Hyperspaces and symmetric products of topological spaces *

by

R. M. Schori (Baton Rouge, Louisiana)

1. Introduction. This paper deals with certain hyperspaces of a topological space X. One such space is the hyperspace of all non-void closed subset of X, denoted by 2^X, and another is the collection of all non-void subsets of X containing at most n points ($n \geqslant 1$) of X. This is denoted by $X(n)$ and is called the n-fold symmetric product of X. Of primary concern will be the case where X is the closed unit interval I, the n-cell I^n, or the Hilbert cube I^∞.

In [1], Borsuk and Ulam introduced the notion of symmetric product and studied it with two distinct approaches. One was to see what topological properties are preserved in the operation of taking a topological space X to its n-fold symmetric product $X(n)$ and the second approach was to study the topological properties of some specific $X(n)$, notably $I(n)$. As an example of the second approach, they proved for $n = 1, 2$, or 3 that $I(n)$ is homeomorphic to $I^n (I(n) \approx I^n)$ and for $n \geqslant 4$ that $I(n)$ is not homeomorphic with any subset of R^n. Ganea [4] continued the study of symmetric products by essentially the first approach and Molski [5] followed the second approach. He showed that $I^2(2) \approx I^4$ and for $n \geqslant 3$ that neither $I^2(n)$ nor $I^n(2)$ is homeomorphic with any subset of R^{2n}. In [3] Bott corrected Borsuk's [2] statement that $S^1(3) \approx S^1 \times S^2$ by showing that actually $S^1(3) \approx S^3$.

In a somewhat different direction, Wu [6] shows that, for n odd, $S^1(n)$ is a homology n-sphere and, for n even, $H^0 = Z$, $H^{n-1} = Z$, and for $j \neq 0$, $n-1$, $H^j = 0$.

Thus, the positive results for giving topological characterizations for a non-trivial n-fold symmetric product $X(n)$ have been limited to spaces $X = S^1$, I, or I^2 and $n \leqslant 3$. This paper gives some techniques in this topological characterization problem for extending the space X to I^m ($m = \infty, 1, 2, \ldots$) and n to $\infty, 1, 2, \ldots$ (notation: $X(\infty) = 2^X$).

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Key techniques come from § 3 on cones where it is proved (Theorem 1) that an interval can be factored out of a double cone (in fact, for $n \geqslant 2$, $C^n(X) \approx CX \times I^{n-1}$) and from theorems in § 4 showing that $I(n)$ ($n = \infty, 2, 3, \ldots$) is a double cone. In addition it is hoped that these techniques will help settle the conjecture (¹) that $2^X \approx I^\infty$ for each Peano space X. In this direction Theorem 9 of this paper says that $I^\infty(n) \approx I^\infty(n) \times \times I^\infty$ ($n = 1, 2, \ldots$). Recent methods of R. D. Anderson suggest that if Y is a contractible complex, then $Y \times I^\infty \approx I^\infty$. These methods might also imply that $I^\infty(n) \times I^\infty \approx I^\infty$.

2. Preliminaries. For a topological space X we now define topologies for 2^X and $X(n)$. For that which immediately follows, let Y stand for either 2^X or $X(n)$. If $G_1, \ldots, G_k$ ($k \geqslant 1$) are open sets in X, let

$$U(G_1, \ldots, G_k) = \{A \in Y\colon A \subset \bigcup_{i=1}^{k} G_i \text{ and } A \text{ intersects each } G_i\}.$$

The collection of all such $U(G_1, \ldots, G_k)$ is a basis for a topology on Y that is called the *Vietoris finite topology*. Note that if X is T_1, then this topology on $X(n)$ coincides with the relativized topology on $X(n)$ where $X(n)$ is viewed as a subspace of 2^X.

If (X, d) is a bounded metric space, then $D\colon 2^X \times 2^X \to R$ defined by

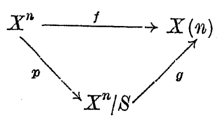
$$D(A, B) = \inf\{\varepsilon\colon A \subset S(B, \varepsilon) \text{ and } B \subset S(A, \varepsilon)\},$$

where $S(C, \varepsilon)$ is the ε-sphere about $C \subset X$, is a metric on 2^X, and thus on $X(n)$, and is called the *Hausdorff metric*. It is well known that if X is compact then the Vietoris finite topology on 2^X coincides with the topology induced by the Hausdorff metric on 2^X.

