Maximal lattice-ordered algebras
of continuous functions

by

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Let $X$ be a compact Hausdorff space and let $D(X)$ be the lattice of continuous functions from $X$ to the extended real line, $\mathbb{R}$, which are real-valued on a dense subset. With a natural definition of operations, $D(X)$ becomes a partial algebra.

A $\Phi$-algebra is a real Archimedean lattice-ordered algebra with positive 1 which is a weak order unit. A $\Phi$-subalgebra of $D(X)$ is a subset of $D(X)$ which is a $\Phi$-algebra under the operations defined above. By Zorn's lemma, every $\Phi$-subalgebra of $D(X)$ is contained in a maximal $\Phi$-subalgebra.

The purpose of this work is a study of $D(X)$ by means of these maximal $\Phi$-subalgebras. $D(X)$ is an object of some importance in representing certain algebraic structures as collections of functions, as the following paragraphs indicate.

M. Henriksen and D. G. Johnson have proved ([8], 2.3) that every $\Phi$-algebra $A$ is isomorphic to a point-separating $\Phi$-subalgebra of $D(\mathcal{M}(A))$, where $\mathcal{M}(A)$ is the (compact) space of maximal $I$-ideals of $A$.

Likewise, each Archimedean vector-lattice is isomorphic to a point-separating $I$-subspace of $D(X)$ for an appropriate choice of $X$ (see [9], 6.8).

The operations on $D(X)$ need not be everywhere defined. However, $D(X)$ can be embedded "isomorphically" in a $\Phi$-algebra $D(X_\infty)$. This proposition leads to the following result: A subalgebra of $D(X)$ is a maximal $\Phi$-subalgebra iff it is a maximal subalgebra.

The idea of locality in maximal $\Phi$-subalgebras turns out to be of some importance. Let $A$ be a maximal $\Phi$-subalgebra of $D(X)$. If $f$ is locally in $A$, then $f$ belongs to $A$.

The structure space, $\mathcal{M}(A)$, of $A$ is obtained as a quotient space of $X$. The embedding of $A$ into $D(\mathcal{M}(A))$ induced by the quotient map

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* This paper consists of a portion of a doctoral dissertation written under the direction of Professor E. C. Weinberg at the University of Illinois. This revision was supported by NSF Grant GP 5801.
is just the Henriksen-Johnson embedding. Locality is used in showing that the (maximal) stationary sets of \( A \) are nowhere dense, closed, and connected (or are isolated points).

The result of the previous paragraph indicates that stationary sets are small. Examples are given to show that stationary sets may be very large. For a familiar 1-dimensional space \( X \), there is a maximal \( \Phi \)-subalgebra of \( D(X) \) with a 1-dimensional stationary set. If \( \kappa \) is any cardinal number, there is a maximal \( \Phi \)-subalgebra of \( D([0, 1]) \) with structure space \([0, 1] \).

If \( X \) is a 1st-countable space, the intersection of all the maximal \( \Phi \)-subalgebras of \( D(X) \) is exactly the set of locally constant functions. The class of uniformly closed (= closed under uniform convergence) \( \Phi \)-algebras is of particular interest. Just as the Stone-Weierstrass theorem shows that \( C(X) \) is the smallest uniformly closed \( \Phi \)-algebra with structure space \( X \), we prove that there is a largest uniformly closed \( \Phi \)-subalgebra, \( U(X) \), of \( D(X) \) with structure space \( X \). \( U(X) \) is characterized as the collection of all elements of \( D(X) \) whose cofinality sets are \( C^* \)-embedded in \( X \). Every uniformly closed maximal \( \Phi \)-algebra is \( U(X) \) for some space \( X \). \( U(X) \) is the intersection of all the maximal \( \Phi \)-subspaces of \( D(X) \). Conditions are given under which \( U(X) = C(X) \), and necessary and sufficient conditions are given for \( U(X) \) to be isomorphic to \( C(Y) \) for some space \( Y \).

Using the characterization of \( U(X) \), it is shown that if \( P \) is a hyper-real prime ideal of a uniformly closed maximal \( \Phi \)-algebra \( A \), then \( A/P \) has a countable cofinal subset. Hence hyper-real quotient fields of uniformly closed maximal \( \Phi \)-algebras are (real closed) \( \eta_i \)-fields. It is known that in the absence of maximality this result need not hold.

Since \( D(Y) = D(Y) \) for any completely regular space \( Y \), no particular effort has been made to eliminate compactness from the hypotheses of certain theorems.

The author would like to express his gratitude to Professor E. C. Weinberg for hours of valuable discussion and penetrating questions on the material of this paper.

