A theorem of Stone-Čech type, and a theorem of Tychonoff type, without the axiom of choice; and their realcompact analogues

by

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§ 1. Introduction. It is a hopeless endeavor, doomed to failure, to attempt to prove either the Stone-Čech compactification theorem or the Tychonoff product theorem without invoking some form of the axiom of choice since, as is well known (see p. 276 of [5], for example, and [9]), the latter theorem implies the axiom of choice and the former implies one of its weaker forms. It is my feeling, however, that the definition of compactness relative to which the theorems of Stone-Čech and Tychonoff are unprovable without the axiom of choice is, from the point of view of topological analysis and the theory of rings of continuous functions, unnatural and unsuitable; the Stone-Čech compactification of a completely regular Hausdorff space, for example, should be obtainable directly from the ring of real-valued continuous functions on the space. It is the object of the present paper to propose, for completely regular Hausdorff spaces, a new definition of compactness, evidently equivalent to the usual one in the presence of the axiom of choice; and to prove the appropriate versions of the Stone-Čech and the Tychonoff theorems on the basis of this definition, without the axiom of choice.

Another argument for the thesis that the definition of compactness proposed below is a "correct" or natural definition stems from the numerous well-known parallels drawn by Hewitt between his realcompact spaces (called *Q-spaces* in [7]) and compact spaces. Nearly every realcompact or "v" theorem given by Hewitt in [7], or by Gillman and Jerison in [5], admits a compact or "β" analogue. The definition proposed here is, in this sense, in the spirit of [7] and [5]. It has been generally known for some time (see [5], p. 158) that the existence of the Hewitt realcompactification of a given completely regular Hausdorff space does not depend upon the axiom of choice; it is scarcely surprising, then, that Hewitt's

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product theorem, to the effect that a product of realcompact spaces is realcompact, can likewise be effected without an appeal to the axiom of choice. A proof of this fact is given in Theorem 4.2 below.

Experience gained presenting the content of this paper before a learned audience discloses that the wily, attentive listener will expend more energy searching for the possible hidden presence of the axiom of choice in the proofs than he will in following the positive, constructive aspects of these proofs. Guided by this fact and by the fact that most of the constructions which follow are available in [5] and are in any event familiar to the educated reader, I have chosen to suppress, in what follows, many of the difficult but standard arguments, and to explain in some detail how and why the axiom of choice is unnecessary. The following worthy principle of exposition is, then, abandoned: An argument or a proof should be omitted from a manuscript, if it is more easily established, or logically less essential, than some other argument or proof which has been omitted.

I am indebted to Arthur Stone and Norman Alling for spirited, constructive heckling when this paper was presented to a topology seminar (University of Rochester, Rochester, New York, May, 1967). I am indebted also to Robert Solovay for a stimulating, informal lecture on the relation of the axiom of choice to various other familiar concepts, both algebraic and topological, and for bringing to my attention the papers [3] and [10], to which reference will be made later. The helpful interest of these men should not be construed, however, as an endorsement on their part of the value or the validity of the present paper.

§ 5 discusses briefly several papers related to this one. It also elaborates upon the following fact, a defect of our (modified) compact spaces concerning which the reader deserves to be warned at once: Without the axiom of choice, it cannot be shown that our spaces admit no unbounded real-valued continuous functions.

§ 2. Ordering the quotient fields. The *ring of real-valued continuous functions on the topological space* X will be denoted by CX. The *ring of bounded real-valued continuous functions on* X is denoted by C^*X. The subset Y of X is said to be *C-embedded* in X if each function in CY is the restriction to Y of some function in CX; the expression "Y is C^*-embedded in X" is defined analogously. The set $f^{-1}(0)$, for $f \in CX$, is called the *zero-set* of f and is usually written Zf. For $\eta > 0$ and $f \in CX$ we write $Z_\eta(f) = \{x \in X\colon |f(x)| \leqslant \eta\}$. If f belongs to any (proper) ideal I in C^*X, then the sets $Z_\eta(f)$ are nonempty (since otherwise we would have $1 = f\cdot(1/f) \in I$). If $A \subset CX$ we write, as in [5], $\mathbf{Z}[A] = \{Zf\colon f \in A\}$; and we say that A is *fixed* if $\bigcap \mathbf{Z}[A] \neq \emptyset$.