The following gives a useful characterization of $X(n)$ and motivates the name "n-fold symmetric product" for the space $X(n)$. Let X^n be the cartesian product of n copies of X and define $f\colon X^n \to X(n)$ by $f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$. Let S be the equivalence relation on X^n associated with f, that is

$$(x_1, \ldots, x_n) S (y_1, \ldots, y_n) \quad \text{iff} \quad f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n).$$

Thus, two points of X^n are identified if the point sets of their corresponding coordinates are equal. Let $p\colon X^n \to X^n/S$ be the natural projection of X^n onto the quotient space X^n/S and denote $p(x_1, \ldots, x_n)$ by $[x_1, \ldots, x_n]$. Define $g\colon X^n/S \to X(n)$ by $g[x_1, \ldots, x_n] = \{x_1, \ldots, x_n\}$. By [1], p. 877, f is continuous.



(¹) I first heard this conjecture in the form of $2^I \approx I^\infty$ from R. D. Anderson.

The following theorem of Ganea ([4], Hilfssatz 2) characterizes $X(n)$ (for T_1 space X) as a quotient of X^n.

THEOREM 0. *If X is T_1, then*

$$g\colon X^n/S \to X(n)$$

is a homeomorphism.

3. Cones. This section is developed independently from hyperspaces and symmetric products but the results are used extensively in the rest of the paper.

If X is a topological space, by the *cone* of X, denoted by CX or $C(X)$, we mean $(X \times I)/R$ where R is the equivalence relation on $X \times I$ defined by

$$(x, s) R (y, t) \quad \text{iff} \quad (x, s) = (y, t) \quad \text{or} \quad s = t = 1.$$

Geometrically, we take $X \times I$ and "shrink" $X \times 1$ to a point. Henceforth, if we say let $CX = (X \times I)/R$ be the cone of X, we mean that R is the equivalence relation defined above. Also, if $x \in X$ and $t \in I$, then $[x, t]$ is the equivalence class in CX with representative (x, t). Define $C^n(X)$ by

(1) $C^1(X) = CX$ and

(2) $C^n(X) = C(C^{n-1}(X))$.

THEOREM 1. *If X is a topological space and $n \geqslant 2$, then*

$$C^n(X) \approx CX \times I^{n-1}.$$

Proof. It is sufficient to prove the theorem for $n = 2$ since the general case follows from successive applications of this special case. Let

$$CX = (X \times I)/P, \qquad CCX = (CX \times I)/Q, \qquad \text{and} \qquad CI = I^2/T$$

be the cones of X, CX, and I, respectively. Define

$$\alpha\colon X \times I^2 \to CX \times I$$

by $\alpha(x, s, t) = ([x, s], t)$. Thus, α is continuous and onto. If R_1 is the equivalence relation associated with α, then

$$(x, s, t) R_1 (x', s', t') \quad \text{iff} \quad (x, s, t) = (x', s', t') \text{ or } s = s' = 1 \text{ and } t = t'.$$

Let

$$\beta\colon CX \times I \to CCX$$

be the natural projection and define

$$\gamma\colon X \times I^2 \to CCX$$

by $\gamma = \beta \circ \alpha$. Thus, γ is continuous and onto and is described by $\gamma(x, s, t)$

$= [[x, s], t]$. Let R be the equivalence relation on $X \times I^2$ associated with γ. It is routine to check that

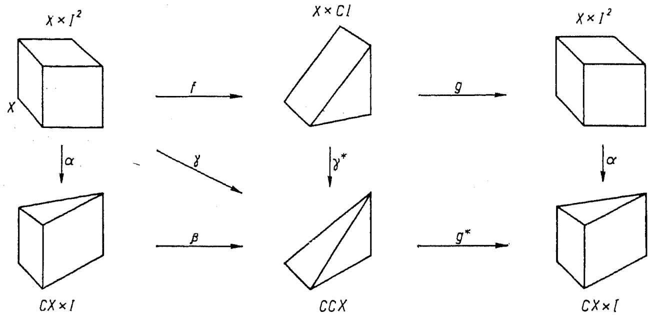
$$(x, s, t) R (x', s', t') \quad \text{iff} \quad (x, s, t) = (x', s', t'), \text{ or } s = s' = 1 \text{ and } t = t', \text{ or } t = t' = 1.$$

Define

$$f\colon X \times I^2 \to X \times CI$$

by $f(x, s, t) = (x, [s, t])$. Thus, f is continuous and onto. If R_2 is the equivalence relation associated with f, we have

$$(x, s, t) R_2 (x', s', t') \quad \text{iff} \quad (x, s, t) = (x', s', t'), \text{ or } x = x' \text{ and } t = t' = 1.$$