1. Preliminaries. An attempt has been made to keep this paper reasonably self-contained. Questions of notation and terminology can be answered by consulting [3] or [5], the latter of which is devoted to a study of the structure of \( \Phi \)-algebras.

In this paper, all given spaces are assumed to be completely regular (and Hausdorff). When not mentioned to the contrary, \( X \) is always a compact space.

The set (field, topological space) of real numbers is denoted by \( R \); and the set of natural numbers, by \( N \). If \( Y \) is a space and \( r \in R \), then \( r \) denotes the constant function on \( Y \) taking the value \( r \).

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1.1. A space \( Y \) is extremally disconnected iff the closure of every open subset of \( Y \) is open. It is known ([5], 1.1.1) that every open subset of an extremally disconnected space is \( C^* \)-embedded and extremally disconnected, and that every dense subset of an extremally disconnected space is extremally disconnected.

Let \( f: Y \to Z \) be a continuous function. Then \( f \) is tight iff every non-empty open subset of \( Y \) contains the non-empty inverse image of an open subset of \( Z \); \( f \) is fitting iff \( f \) is closed, onto, and the inverse image of each point of \( Z \) is compact.

Theorem ([13], section 2.) Every space \( Y \) is the continuous image under a tight fitting map, \( \tau \), of an extremally disconnected space, \( X_\tau \). If \( Y \) is another extremally disconnected space and \( \alpha \) is a tight fitting map of \( Z \) onto \( Y \), then there is a homeomorphism \( \rho \) of \( Y \) onto \( Z \) such that \( \sigma \circ \rho = \tau \). (This is an extension of a theorem of Gleason [2].)

The pair \((X_\tau, \tau)\) is called the minimal projective extension of \( Y \).

1.2. Let \( Y \) be a subset of \( Z \). Then \( Y \) is \( C^* \)-embedded in \( Z \) iff every continuous function on \( Y \) to \([0, 1]\) has a continuous extension over \( Z \). When \( Y \) is dense in \( Z \), \( Y \) is \( C^* \)-embedded in \( Z \) iff every continuous function on \( Y \) to any compact space has a continuous extension over \( Z \). For any (completely regular) space \( Y \), there exists a compactification \( \beta Y \) of \( Y \) characterized by the condition \( Y \) is \( C^* \)-embedded in \( \beta Y \). For a more detailed description of \( \beta Y \), see [3], Chap. 6.

1.3. A \( \Phi \)-algebra is a real Archimedean lattice-ordered algebra with positive 1 which is a weak order unit. Equivalently, a \( \Phi \)-algebra is a real Archimedean lattice-ordered algebra which as a ring is a subdirect sum of totally ordered rings, and which has an identity.

Let \( Y \) by a completely regular space. A continuous function \( f: Y \to R \) is said to be an extended real-valued function iff the coindensity set \( \mathcal{K}(f) = \{ y \in Y : f(y) < R \} \), \( f \) is a dense subset of \( Y \), \( D(Y) \) is the set of all extended real-valued functions on \( Y \). With the partial order defined pointwise, \( D(Y) \) becomes a lattice.

Let \( f \) and \( g \) be extended real-valued functions. If there is a function \( h \) in \( D(Y) \) such that \( h(x) = f(x) + g(x) \) for all \( x \in \mathcal{K}(f) \cap \mathcal{K}(g) \), then \( h \) is the sum, \( f + g \), of \( f \) and \( g \). Since \( \mathcal{K}(f) \cap \mathcal{K}(g) \) is dense, this sum is unique if it exists. Similarly, \( fg \) is the extension of the pointwise product defined on the intersection of the coindensity sets, and it is unique if it exists.

A subset \( A \) of \( D(Y) \) which is a \( \Phi \)-algebra under the operations on \( D(Y) \) is called a \( \Phi \)-subalgebra of \( D(Y) \). Subalgebra, subspace, and 1-subspace are defined similarly.

Note that a \( \Phi \)-subalgebra of \( D(Y) \) is simply an 1-subalgebra with 1.