The reader is reminded that when M is a maximal ideal in CX the quotient field CX/M is linearly ordered by the relation $<$, where by definition

$$f+M < g+M \quad \text{if} \quad f < g \text{ on } Zh \text{ for some } h \in M\,.$$

The field CX/M contains the real field R under the identification $r \to r+M$ with $r \in R$, and the maximal ideal M in CX is called a *real ideal* if $CX/M = R$. These concepts and many related to them are discussed and examined from many points of view in [5], to which the reader is referred for further background.

With the Stone-Čech compactification not at our disposal it is no longer evident that each ring C^*X is isomorphic to a ring of the form CY, so that the definition of $<$ just given does not apply to quotient fields of the form C^*X/M, with M a maximal ideal in C^*X. (Of course, we expect that each such field will be isomorphic to R, in which case an order can easily be described. It is, in fact, precisely this isomorphism which we seek for use in Theorem 3.2*. Unable to achieve a quick, direct proof of this isomorphism, I propose to prove 2.4* below on the basis of the following definition, suggested by a reading of [7].)

2.1*. Definition. Let M be a maximal ideal in C^*X. For $f \in C^*X$ and $g \in C^*X$ we write

$$f+M < g+M$$

provided there exist $\varepsilon > 0$, $\eta > 0$ and $h \in M$ such that $f+\varepsilon < g$ on $Z_\eta h$.

It is not difficult to see that the relation $<$ is well-defined, antisymmetric and transitive. (For example: If $f_1+\varepsilon_1 < f_2$ on $Z_{\eta_1} h_1$ and $f_2+\varepsilon_2 < f_3$ on $Z_{\eta_2} h_2$, then $f_1+\varepsilon < f_3$ on $Z_\eta h$, with $\varepsilon = \varepsilon_1+\varepsilon_2$, $\eta = \min(\eta_1^2, \eta_2^2)$, and $h = h_1^2+h_2^2$.)

The following statements about arbitrary functions f and g in C^*X are verified with similar ease, so that in fact the relation $<$ makes of C^*X/M a partially ordered field:

(a) *if* $f+M > M$ *and* $g+M > M$, *then* $(f+g)+M > M$ *and* $fg+M > M$;

(b) $f+M > g+M$ *if and only if* $(f-g)+M > M$;

(c) *the map* $r \to r+M$ *from* R *into* C^*X/M *is an order-isomorphism embedding* R *as a linearly ordered subfield of the partially ordered field* C^*X/M.

The proof that $<$ linearly orders the field C^*X/M requires two lemmas. For $f \in C^*X$ and $x \in X$ we define

$$f^+x = \max(fx, 0)\,, \qquad f^-x = \max(-fx, 0)\,,$$

so that $f^+ \in C^*X$, $f^- \in C^*X$, $f = f^+ - f^-$, and $f^+ \geqslant 0$, $f^- \geqslant 0$.

2.2*. LEMMA. *If $f \in C^*X$ and M is a maximal ideal in C^*X, then the relations*

$$f^+ + M > M\,, \quad f^- + M > M\,,$$

cannot both hold.

Proof. Otherwise there exist positive numbers ε_k, η_k, and functions h_k in M $(k = 1, 2)$ with

$$f^+ - \varepsilon_1 > 0 \text{ on } Z_{\eta_1} h_1\,, \quad f^- - \varepsilon_2 > 0 \text{ on } Z_{\eta_2} h_2\,.$$

Since $h_1^2 + h_2^2 \in M$, there exists $x \in Z_\eta(h_1^2 + h_2^2)$, where $\eta = \min(\eta_1^2, \eta_2^2)$. Then $x \in Z_{\eta_1} h_1 \cap Z_{\eta_2} h_2$, so that $f^+ x > 0$ and $f^- x > 0$, a contradiction.

In the language of [4], the next lemma asserts simply that each quotient field of the form C^*X/M is quasi-real. The axiom of choice (actually, Zorn's lemma) is used in 8.5 of [4] to obtain a sharpened form of the Artin–Schreier theorem, according to which each quasi-real order on a field extends to a linear order. For our purposes 2.3* alone is not enough, since we are not content to know that $<$ will extend to a linear order; we need to know that it is itself a linear order. This is the content of 2.4*.