We now show that $R_2 \subset R$. Let $a = (x, s, t) R_2 (x', s', t') = b$. If $a = b$, then aRb. Otherwise $x = x'$ and $t = t' = 1$, so $t = t' = 1$ and thus aRb. Thus γ induces a function

$$\gamma^*\colon X \times CI \to CCX$$

and γ^* is described by $\gamma^*(x, [s, t]) = [[x, s], t]$. Thus, γ^* is continuous and onto. If R_3 is the equivalence relation associated with γ^*, then

$$(x, [s, t]) R_3 (x', [s', t']) \quad \text{iff} \quad (x, [s, t]) = (x', [s', t']), \text{ or } s = s' = 1 \text{ and } t = t'.$$

Let $g'\colon CI \to I^2$ be a homeomorphism such that $g'[s', t'] = (1, t)$ iff $s' = 1$ and $t = t'$. Define

$$g\colon X \times CI \to X \times I^2$$

by $g = \mathrm{id} \times g'$ and thus, g is a homeomorphism. Define R_4 on $X \times I^2$ by $g(a) R_4 g(b)$ iff aR_3b. It is easy to verify that $R_1 = R_4$ and thus let

$$\alpha\colon X \times I^2 \to CX \times I$$

be the natural projection.

Since aR_3b iff $g(a) R_1 g(b)$, the homeomorphism g induces a homeomorphism

$$g^*\colon CCX \to CX \times I. \quad \square$$

Henceforth, CCX will be regarded as $(X \times I^2)/R$ where R is defined above.

Let S be an equivalence relation on a space X. Then the induced equivalence relation on $CX = (X \times I)/R$, denoted S_1, is defined by $[x, s] S_1 [x', s']$ iff $[x, s] = [x', s']$, or xSx' and $s = s'$. The induced equivalence relation on $CCX = (X \times I^2)/R$, denoted by S_2, is defined by

$$[x, s, t] S_2 [x', s', t'] \quad \text{iff} \quad [x, s, t] = [x', s', t']; \quad \text{or} \quad xSx', \ s = s', \text{ and } t = t'.$$

THEOREM 2. *Let S be an equivalence relation on a space X and let S_1 and S_2 be the induced equivalence relations on CX and CCX, respectively. Then*

$$(CCX)/S_2 \approx (CX)/S_1 \times I.$$

Proof. By Theorem 1 we have

$$CC(X/S) \approx C(X/S) \times I.$$

We now show that $CC(X/S) \approx (CCX)/S_2$ and $C(X/S) \approx (CX)/S_1$. Define

$$\nu\colon CCX \to CC(X/S) \quad \text{and} \quad \mu\colon CX \to C(X/S)$$

by $\nu[x, s, t] = [[x], s, t]$ and $\mu[x, s] = [[x], s]$. Thus, ν and μ are continuous and onto and it is an easy step to show that S_2 and S_1 are the equivalence relations associated with ν and μ, respectively. Thus

$$(CCX)/S_2 \approx CC(X/S) \text{ and } C(X/S) \approx (CX)/S_1. \quad \square$$

THEOREM 3. (i) *If $n \geqslant 1$, then*

$$A_n = \{(x_i) \in I^n\colon \text{for some } j, x_j = 0\} \text{ is homeomorphic to } I^{n-1}.$$

(ii) *If $n \geqslant 2$ and $B_n = \{(x_i) \in I^n\colon$ for some $j, x_j = 0$ and for some $k, x_k = 1\}$, then $A_n \approx CB_n$ and $I^n \approx CA_n \approx CCB_n$.*

Proof. The proof of (i) is, among other places, contained in the folk-lore of combinatorial topology.

For the proof of (ii) we start by mentioning that if X is a subset of a linear space E, $q \in E \setminus X$, $(1-s)x + sq \notin X$ for each $x \in X$ and $s \in (0, 1)$, and

$$E(X, q) = \{(1-s)x + sq\colon x \in X \text{ and } 0 \leqslant s \leqslant 1\},$$

then $CX = (X \times I)/R_0 \approx E(X, q)$ and $f\colon CX \to E(X, q)$ defined by $f[x, s] = (1-s)x + sq$ is a homeomorphism. Thus, if $E = E^n$, $X = B_n$, and

$q = (0, 0, \ldots, 0)$, the above conditions are satisfied and hence $CB_n \approx E^n(B_n, q)$. But $A_n = E^n(B_n, q)$ and thus $CB_n \approx A_n$ where $f\colon CB_n \to A_n$ defined by $f[x, s] = (1-s)x + sq = (1-s)x$ is a homeomorphism.