Proposition. ([6], 2.2.) \( D(Y) \) is a \( \Phi \)-algebra iff every dense cozero subset of \( Y \) is \( C^* \)-embedded.
1.4. An (algebra) ideal $I$ of a $\Phi$-algebra $A$ is an $l$-ideal if $a \in A$, $b \in I$, and $|a| < |b|$ together imply $a \in I$. Let $\mathcal{K}(A)$ denote the set of all maximal $l$-ideals of $A$. If $I$ is an $l$-ideal of $A$, let $\mathcal{K}(I) = \{ M \in \mathcal{K}(A) : I \subseteq M \}$. If $S \subseteq \mathcal{K}(A)$, let $k(S) = \bigcap S$. The topological space with underlying set $\mathcal{K}(A)$ and closure operator $S \mapsto k(S)$ is called the structure space of $A$. For $a \in A$, let $\mathcal{M}(a)$ be the set of all maximal $l$-ideals of $A$ containing $a$. Then the collection $\{ \mathcal{M}(a) : a \in A \}$ forms a base for the closed sets of $\mathcal{K}(A)$.

If $I$ is an $l$-ideal of a $\Phi$-algebra $A$ and if $a \in A$, then $I(a)$ denotes the image of $a$ under the natural projection of $A$ onto $A/I$.

Let $A$ be a lattice-ordered ring and $I$ an $l$-ideal of $A$. Then $A/I$ can be made into a lattice-ordered ring as follows: call $I(a)$ (positive if $a^+ = (a + 0)$ belongs to $I$) (see [3], Chapt. 5).

The following fundamental representation theorem is due to M. Henriksen and D. G. Johnson.

**Theorem.** ([6], 2.3.) Every $\Phi$-algebra $A$ is isomorphic to an algebra $A'$ of extended real-valued functions on $\mathcal{K}(A)$. Moreover,

(a) $\mathcal{K}(A)$ is a compact (Hausdorff) space, and

(b) if $S$ and $T$ are disjoint closed subsets of $\mathcal{K}(A)$, then there is an $a' \in A'$ such that $a'[S] = \{ 0 \}$ and $a'[T] = \{ 1 \}$.

**Proof.** We give only the definition of the isomorphism $a \rightarrow a'$ of $A$ into $D(\mathcal{K}(A))$. For each $0 \leq a \in A$, let $a'(M) = \inf \{ r \in \mathbb{R} : M(a) < r \} \quad (M \in \mathcal{K}(A)).$

For arbitrary $a \in A$, let $a'(M) = (a')'(M) - (a')'(M)$.  

1.5. **Theorem.** ([5], 3.3.) If $A$ is a $\Phi$-algebra and $B$ is a subalgebra, then the least sublattice of $A$ containing $B$ is a subalgebra.

1.6. The next theorem is a slightly weakened version of a theorem of [6], and follows immediately from the proof of [6], 5.2.

**Theorem.** ([6], 5.2.) A $\Phi$-algebra $A$ is isomorphic to $C(Y)$ for some space $Y$ iff

(a) $\mathcal{K}(A) = \bigcap \{ \mathcal{K}(f) : f \in A \}$ is dense in $\mathcal{O}(A)$, and

(b) $A$ is uniformly closed (i.e., closed under uniform convergence), and

(c) $A$ is closed under inversion (i.e., if $a \in A$ with $Z(a)$ disjoint from $\mathcal{K}(A)$, then $a$ is invertible in $A$).

In this case, $Y$ may be taken to be $\mathcal{K}(A)$.

2. **Maximal $\Phi$-algebras.** A maximal $\Phi$-subalgebra of $D(Y)$ is a $\Phi$-subalgebra of $D(Y)$ which is not properly contained in any $\Phi$-subalgebra of $D(Y)$.

In many of the arguments throughout this paper, we shall want to know whether a particular function $f$ belongs to a given maximal $\Phi$-subalgebra $A$. The results of this section permit us to ignore the lattice operations in attempting to adjoin $f$ to $A$, and to concentrate on "local belongingness".

2.1. Let $(Y, \tau)$ be the minimal projective extension of $Y$. It is easy to show that the map $\tau^* : D(Y) \rightarrow D(Y, \tau)$ given by $\tau^*(f) = f \tau$ is a lattice isomorphism which preserves sums and products when they exist. Moreover, if $\tau^*(f) + \tau^*(g)$ belongs to $\tau^*(D(Y))$, then $f + g$ is defined in $D(Y)$; similarly for products. Since $Y, \tau$ is extremally disconnected, it follows from 1.3 that $D(Y, \tau)$ is a $\Phi$-algebra.

**Theorem.** A subalgebra $A$ of $D(Y)$ is a maximal subalgebra iff $A$ is a maximal $\Phi$-subalgebra.

**Proof.** It clearly suffices to prove that every subalgebra of $D(Y)$ is contained in a $\Phi$-subalgebra.

Let $A$ be a subalgebra of $D(Y)$. Then $\tau^*[A]$ is a subalgebra of $\tau^*[D(Y)]$, and $\tau^*[D(Y)]$ is a sublattice of $D(Y, \tau)$; hence the sublattice, $E$, of $D(Y, \tau)$ generated by $\tau^*[A]$ lies in $\tau^*[D(Y)]$. By 1.5, $E$ is a $\Phi$-algebra. Clearly, then, $\tau^*[E]$ is a $\Phi$-subalgebra of $D(Y)$ containing $A$.