2.3*. LEMMA. *If $f \in C^*X$ and M is a maximal ideal in C^*X, then either $f \in M$ or $f^2 + M > M$.*

Proof. If $f \notin M$, then $f^2 \notin M$, so there exists $g \in C^*X$ and $h \in M$ such that

$$1 = f^2 g + h\,.$$

If the relation $f^2 + M > M$ fails, we can show that g is unbounded on X by showing that for each positive integer n there is a point x in X such that $gx \geqslant n-1$. Indeed, if n is given, we need only choose x so that

$$f^2 x \leqslant 1/n \quad \text{and} \quad x \in Z_{1/n} h;$$

for in this case we have

$$(gx)/n \geqslant (f^2 x)(gx) = 1 - hx \geqslant 1 - 1/n = (n-1)/n\,.$$

COROLLARY. *If $0 \leqslant f \in C^*X$ and M is a maximal ideal in C^*X, then either $f \in M$ or $f + M > M$.*

Proof. From the least upper bound property of R it follows without the axiom of choice that f admits a square root $f^{1/2}$ in C^*X. Since $f \in M$ if and only if $f^{1/2} \in M$, the lemma above yields the result.

2.4*. THEOREM. *If M is maximal ideal in C^*X, then the relation $<$ is a linear order on the quotient field C^*X/M. So ordered, C^*X/M is order-isomorphic to the real field.*

Proof. According to 2.1*(b) the linearity of $<$ will be established if it is shown that for each $f \in C^*X$ either $f + M > M$ or $f + M < M$ or $f \in M$. To this end let f be given and write $f = f^+ - f^-$ as in 2.2*. If both $f^+ \in M$ and $f^- \in M$, then $f \in M$. Otherwise, according to Corollary 2.3* and Lemma 2.2*, exactly one of the two relations $f^+ + M > M$, $f^- + M > M$ is valid. In the former case we have $f^- \in M$ so that

$$f + M = f^+ + M > M\,,$$

and in the latter case we have $f + M < M$.

The relation $<$, just shown to be linear on C^*X/M, is easily seen to be Archimedean. (If $f < \boldsymbol{n}$ in C^*X, we cannot claim $f + M < \boldsymbol{n} + M$. But the relation $f + M < (\boldsymbol{n}+\boldsymbol{1}) + M$ is valid, and easily established.)

Thus C^*X/M is an Archimedean (linearly) ordered field containing R, and the theorem follows. (The familiar fact that each Archimedean ordered field admits an Archimedean completion, which is necessarily order-isomorphic to R, is proved in detail, for example, on pages 35-45 of [8]. The proof occasionally requires the term-by-term construction of a sequence but this procedure, because it can be effected by successive appeals to the well-ordering of the positive integers, does not require any form of the axiom of choice.)

§ 3. The Stone-Čech theorem and its Hewitt analogue.

To avoid confusion we shall permit the word *compact* to retain its usual meaning (that is, we say that a space is compact if each of its families of closed subsets with the finite intersection property has non-void intersection). The spaces we propose to consider will be called *compact**, and (as in § 2 above) statements dealing with them will in general be adorned by the asterisk *. The definition of realcompactness is the standard one, due to Hewitt [7].

3.1. DEFINITION. A space X is *realcompact* if it is a completely regular Hausdorff space for which each real maximal ideal in CX is fixed.

3.1*. DEFINITION. A space X is *compact** if it is a completely regular Hausdorff space for which each maximal ideal in C^*X is fixed.

The reader is now invited, before he reads §§ 3 and 4, to prove for ihimself that the product of compact* spaces is compact*. He is allowed to use the fact that the product of completely regular Hausdorff spaces is another such space in the product topology, because this result does not invoke the axiom of choice; for the same reason he may appeal to the usual one-to-one correspondence between maximal ideals in CX and filters of zero-sets in X which are maximal with respect to the finite intersection property. My own attempts to prove this result "directly" have been unsuccessful, and this lends some interest to the proof of Theorem 4.2* and the theorem of Stone–Čech type upon which it depends.

If the reader can respond successfully to the invitation just offered, he will probably find little of interest in the rest of this paper.

3.2. THEOREM. *For each completely regular Hausdorff space X there is a realcompact space vX in which (a homeomorph of) X is dense and C-embedded; the space vX may be chosen homeomorphic with a closed subset of a product of lines.*

3.2*. THEOREM. *For each completely regular Hausdorff space X there is a compact* space βX in which (a homeomorph of) X is dense and C^*-embedded; the space βX may be chosen homeomorphic with a closed subset of a product of closed intervals* $[0,1]$.

Proof. The proof of 3.2 is considerably simpler than the proof of 3.2*; we present here only the latter.