Likewise, $I^n = E^n(A_n, p)$ for $p = (1, 1, \ldots, 1)$ and thus $CA_n \approx I^n$ where $g\colon CA_n \to I^n$ defined by $g[y, t] = (1-t)y + tp$ is a homeomorphism. Thus $h\colon CCB_n = (B_n \times I^2)/R \to I^n$ defined by

$$h[x, s, t] = g\big[f[x, s], t\big] = (1-t)f[x, s] + tp = (1-t)(1-s)x + tp$$

is a homeomorphism. □

4. Back to hyperspaces. Let $I_0(n)$ be the subspace of $I(n)$ consisting of all elements of $I(n)$ containing 0, and let $I_0^1(n)$ be the subspace of $I(n)$ consisting of all elements of $I(n)$ containing both 0 and 1.

THEOREM 4. $I_0(n) \approx CI_0^1(n)$ $(n = \infty, 2, 3, \ldots)$.

Proof. Let $CI_0^1(n) = \big(I_0^1(n) \times I\big)/R$ be the cone of $I_0^1(n)$. Define $h\colon CI_0^1(n) \to I_0(n)$ by $h[A, t] = (1-t)A$ where $A \in I_0^1(n)$ and $t \in I$. Thus if $[A, t] \in CI_0^1(n)$, then A is a set in $I = [0, 1]$ that contains both 0 and 1 and h shrinks A to the left to fit the interval $[0, 1-t]$.

To show that h is a homeomorphism we define $g\colon I_0(n) \to CI_0^1(n)$ as follows. For $A \in I_0(n)$, let $s = \sup A$, let $g(A) = [(1/s)A, 1-s]$ if $s \neq 0$, and let $g(A) = [A, 1]$ if $s = 0$. Note that $[A, 1] = [B, 1]$, for any $A, B \in I_0^1(n)$, is the cone point of $CI_0^1(n)$. Thus, $g \circ h = \mathrm{id}$ and $h \circ g = \mathrm{id}$ and since the continuity of both is clear, h is a homeomorphism. □

THEOREM 5. $I(n) \approx CI_0(n)$ $(n = \infty, 1, 2, \ldots)$.

Proof. Let $CI_0(n) = \big(I_0(n) \times I\big)/R$ be the cone of $I_0(n)$. Define $h\colon CI_0(n) \to I(n)$ by $h[A, t] = (1-t)A + t$ where $A \in I_0(n)$ and $t \in I$. Thus if $[A, t] \in CI_0(n)$, then A is a set in $I = [0, 1]$ that contains 0, and h shifts A to the right t units and shrinks it to fit the interval $[t, 1]$. Thus the sets containing t in $[t, 1]$ come from the sets containing 0 in $[0, 1]$ at the tth level in $CI_0(n)$.

To show that h is a homeomorphism we define $g\colon I(n) \to CI_0(n)$ as follows. For $A \in I(n)$, let $s = \inf A$, let $g(A) = \big[\big(1/(1-s)\big)(A-s), s\big]$ if $s \neq 1$, and let $g(A) = [A, 1]$ if $s = 1$. Thus $g \circ h = \mathrm{id}$ and $h \circ g = \mathrm{id}$ and since both are clearly continuous, h is a homeomorphism. □

Theorem 5 is actually a special case of the following statement that is called a corollary. The natural order has been reversed to keep from obscuring the proof of Theorem 5.

COROLLARY. *A hyperspace of a cone is a cone. That is, if $Y = CX = (X \times I)/R$ is the cone of a space X, and $n = \infty, 1, 2, \ldots$, then*

$$Y(n) \approx CY_0(n) \quad \textit{where} \quad Y_0(n) = \{A \in Y(n)\colon A \cap [X \times 0] \neq \emptyset\}.$$

Proof. We will show how to modify the proof of Theorem 5. If $y \in A \in Y_0(n)$, then $y = [x, s]$ where $x \in X$ and $s \in I$. If $A \in Y_0(n)$ and $t \in I$, let

$$tA = \{[x, ts]\colon [x, s] \in A\} \quad \text{and} \quad A + t = \{[x, s+t]\colon [x, s] \in A\}.$$

Then $h\colon CY_0(n) \to Y(n)$ defined precisely as in Theorem 5 is the required homeomorphism and the result follows. □

If R is an equivalence relation on a set X and $A \subset X$, then R induces an equivalence relation R_A on A where R_A is defined by aR_Ab iff aRb. However, instead of using the symbol R_A we will use R and understand that it is really R_A.