The proof of 2.1 with the Henriksen–Isbell result (1.5) replaced by its analog for vector-lattices (easily proved by induction using the well-known identities $a(a+b) = ab + ab, (a+b)+c = (a+c)+(b+c)$ (for $0 \leq a \in \mathbb{R}$ and $a, b, c$ in a (real) vector lattice and their duals) can be used to show that maximal subspace and maximal $l$-subspace of $D(Y)$ are the same.

2.2. Since maximal subalgebras of $D(Y)$ contain the constant functions, we have the following corollary.

**Corollary.** If $A$ is a maximal $\Phi$-subalgebra of $D(Y)$, and $f \in D(Y) \sim A$, then for some $n \in N$, there exist $m_i \in A$ $(0 < i < n)$ such that $\sum m_i f^i$ is not defined.

2.3. Let $A$ be a $\Phi$-subalgebra of $D(Y)$ and let $f \in D(Y)$. Then $f$ is in $A$ at $x \in Y$ if and only if there exists $m_i \in A$ such that $f^i = m_i f$ for all $1 < i < n$.

**Theorem.** Let $X$ be a compact space. Then every maximal $\Phi$-subalgebra of $D(X)$ is local.

**Proof.** Let $A$ be a maximal $\Phi$-subalgebra of $D(X)$ and suppose that $f$ is in $A$ at each $x \in X$. By a compactness argument, there exist a finite open cover $(U_i ; 1 \leq i \leq n)$ of $X$ and a subset $(a_i : 1 \leq i \leq n)$ of $A$ such that $f|_{U_i} = a_i f$ for all $1 \leq i \leq n$.

Let $g_1, \ldots, g_n \in A$. Define $h : X \rightarrow gR$ by

$$h(x) = \sum (g_j a_j'(x); 0 \leq j \leq k)$$

for $x \in U_i$. 


3. Structure spaces. In this section, $A$ is assumed to be a $\Phi$-subalgebra of $D(X)$ containing the constant functions.

3.1. The following proposition generalizes a result of M. Henriksen and D. G. Johnson ([5], 2.5).

Proposition. For $x \in X$, the set
\[ M_x = \{ f \in A : fg(x) = 0 \text{ for all } g \in A \} \]
is a maximal $l$-ideal of $A$. Every maximal $l$-ideal of $A$ is of this form. Finally, $M_x \neq M_y$ if and only if there is some $f \in A$ such that $f(x) \neq f(y)$.

Proof. Clearly $M_x$ is a proper $l$-ideal of $A$.

Note that if $f \in A$ vanishes on a neighborhood of $x$, then $f \in M_x$, and that if $f \in M_x$, then $f(x) = 0$.

Suppose that $I$ is a proper $l$-ideal of $A$ such that for all $x \in X$, there exist $f, g \in A$ with $fg(x) > 0$. By compactness of $X$ there are $f_1, \ldots, f_n$ in $I$ such that $(\sum_i f_i(x))^2 \geq 0$ is positive for all $x \in X$, so $I = A$. Hence every proper $l$-ideal of $A$ is contained in some $M_x$.

Suppose that $M_y$ properly contains $M_x$. Then there is $g \in A$ such that $fg(x) = 1$ and $(fg)(x) = 0$, so $(fg)(1/2) - 1/2$ belongs to $M_x - M_y$, a contradiction. Hence no $M_x$ properly contains any $M_y$.

In view of the preceding paragraph, this implies that each $M_x$ is a maximal $l$-ideal. If $M_x \neq M_y$, then there are $f \in M_x$ and $g \in A$ such that $fg(y) \neq 0$.

If $f$ belongs to $A$ with $f(x) \neq f(y)$, we may suppose that $f(x) = 1$ and $f(y) = -1$; then $f$ is $0 \in M_x$; clearly $f \notin M_y$.

3.2. A subset $S$ of $D(X)$ is said to separate points of $Y \subset X$ if whenever $x$ and $y$ are distinct points of $Y$, then there exists $x \in S$ and $y \neq (x) \neq (y)$.