Let $\mathcal{F} = CX \cap [0,1]^X$ and define (as usual) the evaluation map e from X into the product space $P = \prod_{f\in\mathcal{F}} [0,1]_f$ as follows: $(ex)_f = fx$. Because X is a completely regular Hausdorff space, e is a homeomorphism. Now define, as did Čech in [2], $\beta X = \mathrm{cl}_P eX$. Since all else is obvious, we need check only that βX is compact*; this we do, using ideas drawn from Chapter 11 of [5].

For each $f \in C^*X$ we denote by f' that unique element of $C^*(\beta X)$ for which $f'(ex) = fx$ whenever $x \in X$. The fact that eX is C^*-embedded in βX guarantees that $\{f'\colon f \in C^*X\} = C^*(\beta X)$; and the fact that eX is dense in βX guarantees that $f'q = q_f$ whenever $f \in \mathcal{F}$ and $q \in \beta X$. Now let M' be a maximal ideal in $C^*(\beta X)$, so that M' has the form $M' = \{f'\colon f \in M\}$ for some maximal ideal M in C^*X, and let φ be the homomorphism which associates with each f in C^*X that real number r for which $f+M = r+M$. Denoting by $\bar\varphi$ the restriction of φ to $\mathcal{F}$, we have by 2.4* that $\bar\varphi \in [0,1]^{\mathcal{F}}$. (Without 2.4* we would be unable to conclude that for each $f \in \mathcal{F}$ there exists $r \in [0,1]$ for which $\bar\varphi f = r+M$.) We will show that M' is fixed by proving that $\bar\varphi \in \beta X$. To this end, let

$$U = \{q \in [0,1]^{\mathcal{F}}\colon |q_{f_k} - \bar\varphi_{f_k}| < \varepsilon\} \qquad (1 \leqslant k \leqslant n)$$

be a basic neighborhood in $[0,1]^{\mathcal{F}}$ of $\bar\varphi$ and write

$$f = \sum_{k=1}^{n} (f_k - \bar\varphi_{f_k})^2/n,$$

so that $f \in \mathcal{F}$ and $\bar\varphi f = 0$, i.e., $f \in M$. It does not follow that $Zf \neq \emptyset$, but evidently $Z_\eta f \neq \emptyset$ for each $\eta > 0$ (since otherwise f is invertible in C^*X.) If x is a point in X for which $|fx| < \varepsilon^2/n$, then

$$|f_k x - \bar\varphi_{f_k}| < \varepsilon \qquad (1 \leqslant k \leqslant n),$$

so that

$$|(ex)_{f_k} - \bar\varphi_{f_k}| < \varepsilon \qquad (1 \leqslant k \leqslant n),$$

that is, $ex \in U \cap eX$. Thus each neighborhood of $\bar\varphi$ in $[0,1]^{\mathcal{F}}$ hits eX, so $\bar\varphi \in \beta X$.

Now let $h' \in M'$ (with, say, $|h'| \leqslant 1$ on βX). Then $h'(\bar\varphi) = \bar\varphi_h = \varphi h = 0$, so that $\bar\varphi \in \bigcap \boldsymbol{M}[M']$. The proof that βX is compact* is complete.

The symbol e having served its purpose, we shall suppress it henceforth. We shall, then, regard X itself as a subset of vX and of βX, and we shall write $X \subset vX$ and $X \subset \beta X$.

Any worthy treatment of the classical Stone-Čech theory, necessarily effected with the aid of the axiom of choice, includes a theorem asserting that a compact space is its own compactification; and a theorem asserting that a continuous function into any compact Hausdorff space whatever extends to a continuous function whose domain is the constructed Stone–Čech compactification. Likewise Hewitt in [7], and following him Gillman–Jerison in [5], show that realcompact spaces are (in our terminology) their own Hewitt realcompactifications; and that a continuous function on X to any realcompact space whatever extends to a continuous function with domain vX. We wish to prove these four theorems (the "compact" statements modified so that they deal with our compact* spaces) without the axiom of choice. The desired theorem of Tychonoff type will then follow with ease.

3.3. THEOREM. (a) *If X is realcompact, then $X = vX$;* (b) *Each realcompact space is (homeomorphic with) a closed subset of a product of real lines.*

3.3*. THEOREM. (a) *If X is compact*, then $X = \beta X$;* (b) *Each compact* space is (homeomorphic with) a closed subset of a product of intervals* $[0,1]$.

Proof. In the presence of 3.2 and 3.2*, each part (b) will follow from the corresponding part (a). Again we will prove 3.3*(a), the necessary modifications for 3.3(a) being straightforward.

