B'-B|\leqslant 2$. As in the proof of Lemma 6.9, $|B'\cap h(B')| = |B\cap h(B)|+1 \leqslant 2$, and $|B'\cap h(B')|\leqslant 2$.

Now we only have to show that for all $f \in \mathcal{F}^*-\{h, h^{-1}\}$ $|B' \cap f(B')| \leqslant 2$. Suppose the contrary. Then, as in the proof of Lemma 6.9, for some $f \in \mathcal{F}^*-\{h, h^{-1}\}$ and $b_1, \ldots, b_5 \in B$, at least one of the following equations holds (where each side of each equation represents a triple of distinct points in R^2):

(1) $f(q, b_1, b_2)=(b_3, b_4, b_5)$,

(2) $\quad" \quad =(q, b_3, b_4)$,

(3) $\quad" \quad =(b_3, q, b_4)$,

(4) $f(h^{-1}(q), b_1, b_2)=(b_3, b_4, b_5)$,

(5) $\quad" \quad =(h^{-1}(q), b_3, b_4)$,

(6) $\quad" \quad =(b_3, h^{-1}(q), b_4)$,

(7) $f(q, b_1, b_2)=(h^{-1}(q), b_3, b_4)$,

(8) $f(q, b_1, b_2)=(b_3, h^{-1}(q), b_4)$,

(9) $f(q, h^{-1}(q), b_1)=(b_2, b_3, b_4)$,

(10) $\quad" \quad =(q, b_2, b_3)$,

(11) $\quad" \quad =(b_2, q, b_3)$,

(12) $\quad" \quad =(b_2, b_3, q)$,

(13) $\quad" \quad =(h^{-1}(q), b_2, b_3)$,

(14) $\quad" \quad =(b_2, h^{-1}(q), b_3)$,

(15) $\quad" \quad =(b_2, b_3, h^{-1}(q))$,

(16) $\quad" \quad =(q, h^{-1}(q), b_2)$,

(17) $\quad" \quad =(q, b_2, h^{-1}(q))$,

(18) $\quad" \quad =(h^{-1}(q), q, b_2)$,

(19) $\quad" \quad =(b_2, q, h^{-1}(q))$,

(20) $\quad" \quad =(h^{-1}(q), b_2, q)$,

(21) $\quad" \quad =(b_2, h^{-1}(q), q)$.

Because $q \notin C_1$, (1), (3), (8), (9), (11), (14) do not hold. Because $q \notin C_2$, (4), (6), (10), (13) do not hold. Because $q \notin C_4$, (2) does not hold. Because $q \notin C_5$ (7) does not hold. Because $q \notin C_6$, (5) does not hold. Because $q \notin C_8$, (17), (19) do not hold. Because $q \notin C_9$, (20), (21) do not hold. If (12) holds then $|q-b_1|=|q-b_2|$ and since $q \notin C_3$, this implies that $b_1=b_2$: thus f interchanges b_1 and q which is impossible because $f \in F^*$. Hence (12) does not hold. Similarly, if (15) holds then $|h^{-1}(q)-b_1|=|h^{-1}(q)-b_3|$ and, since $q \notin C_7$, this implies that $b_1=b_3$; thus f interchanges b_1 and $h^{-1}(q)$ which is impossible since $f \in F^*$. Hence (15) does not hold. Finally, (16), (18) do not hold because (16) implies that f fixes q and $h^{-1}(q)$, and (18) implies that f interchanges q and $h^{-1}(q)$, and neither is possible since $f \in F^*$. We now have a contradiction so the proof is complete.

THEOREM 6.11. *Let n be an integer, $n \geqslant 3$. There exists a set $A \subset R^2$ such that for all $f \in \mathcal{F}_\pi$ $|A \cap f(A)|=n$.*

Proof. Form a transfinite sequence $(f_\xi\colon 1\leqslant \xi < c)$ the terms of which are members of $\mathcal{F}_\pi$ and such that every $f \in \mathcal{F}_\pi$ occurs exactly n times in the sequence.

We now define by induction a transfinite sequence $(A_\xi\colon 1 \leqslant \xi < c)$ of subsets of R^2 such that for all $f \in \mathcal{F}_\pi$ and η, ζ with $1 \leqslant \eta \leqslant \zeta < c$

(i) $A_\eta \subset A_\zeta$,

(ii) $$|\{\xi:\ \xi \leqslant \zeta,\ f_\xi = f\}| \leqslant |A_\zeta \cap f(A_\zeta)|,$$

(iii) $$|A_\zeta \cap f(A_\zeta)| \leqslant n,$$

(iv) $$|A_1| \leqslant 2 \quad \textit{and} \quad |A_\zeta - \bigcup\{A_\xi:\ \xi < \zeta\}| \leqslant 2.$$

Choose any point $a \in R^2$ and put $A_1 = \{a, f_1^{-1}(a)\}$. Then (i)-(iv) hold for all $f \in \mathcal{F}_\pi$ and $\eta = \zeta = 1$. Let $\varkappa$ be any ordinal number with $1 \leqslant \varkappa < c$. Suppose that A_ξ is defined for all $\xi < \varkappa$ and that (i)-(iv) hold for all $f \in \mathcal{F}_\pi$ and $1 \leqslant \eta \leqslant \zeta < \varkappa$.