Let S be an equivalence relation on a set Y and define $T = R \times S$ on $X \times Y$ by

$$(x, y)T(x', y') \quad \text{iff} \quad xRx' \text{ and } ySy'.$$

The natural function

$$\sigma\colon (X \times Y)/T \to X/R \times Y/S$$

defined by $\sigma[x, y] = ([x], [y])$ is a bijection and consequently we will often equate the sets $(X \times Y)/T$ and $X/R \times Y/S$.

Define the equivalence relation Δ on a set X by $x\Delta y$ iff $x = y$. We will often equate X/Δ and X.

THEOREM 6. (a) *$I(n) \approx I_0(n) \times I$ $(n = \infty, 1, 2, \ldots)$ and*

(b) *if $n = 1, 2, \ldots$, there is a homeomorphism $H\colon I^n \to I^n$, and equivalence relations S and S_α on I^n and S_β on I^{n-1} such that H induces a homeomorphism*

$$H^*\colon I^n/S \to I^n/S_\alpha = I^{n-1}/S_\beta \times I$$

where $I^n/S \approx I(n)$ and $I^{n-1}/S_\beta \approx I_0(n)$.

Proof. (a) By Theorem 4, $I_0(n) \approx CI_0^1(n)$ and hence by Theorem 5, $I(n) \approx CCI_0^1(n)$. Then by Theorems 1 and 4 we have

$$I(n) \approx CI_0^1(n) \times I \approx I_0(n) \times I.$$

(b) Let A_n and B_n be defined as in Theorem 3. Define S on I^n (and therefore also on A_n and B_n) by

$$(x_1, \ldots, x_n)S(y_1, \ldots, y_n) \quad \text{iff} \quad \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}.$$

By Theorem 0, $I(n) \approx I^n/S$ and $I_0(n) \approx A_n/S$. Let S_1 and S_2 be the equivalence relations on $CB_n = (B_n \times I)/R_0$ and $CCB_n = (B_n \times I^2)/R$, respectively, induced by S. By the proof of Theorem 3 let

(1) $$f\colon CB_n \to A_n \quad \text{and} \quad h\colon CCB_n \to I^n$$

be the homeomorphisms defined by $f[x, s] = (1-s)x$ and $h(x, s, t) = (1-t)(1-s)x + tp$ where $p = (1, 1, \ldots, 1)$. We now show that

(a) *S_1 is the equivalence relation on CB_n associated with f, and*
(b) *S_2 is the equivalence relation on CCB_n associated with h.*

We shall prove only (b) since (a) is a simplified version of (b).

Suppose $[x, s, t]S_2[x', s', t']$. We have $[x, s, t]S_2[x', s', t']$ iff $[x, s, t] = [x', s', t']$; or xSx', $s = s'$, and $t = t'$ iff $(x, s, t) = (x', s', t')$; or $s = s' = 1$ and $t = t'$; or $t = t' = 1$; or $\{x_i\} = \{x'_i\}$, $s = s'$, and $t = t'$. Thus in all cases

$$\{(1-t)(1-s)x_i+t\} = \{(1-t')(1-s')x'_i+t'\}$$

and hence

$$h[x, s, t]Sh[x', s', t'] .$$

Suppose now that $h[x, s, t]Sh[x', s', t']$. Thus

$$A = \{(1-t)(1-s)x_i+t\} = \{(1-t')(1-s')x'_i+t'\} = B$$

and since $\{x_i\}$ and $\{x'_i\}$ both contain 0, $t = \min A = \min B = t'$. If $t = t' = 1$, $[x, s, t] = [x', s', t']$ and hence $[x, s, t]S_2[x', s', t']$. If $t = t' \neq 1$, then since $\{x_i\}$ and $\{x'_i\}$ both contain 1, we have $(1-t)(1-s)+t = \max A = \max B = (1-t)(1-s')+t$ and hence $s = s'$. If $s = s' = 1$, then $[x, s, t] = [x', s', t']$ and hence $[x, s, t]S_2[x', s', t']$. If $s = s' \neq 1$, then $\{x_i\} = \{x'_i\}$ and hence in all cases we have $[x, s, t]S_2[x', s', t']$.