Suppose then that $p \in \beta X \setminus X$, and set

$$M = \{f \in C^*(\beta X)\colon fp = 0\}.$$

The map $f \to \tilde f$, where $\tilde f$ denotes the restriction of f to X, is an isomorphism from $C^*(\beta X)$ onto C^*X which associates with M a maximal ideal $\tilde M$ in C^*X. Since X is compact*, there is a point x in X with $x \in \bigcap \boldsymbol{Z}[\tilde M]$. From the complete regularity of βX there is a function $g \in C^*(\beta X)$ with $gx \neq 0$ and $gp = 0$. But then $\tilde g \in \tilde M$ (since $g \in M$) and $\tilde g \notin \tilde M$ (since $\tilde g x \neq 0$).

The proof of the familiar assertion that a maximal ideal M in CX is real if and only if the collection $\boldsymbol{Z}[M]$ has the countable intersection property seems to require a weak form of the axiom of choice. The statement can be recast slightly, and the axiom of choice avoided, as follows.

3.4. THEOREM. *For a maximal ideal M in CX the following conditions are equivalent*:

(a) *M is real*;

(b) *if $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions drawn from M, then $\bigcap_{n=1}^{\infty} Z(f_k) \neq \emptyset$.*

Proof. (a) $\Rightarrow$ (b). If (b) fails for $\{f_k\}_{k=1}^{\infty}$, we let $g_k = f_k^2 \wedge \mathbf{1}$, so that $Z(g_k) = Z(f_k)$ and $g_k \in M$. With $g = \sum_{k=1}^{\infty}(g_k/2^k)$ we have, as in 5.14 of [5], that $g > 0$ on X so that $g+M > 0+M$; yet for each $\varepsilon > 0$ there is a function $h \in M$ with $g < \varepsilon$ on Zh, so that $g+M = 0+M = M$ if CX/M is the real field.

(b) $\Rightarrow$ (a). If M is not real, there exists f in CX with $f+M > \mathbf{k}+M$ for each positive integer k, and with $f \geqslant 0$ on X. Defining $f_k = (f\wedge \mathbf{k})-\mathbf{k}$ we have $f_k \in M$ for each positive integer k. (To check this for a given k, find $h_k \in M$ with $f > k$ on $Z(h_k)$. Then $f_k \equiv 0$ on $Z(h_k)$.) Then

$$\bigcap_{k=1}^{\infty} Z(f_k) = \{x \in X\colon fx \geqslant k \text{ for each } k\} = \emptyset .$$

3.5. THEOREM. *If X is dense and C-embedded in the realcompact space X', and if f maps X continuously into the realcompact space Y, then some continuous functions $\tilde{f}$ from X' into Y agrees with f on X.*

3.5*. THEOREM. *If X is dense and C^*-embedded in the compact* space X', and if f maps X continuously into the compact* space Y, then some continuous function $\tilde{f}$ from X' into Y agrees with f on X.*

Proof. For the sake of variety, and because it is by a slight margin the more difficult, we give the proof of 3.5, on the basis of which the reader can easily prove 3.5* for himself.

Given the continuous function f from X into Y, we associate with each $g \in CY$ that (unique) continuous function $(g\circ f)'$ defined on X' whose restriction to X is $g\circ f$.

If $p \in X'\backslash X$, we define

$$M(p) = \{g \in CY\colon (g\circ f)'p = 0\} .$$

If $g_1 \in CY$ and $g_2 \in CY$, we have

$$(g_1+g_2)\circ f \equiv g_1\circ f + g_2\circ f \quad \text{and} \quad (g_1\circ f)\cdot(g_2\circ f) \equiv (g_1 g_2)\circ f$$

on X, so that $M(p)$ is an ideal in CY. The following computation that $M(p)$ is maximal is routine (though the assertion itself is a trifle surprising, since one is not accustomed to constructing maximal ideals

without the axiom of choice): If $g \notin M(p)$, so that $(g\circ f)'p = r$ for some real number $r \neq 0$, then the function $\mathbf{1}$ can be expressed in the form

$$\mathbf{1} = \frac{\mathbf{r}-g}{\mathbf{r}} + \frac{g}{\mathbf{r}};$$

thus no (proper) ideal in CY can contain both $M(p)$ and g.