Put $A_\varkappa^* = \bigcup\{A_\xi:\ \xi < \varkappa\}$. Then by definition

(i*) $$A_\eta \subset A_\varkappa^*$$

for all $\eta < \varkappa$; and by (i), (ii), (iii), for all $f \in \mathcal{F}_\pi$

(ii*) $$|\{\xi:\ \xi < \varkappa,\ f_\xi = f\}| \leqslant |A_\varkappa^* \cap f(A_\varkappa^*)|,$$

(iii*) $$|A_\varkappa^* \cap f(A_\varkappa^*)| \leqslant n.$$

We now define $A_\varkappa$ and show that (i)-(iv) hold for all $f \in \mathcal{F}_\pi$ and $1 \leqslant \eta \leqslant \zeta = \varkappa$.

If $|A_\varkappa^* \cap f_\varkappa(A_\varkappa^*)| = n$ put $A_\varkappa = A_\varkappa^*$. Let $f \in \mathcal{F}_\pi$ and $1 \leqslant \eta \leqslant \zeta = \varkappa$. Then (i) follows from (i*). If $f \neq f_\varkappa$ then

$$\{\xi:\ \xi < \varkappa,\ f_\xi = f\} = \{\xi:\ \xi \leqslant \varkappa,\ f_\xi = f\}$$

and (ii) follows from (ii*). If $f = f_\varkappa$, then (ii) follows because $|A_\varkappa^* \cap f(A_\varkappa^*)| = n$ and, by the definition of (f_ξ), $|\{\xi:\ \xi \leqslant \varkappa,\ f_\xi = f_\varkappa\}| \leqslant n$. Also (iii) follows from (iii*), and (iv) is obvious since $A_\varkappa = A_\varkappa^*$.

Now suppose that $|A_\varkappa^* \cap f_\varkappa(A_\varkappa^*)| < n$. By (iii) and the definition of $A_\varkappa^*$ we have $|A_\varkappa^*| < c$. Hence, by Lemma 6.9 with $h = f_\varkappa$ and $B = A_\varkappa^*$, we can choose $A_\varkappa \supset A_\varkappa^*$ such that

(a) $$|A_\varkappa - A_\varkappa^*| \leqslant 2,$$

(b) $$|A_\varkappa \cap f_\varkappa(A_\varkappa)| > |A_\varkappa^* \cap f_\varkappa(A_\varkappa^*)|,$$

(c) $$\textit{for all } f \in \mathcal{F}_\pi \quad |A_\varkappa \cap f(A_\varkappa)| \leqslant n.$$

Let $f \in \mathcal{F}_\pi$ and $1 \leqslant \eta \leqslant \zeta = \varkappa$. Then (i) follows from (i*) and the fact that $A_\varkappa \supset A_\varkappa^*$. If $f \neq f_\varkappa$, then, as before,

$$\{\xi:\ \xi < \varkappa,\ f_\xi = f\} = \{\xi:\ \xi \leqslant \varkappa,\ f_\xi = f\}$$

and (ii) follows from (ii*). If $f = f_\varkappa$, then

$$|\{\xi:\ \xi \leqslant \varkappa,\ f_\xi = f\}| = |\{\xi:\ \xi < \varkappa,\ f_\xi = f\}| + 1 \leqslant |A_\varkappa^* \cap f(A_\varkappa^*)| + 1$$

by (ii*), and therefore (ii) follows from (b). Also (iii) follows from (c), and (iv) follows from (a).

The definition of $(A_\xi:\ 1 \leqslant \xi < c)$ is complete and (i)-(iv) hold for all $f \in \mathcal{F}_\pi$ and $1 \leqslant \eta \leqslant \zeta < c$. From (i), (ii), (iii) we deduce that, putting $A = \bigcup\{A_\xi:\ \xi < c\}$,

$$|\{\xi:\ \xi < c,\ f_\xi = f\}| \leqslant |A \cap f(A)| \leqslant n.$$

Since $|\{\xi:\ \xi < c,\ f_\xi = f\}| = n$ the proof is complete.

Our final theorem may be proved by the same argument, but using Lemma 6.10 instead of Lemma 6.9.

THEOREM 6.12. *There is a set* $A \subset R^2$ *such that for all* $f \in \mathcal{F}^*$ $|A \cap f(A)| = 2$.

It is interesting to note that $\mathcal{F}^*$ is the set of all isometries (in the group of all isometries of R^2) that do not have order 1, 2 or 6.

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