Thus h: $CCB_n \to I^n$ induces a homeomorphism

(2) $$h^*:\ (CCB_n)/S_2 \to I^n/S \approx I(n)$$

and f: $CB_n \to A_n$ induces a homeomorphism

$$F:\ (CB_n)/S_1 \to A_n/S \approx I_0(n) .$$

Define S_1^* on $CB_n \times I$ by $S_1^* = S_1 \times \Delta$. From the statement and proof of Theorem 1, let $X = B_n$ and let

(3) $$\varphi = g^*:\ CCB_n \to CB_n \times I .$$

From the proof of Theorem 2, φ induces a homeomorphism

(4) $$\varphi^*:\ (CCB_n)/S_2 \to (CB_n)/S_1 \times I = (CB_n \times I)/S_1^* .$$

Define

(5) $$f_1:\ CB_n \times I \to A_n \times I$$

by $f_1 = f \times \mathrm{id}$ and define S^* on $A_n \times I$ by $S^* = S \times \Delta$. Then f_1 induces

(6) $$f_1^*:\ (CB_n \times I)/S_1^* = (CB_n)/S_1 \times I \to A_n/S \times I = (A_n \times I)/S^* .$$

By Theorem 3, let p: $A_n \to I^{n-1}$ be a homeomorphism, define

(7) $$p_1:\ A_n \times I \to I^{n-1} \times I = I^n$$

by $p_1 = p \times \mathrm{id}$, define S_β on I^{n-1} by $p(x)S_\beta p(y)$ iff xSy, and define S_α on I^n by $S_\alpha = S_\beta \times \Delta$. Thus, p_1 induces a homeomorphism

(8) $$p_1^*:\ (A_n \times I)/S^* = A_n/S \times I \to I/S_\alpha = I^{n-1}/S_\beta \times I .$$

Thus by (1)-(8), $H = p_1 \circ f_1 \circ \varphi \circ h^{-1}$: $I^n \to I^n$ is a homeomorphism and induces a homeomorphism

$$H^* = p_1^* \circ f_1^* \circ \varphi^* \circ h^{*-1}:\ I^n/S \to I^n/S_\alpha = I^{n-1}/S_\beta \times I$$

where $I^n/S \approx I(n)$ and $I^{n-1}/S_\beta \approx I_0(n)$. □

THEOREM 7. *$I^m(n) \approx I^{(n-1)m}/R \times I^m$ $(m = \infty, 1, 2, \ldots;\ n = 1, 2, \ldots)$ for a properly defined equivalence relation R.* (That is, $I^m(n)$ contains I^m as a factor.)

Proof. Let $I_{ij} = I$ for all i and j, let $I_j^m = \prod_{i=1}^{m} I_{ij}$ and let $I_i^n = \prod_{j=1}^{n} I_{ij}$. Then

$$I^{mn} = \prod_{j=1}^{n} I_j^m = \prod_{i=1}^{m} I_i^n .$$

Also, $x_j^m = (x_{1j}, x_{2j}, \ldots, x_{mj})$ for $x_j^m \in I_j^m$. By Theorem 0

$$I^m(n) \approx I^{mn}/R_1 = \Big(\prod_{j=1}^{n} I_j^m\Big)/R_1$$

where $(x_1^m, \ldots, x_n^m)R_1(y_1^m, \ldots, y_n^m)$ iff $\{x_1^m, \ldots, x_n^m\} = \{y_1^m, \ldots, y_n^m\}$.

$$\overset{\textstyle I^m}{I^n\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix}} \qquad \overset{\textstyle I^m}{I^n\begin{pmatrix} y_{11} & y_{12} & \ldots & y_{1n} \\ y_{21} & y_{22} & \ldots & y_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ y_{m1} & y_{m2} & \ldots & y_{mn} \end{pmatrix}}$$

Let H: $I^n \to I^n$, S, S_α, and S_β be as in Theorem 6. Define H_i: $I_i^n \to I_i^n$ by $H_i = H$ and define

$$H_m:\ \prod_{i=1}^{m} I_i^n \to \prod_{i=1}^{m} I_i^n$$

by $H_m = \prod_{i=1}^{m} H_i$. If $x_i^n \in I_i^n$, then $x_i^n = (x_{i1}, \ldots, x_{in})$. Define S^m on $\prod_{i=1}^{m} I_i^n$ by $S^m = \prod_{i=1}^{m} S_i$ where $S_i = S$, which says that for each $i = 1, \ldots, m$, $\{x_{1n}, \ldots, x_{in}\} = \{y_{i1}, \ldots, y_{in}\}$.