The proof that $M(p)$ is real will be based on Theorem 3.4. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of functions in $M(p)$ with, say, $0 \leqslant g_k \leqslant 1/2^k$. Defining

$$h_k = (g_k\circ f)'$$

and

$$M^p = \{h \in C(X')\colon hp = 0\}$$

we see that each $h_k \in M^p$, so that $\sum_{k=1}^{\infty} h_k$, which we will call h, also lies in M^p. It follows that $Zh \cap X \neq \emptyset$ since otherwise the restriction of h to X has a continuous real-valued reciprocal unbounded on each neighborhood in X' of p, contrary to the fact that X is C-embedded in X'. If $x \in Zh \cap X$, then $fx \in \bigcap_{k=1}^{\infty} Z(g_k)$.

We have shown that for each $p \in X'\backslash X$ the ideal $M(p)$ in CY is maximal and real. Because Y is realcompact the ideal $M(p)$ is fixed and with $y \in \bigcap \boldsymbol{Z}[M(p)]$ we define $\tilde{f}p = y$. (The axiom of choice is not required in this selection process because, $M(p)$ being maximal in CY, the set $\bigcap \boldsymbol{Z}[M(p)]$ contains exactly one point.) The continuity of $\tilde{f}$ is now easily verified. The reader may achieve it directly, or he may refer to 6H of [5], according to which any function $\tilde{f}$, continuous on each set of the form $X \cup \{p\}$, where X is dense in the domain of $\tilde{f}$, is automatically continuous at each point of its domain. Our proof is complete.

The Hewitt realcompactification, and the classical Stone–Čech compactification, associated with a given completely regular Hausdorff space are well-known and easily shown to be unique. It is worthy of formal mention at this point that our spaces vX and βX, which were constructed without the axiom of choice and which satisfy the hypotheses of 3.5 and 3.5*, respectively, are likewise unique. The proofs follow from a familiar application of 3.5 and 3.5*, and they are therefore omitted.

3.6. THEOREM. *If $v'X$ is a realcompact space in which X is dense and C-embedded, then there is a homeomorphism from vX onto $v'X$ leaving X fixed pointwise.*

3.6*. THEOREM. *If $\beta'X$ is a compact* space in which X is dense and C^*-embedded, then there is a homeomorphism from βX onto $\beta'X$ leaving X fixed pointwise.*

§ 4. A theorem of Tychonoff type and its realcompact analogue.

4.1. THEOREM. *The real line R is realcompact.*

4.1*. THEOREM. *The closed interval $[0,1]$ is compact*.*

Proof of 4.1. It is easy to show that each Lindelöf space is realcompact, but unfortunately the proof that R is Lindelöf seems to require the use of a weak form of the axiom of choice. We can, however, establish the following assertion without using the axiom of choice: Each open cover of R admits a countable open refinement. To see this, let $\{B_n\}_{n=1}^{\infty}$ be any countable base for R and let $\mathfrak{U}$ be any open cover. Then

$$\{B_n:\ B_n \subset U \text{ for some } U \text{ in } \mathfrak{U}\}$$

is a refinement of $\mathfrak{U}$ which, being a subset of the countable family $\{B_n\}_{n=1}^{\infty}$, is itself countable.

To complete the proof that R is realcompact, let M be a maximal ideal in CR. If $\bigcap \boldsymbol{Z}[M] = \emptyset$, then the family $\{R\backslash Zf:\ f \in M\}$ is an open cover of R which, by the preceding paragraph, admits a countable open refinement $\{A_n\}_{n=1}^{\infty}$. (Our notation indicates that this countable refinement is countably infinite; if it is not, the notation can be modified appropriately.) We claim that for each positive integer n the function f_n, defined explicitly by the relation $f_n(x) = \varrho(x, R\backslash A_n)$, where ϱ is the usual metric on R, lies in M. When n is given we have $Z(f_n) = R\backslash A_n \supset Zf$ for some $f \in M$, so that $f_n \in M$ as desired. From Theorem 3.4 we now see that each maximal ideal in CR, if not fixed, is not real, i.e., R is realcompact.

Proof of 4.1*. Let M be a maximal ideal in $C^*[0,1]$. We must show $\bigcap \boldsymbol{Z}[M] \neq \emptyset$. For each $h \in M$ and $\varepsilon > 0$ we know that $Z_\varepsilon h \neq \emptyset$, and indeed the family $\{Z_\varepsilon h:\ h \in M,\ \varepsilon > 0\}$ is a family of closed subsets of $[0,1]$ with the finite intersection property. Since $\bigcap \boldsymbol{Z}[M] = \bigcap \{Z_\varepsilon h:\ h \in M, \varepsilon > 0\}$ it will evidently suffice, in order to show that $[0,1]$ is compact*, to show that $[0,1]$ is compact (in the usual sense). The usual proof does not require the axiom of choice: Given an open cover $\mathfrak{U}$ of $[0,1]$ one defines

$$S = \{r \in [0,1]:\ [0,r] \text{ is covered by finitely many elements of } \mathfrak{U}\}$$

and, with $s = \sup S$, one shows $s = 1$ and $s \in S$.