Thus, as equivalence relations on I^{mn} we have $R_1 \subset S^m$. Define R_α on I^{mn} by $H_m(x)R_\alpha H_m(y)$ iff xR_1y and define S_α^m on I^{mn} by $S_\alpha^m = \prod_{i=1}^{m} S_{\alpha i}$ where $S_{\alpha i} = S_\alpha$. By Theorem 6 we have xS^my iff $H_m(x)S_\alpha^m H_m(y)$. Thus, $R_1 \subset S^m$ implies $R_\alpha \subset S_\alpha^m$. Define S_β^m on $I^{(n-1)m}$ by $S_\beta^m = \prod_{i=1}^{m} S_{\beta i}$ where

$S_{\beta i} = S_\beta$. Again by Theorem 6, if $u, u' \in I^{(n-1)m}$ and $v, v' \in I^m$, we have $(u, v) S_\alpha^m (u', v')$ iff $u S_\beta^m u'$ and $v = v'$. Hence, since $R_\alpha \subset S_\alpha^m$, we have that $(u, v) R_\alpha (u', v')$ implies $u S_\beta^m u'$ and $v = v'$. Thus, define R on $I^{(n-1)m}$ by uRu' iff $(u, v) R_\alpha (u', v)$ for some $v \in I^m$. Hence $(u, v) R_\alpha (u', v')$ iff uRu' and $v = v'$ and thus

$$H_m\colon I^{mn} \to I^{mn}$$

induces the homeomorphism

$$H_m^*\colon I^{mn}/R_1 \to I^{mn}/R_\alpha = I^{(n-1)m}/R \times I^m$$

where $I^m(n) \approx I^{mn}/R_1$. □

If $m \geq 0$, let P^m be projective m-space.

THEOREM 8. *$I^m(2) \approx C(P^{m-1}) \times I^m$ $(m = 1, 2, \ldots)$.*

Proof. From the proof of Theorem 6(b) take I^n, A_n, and the equivalence relation S for the case $n = 2$. We see that $(x, y) S (x', y')$ iff $\{x, y\} = \{x', y'\}$ reduces to the case that the point (x, y) is identified with the point (y, x). Thus for S on A_2 we have $(x, 0)$ identified with $(0, x)$. We are going to slightly modify the proof of Theorem 6(b) for the case $n = 2$. Instead of p being a homeomorphism of A_2 onto I^1 let p be a homeomorphism of A_2 onto $J = [-1, 1]$ such that if $p(x, 0) = u \in J$, then $p(0, x) = -u$. Thus the induced equivalence relation S_β on J identifies each $u \in J$ with $-u$. Furthermore $p_1\colon A_2 \times I \to J \times I$ and $H\colon I^2 \to J \times I$ and for S_α on $J \times I$ we have the point (u, v) identified with $(-u, v)$. Thus $H^*\colon I^2/S \to (J \times I)/S_\alpha = J/S_\beta \times I$. (Incidently, $J/S_\beta \approx I$.)

We now carry these modifications over to the proof of Theorem 7. For R_1 on $I^m \times I^m$, the point (x, y) is identified with (y, x). We have $H_m\colon I^{2m} \to (J \times I)^m = J^m \times I^m$ and the induced equivalence relation R_α on $J^m \times I^m$ identifies (u, v) with $(-u, v)$ for $(u, v) \in J^m \times I^m$. Thus R defined on J^m identifies u with $-u$. Hence, we have

$$H_m^*\colon I^m(2) \approx I^{2m}/R_1 \to J^m/R \times I^m .$$

We now claim that $J^m/R \approx C(P^{m-1})$. The boundary of J^m, $\mathrm{Bd}J^m$, is topologically a $(m-1)$-sphere, S^{m-1}. Since R identifies diametrically opposite points on $\mathrm{Bd}J^m \approx S^{m-1}$ we have $(\mathrm{Bd}J^m)/R \approx P^{m-1}$. Furthermore $J^m \approx C\mathrm{Bd}J^m$ where the point $(0, 0, \ldots, 0) \in J^m$ is identified as the cone point and R on J^m agrees with the equivalence relation induced on $C\mathrm{Bd}J^m \approx J^m$ by R on $\mathrm{Bd}J^m$. Thus,

$$J^m/R \approx (C\mathrm{Bd}J^m)/R \approx C\big((\mathrm{Bd}J^m)/R\big) \approx C(P^{m-1}) . \quad \square$$

Corollaries 1 and 2 below constitute Molski's [5] Theorem 1 and corollary to Theorem 1, respectively.