The reader may wonder, as he reads the next proof, whether we are asserting that the product of non-void sets is nonvoid, an assertion equivalent to the axiom of choice. We certainly do not claim to have proved such a statement; it seems possible that the product of non-void compact* spaces considered in the next proof is in fact void; we assert simply that it is compact*. On the other hand, it is well to remember that certain product spaces are not empty. If $X_a = [0,1]$ for each a in the index set A, then the product space $\prod_{a\in A} X_a$ is non-void. It contains, for example, the point x defined by the rule $x_a = 0$ for each $a \in A$.

4.2. THEOREM. *Let A be a set, let X_a be a realcompact space for each $a \in A$, and define $X = \prod_{a\in A} X_a$. Then X is realcompact.*

4.2*. THEOREM. *Let A be a set, let X_a be a compact* space for each $a \in A$, and define $X = \prod_{a\in A} X_a$. Then X is compact*.*

Proof. We give only the proof for 4.2*, the other being very similar indeed.

The space X, like every product of completely regular Hausdorff spaces, is itself completely regular Hausdorff. The space βX has been defined in 3.2*, and shown to be compact*. Thus it will suffice to show that $X = \beta X$.

As in 6.8 of [5] we denote (for each a) by π_a the projection from X into X_a (it is possible that $\pi_a X \subsetneq X_a$); and we denote by $\tilde{\pi}_a$ its continuous extension from βX to X_a. Now let f be the function defined on βX by the relation

$$(fp)_a = \tilde{\pi}_a(p) .$$

Evidently f maps βX into the product space X, and f is continuous because its composition with each of the projection mappings π_a is the continuous map $\tilde{\pi}_a$. Furthermore, $fx = x$ for each $x \in X$. Thus f retracts βX onto its dense subset X, and therefore $X = \beta X$.

The final theorems of this paper characterize the spaces we have been studying. Theorem 4.4 is familar (see [7] or Chapter 11 of [5]) and Theorem 4.4* is the compact* analogue of a familiar theorem about compact Hausdorff spaces, evidently equivalent to it in the presence of the axiom of choice. We need a lemma.

4.3. LEMMA. *Let F be a closed subset of the realcompact space X. Then F is realcompact.*

4.3*. LEMMA. *Let F be a closed subset of the compact* space X. Then F is compact*.*

Proof. For the realcompact case we construct, just as in 4.2, a retraction f from vF onto F. Its value at $p \in vF$ is $fp = \tilde{i}p$, where $\tilde{i}$ is the extension guaranteed by 3.5 of the identity function i mapping F into X; the fact that $\tilde{i}F = F$ follows from the continuity of $\tilde{i}$ and the fact that F is closed in X.

The proof of 4.3* is similar.

4.4. THEOREM. *Let X be a completely regular Hausdorff space. Then X is realcompact if and only if X is (homeomorphic to) a closed subset of a product of real lines.*

4.4*. THEOREM. *Let X be a completely regular Hausdorff space. Then X is compact* if and only if X is (homeomorphic to) a closed subset of a product of closed intervals* $[0,1]$.

Proof of 4.4. The "if" implication is given by Theorem 4.1 and 4.3. The "only if" implication is 3.3(b).

The proof of 4.4* is similar.

§ 5. Concluding remarks. The theorem of Tychonoff type given above may be viewed as a result provable without the axiom of choice because we have in effect incorporated one use of this axiom into our definition and shown that every other use (in the usual proofs) is avoidable. Another theorem in this same general spirit is offered by Peter A. Loeb in [11].

I have already alluded to Kelley's [9], where it is shown that the axiom of choice can be deduced from Tychonoff's theorem (for compact spaces in the usual sense). Without the axiom of choice it is impossible to impose upon each member X_α of a set of nonvoid sets $\{X_\alpha: \alpha \in A\}$ a compact Hausdorff topology, although if a distinguished point p_α is given in X_α (by an application of the axiom of choice, say) then one can make X_α Alexandroff's one-point compactification of the discrete space $X_\alpha\backslash\{p_\alpha\}$; Halpern has shown in [6], in effect, that the axiom of choice does not follow from Tychonoff's theorem for compact Hausdorff spaces. (A closely related question had been posed by J. D. Weston in [17].) J. Łoś and C. Ryll-Nardzewski observe in [12] that Kelley's proof shows that the Tychonoff theorem for compact Hausdorff spaces implies the axiom of choice for non-void compact Hausdorff spaces.

Conditions equivalent to the axiom of choice are given in the well-known Rubin-Rubin book [13]; conditions equivalent to one of its weaker consequences in [1]; see also [18].

Sierpiński has shown in [15] how one can construct without the axiom of choice, if he is given a free maximal ideal in CN (with N the countably infinite discrete space) a subset of the real line which is not Lebesgue measurable. A nice exposition of Sierpiński's argument, and an extension of his theorem, is given by Semadeni in [14].

The reader probably noticed in passing that an argument contained in the proof of Theorem 4.1* above shows that in general a completely regular compact Hausdorff space is compact*. One can also show, again without the axiom of choice, that a compact Hausdorff space is normal. (One establishes regularity first, by considering, for a given $p \in X$, the family

$$\mathfrak{U} = \{U \subset X: U \text{ is open and } p \notin \mathrm{cl}_X U\},$$

which covers each closed set not containing p. A similar gambit now establishes normality.) Läuchli constructs in [10] a model, in which the axiom of choice fails, containing a locally compact, normal Hausdorff space with more than one point, each of whose continuous real-valued functions is constant. It follows that Urysohn's lemma cannot be proved without the axiom of choice, and in fact that a compact Hausdorff space need not be completely regular.

Feferman in [3] has shown that it is consistent with Zermelo–Fraenkel set theory that each ultrafilter on N is fixed. Solovay has shown (unpublished) that in Feferman's model the principle of dependent choices (from which follow both the axiom of choice for countable families of non void sets and Urysohn's lemma) is valid. Feferman's result, although it is in consonance with those of the present paper, points up a deficiency in the concept we have used to replace compactness: Our compact* spaces cannot be shown to be pseudocompact (i.e., we cannot show that each real-valued continuous function on each compact* space is bounded). For N will be compact* if each ultrafilter on it is fixed, yet there surely exists an unbounded real-valued continuous function on N.

References

[1] R. L. Blair and M. L. Tomber, *The axiom of choice for finite sets*, Proc. Amer. Math. Soc. 11 (1960), pp. 222-226.

[2] Eduard Čech, *On bicompact spaces*, Annals of Math. 38 (1937), pp. 823-844.

[3] S. Feferman, *Some applications of the notions of forcing and generic sets*, Fund. Math. 56 (1964/65), pp. 325-345.

[4] N. J. Fine, L. Gillman, J. Lambek, *Rings of quotients of rings of functions*, Montreal 1966.

[5] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Princeton, N. J., 1960.

[6] J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. 55 (1964), pp. 57-66.

[7] Edwin Hewitt, *Rings of real-valued continuous functions I*, Trans. Amer. Math. Soc. 64 (1948), pp. 45-99.

[8] Edwin Hewitt and Karl Stromberg, *Real and abstract analysis*, New York 1955.

[9] John L. Kelley, *The Tychonoff theorem implies the axiom of choice*, Fund. Math. 37 (1950), pp. 75-76.

[10] H. Läuchli, *Auswahlaxiom in der Algebra*, Commentarii Math. Helvetici 37 (1962-63), pp. 1-18.

[11] Peter A. Loeb, *A new proof of the Tychonoff theorem*, Amer. Math. Monthly 72 (1965), pp. 711-717.

[12] J. Łoś and C. Ryll-Nardzewski, *Effectiveness of the representation theory for Boolean algebras*, Fund. Math. 41 (1955), pp. 49-56.

[13] H. Rubin and J. Rubin, *Equivalents of the axiom of choice*, Amsterdam 1963.

[14] Z. Semadeni, *Periods of measurable functions and the Stone–Čech compactification*, Amer. Math. Monthly 71 (1964), pp. 891-893.

[15] W. Sierpiński, *Fonctions additives non complètement additives et fonctions non mésurables*, Fund. Math. 30 (1938), pp. 96-99.

[16] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), pp. 375-481.

[17] J. D. Weston, *A generalization of Ascoli's theorem*, Mathematika 6 (1959), pp. 19-24.

[18] E. S. Wolk, *On theorems of Tychonoff, Alexander and R. Rado*, Proc. Amer. Math. Soc. 18 (1967), pp. 113-115.

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