COROLLARY 1. *$I^1(2) \approx I^2$ and $I^2(2) \approx I^4$.*

Proof. Since $P^0 = \{\text{one point}\}$ and $P^1 \approx S^1$, we have $C(P^0) \approx I$ and $C(P^1) \approx I^2$ and the results follow.

COROLLARY 2. *If M is a 2 manifold, then $M(2)$ is a 4-manifold.*

Proof. In the operation of taking I^2 to $I^2(2) \approx I^4$ both boundary points and interior points are preserved and the result follows.

COROLLARY 3. *If $m \geq 3$, then $I^m(2)$ is not a manifold with boundary.*

Proof. It can be shown by a straight forward argument using local homology theory that if K is a complex and M is a manifold with boundary such that $C(K) \times M$ is a manifold with boundary, then K has the same homology groups as a sphere or cell (²). The result follows. □

THEOREM 9. *$I^\infty(n) \approx I^\infty(n) \times I^\infty$ $(n = 1, 2, \ldots)$.*

Proof. By Theorem 7, $I^\infty(n)$ contains I^∞ as a factor. That is, there is a space X such that $I^\infty(n) \approx X \times I^\infty$. Thus

$$I^\infty(n) \approx X \times I^\infty \times I^\infty \approx I^\infty(n) \times I^\infty . \quad \square$$

From the proof of Theorem 8 we see that $I^\infty(2) \approx J^\infty/R \times I^\infty$ where $J = [-1, 1]$ and R identifies u in J^∞ with $-u$.

QUESTION 1. *Is $J^\infty/R \approx I^\infty$? If not, is $J^\infty/R \times I^\infty \approx I^\infty$?*

By Theorem 6 we have $2^I \approx 2_0^I \times I$ where 2_0^I is the space of all closed subsets of I that contain 0.

QUESTION 2. *Is $2_0^I \approx 2^I$?*

If so, then $2^I \approx 2^I \times I$, and thus, for $n \geq 0$,

$$2^I \approx 2^I \times I^n .$$

QUESTION 3. *If $2^I \approx 2^I \times I$, is it true that*

$$2^I \approx 2^I \times I^\infty ?$$

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(²) This result and its proof were communicated to me by M. C. McCord. The proof will be omitted in this paper since it involves notions and techniques that are independent of the rest of this paper.

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LOUISIANA STATE UNIVERSITY
Baton Rouge, Louisiana

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Added in proofreading. Raymond Wong has answered the first part of question 1 in the negative. The proof is very easy. The natural projection p of J^∞ onto J^∞/R is a 2-fold covering map when restricted to $J^\infty/0$. However $I^\infty/\{\text{point}\}$ is simply connected which would contradict the assumption that $J^\infty/R \approx I^\infty$.

Some remarks concerning the mappings of the inverse limit into an absolute neighborhood retract and its applications to cohomotopy groups

by

S. Godlewski (Warszawa)

If $\{X_\alpha, \pi_\alpha^\beta\}$ is an inverse system (see [2], p. 213) of compact metric spaces and $Y \in \mathrm{ANR}$ (see [1], p. 100), we define a map

$$\Phi\colon [\varprojlim\{X_\alpha, \pi_\alpha^\beta\}, Y] \to \varinjlim\{[X_\alpha, Y], \pi_\alpha^{\beta\#}\},$$

where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$. We show that Φ is an isomorphism preserving some structures in the set of homotopy classes: the "dependence" structure, the group structure if Y is a topological group, and the nth cohomotopy group structure if $\dim X_\alpha \leqslant 2n-1$.

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§ 1. Definitions and notations. Let us denote by 2^N the family of all subsets of a set N. A function $\lambda\colon 2^N \to 2^N$ satisfying conditions:

$$A \subset \lambda(A) \quad \textit{for every set } A \subset N,$$
$$\textit{if} \quad A \subset B \subset N, \quad \textit{then} \quad \lambda(A) \subset \lambda(B),$$
$$\lambda(\lambda(A)) = \lambda(A) \quad \textit{for every set } A \subset N$$

is said to be the *dependence operation* in the set N, and the set N in which a such operation is defined is said to be a *dependence domain* (see [1], p. 66).

Let N_1 and N_2 be two dependence domains with dependence operations λ_1 and λ_2, respectively. A function $f\colon N_1 \to N_2$ satisfying the condition

$$f(\lambda_1(A)) \subset \lambda_2(f(A)) \quad \text{for every set } A \subset N_1$$

will be called a *λ-morphism*. A one-to-one λ-morphism for which the inverse function is a λ-morphism is said to be a *λ-isomorphism* (see [1], p. 66).