Let $r$ be the relation on $X$ defined by $(x, y) \in r$ if $x$ does not separate the points of $(x, y)$. Clearly $r$ is an equivalence relation. The equivalence classes of $r$ are called stationary sets of $A$. (Note that here “stationary set” is used instead of the customary “maximal stationary set”.)

Theorem. The structure space, $\mathcal{M}(A)$, of $A$ is homeomorphic to the quotient topological space $X/r$.

Proof. Write $Y$ for $X/r$, and let $q$ be the natural projection of $X$ onto $Y$. Since $M_x = M_x$ iff $(x, y) \in r$, $q((x)) = M_x$ defines a bijection $r : Y \rightarrow \mathcal{M}(A)$. Now, $Y$ is a continuous image of $X$, so $Y$ is quasi-compact.

Let $\mathcal{M}(A) = (M \in \mathcal{M}(A) : a \in M)$ be a basic closed subset of $\mathcal{M}(A)$.

Then
\[ \varphi^{-1}(\mathcal{M}(A)) = \varphi^{-1}(\{ f \in \mathcal{M}(A) : a \in M_x \}) = \{ x \in X : a \notin M_x \} = \bigcap \{ \{ ab : b \in A \} \}
\]
which is closed. Hence $r$ is continuous. Since $\mathcal{M}(A)$ is Hausdorff and $Y$ is quasi-compact, $r$ is a homeomorphism.

This theorem can be used to obtain the structure space of a $\Phi$-subalgebra $B$ of $D(Y)$ for an arbitrary completely regular space $Y$. Every function in $D(Y)$ has a unique extension to an element of $D(BY)$—this yields an isomorphism of $B$ into $D(BY)$. Let $1_B$ be the identity element of $B$. Since $1_B$ is idempotent, the only values that it can assume are 0, 1, or $\infty$. Since $1_B(\infty)$ is thus open (and nowhere dense), $1_B$ can assume only the values 0 and 1. Hence the set
\[ \{ x \in \beta Y : 1_B(x) = 1 \} = \{ x \in \beta Y : 1_B(x) = 0 \}
\]
is an open-closed subset of $\beta Y$ on the complement of which every function in $B$ vanishes. Hence $B$ can be considered as a $\Phi$-algebra of extended real-valued functions on this set.

3.3. If $A$ is a $\Phi$-subalgebra of $D(X)$ and $\varphi$ is the natural projection of $X$ onto $YT$, then $A$ can be considered as a $\Phi$-subalgebra of $D(YT)$ as follows: if $\varphi(x) = \varphi(y)$, then $\varphi(a)(x) = \varphi(a)(y)$ for all $a \in A$, so defining $\varphi' : \varphi^{-1}(\beta Y) \rightarrow D(YT)$ by $\varphi'(\varphi(a)) = \varphi(a)$ yields an isomorphism $\varphi' : A \rightarrow A'$ of $A$ into $D(YT)$.

The homeomorphism $\tau : \mathcal{M}(A) \rightarrow YT$ induces an isomorphism $\tau^{-1} : D(YT) \rightarrow D(\mathcal{M}(A))$ by $\tau^{-1}(\tau(f)) = f$. Then $\tau : \tau^{-1}(\tau(f)) = A'$ is an embedding of $A$ into $D(\mathcal{M}(A))$. It is natural to ask whether this embedding agrees with the Henriksen–Johnson embedding, $a \rightarrow a'$, of $A$ into $D(\mathcal{M}(A))$ (see 1.4).

Theorem. The homeomorphism of theorem 3.2 induces the Henriksen–Johnson embedding of $A$ into $D(\mathcal{M}(A))$; i.e., in the notation of the above paragraphs, $a' = (\tau^{-1})'(a')$.

Proof. Let $0 \leq a \in A$, $M_x \in \mathcal{M}(A)$. Recall that $a'(M_x) = \text{inf}(r \in R : M_x(a) < r)$. If $M_x(a) \leq r$ for some $r \in R$, then there is $m \in M_x$ such that $a < r + m$, so $r = r + m(a) = r + m(a) < a \leq a'(r + m(a)) = (\tau^{-1})'(a')(M_x)$. Hence $(\tau^{-1})'(a') \leq a'$. If $r \in R$ with $r < a'(M_x)$, then $r < M_x(a)$, so there is $m \in M_x$ with $r + m < a$; then $r = r + m(a) < a(x) = (\tau^{-1})'(a')(M_x)$. Hence $a' \geq (\tau^{-1})'(a')$.

We will usually not distinguish notationally between elements of $A$ and their images in $D(X/T)$.

3.4. It is clear that if $A$ is a maximal $\Phi$-subalgebra of $D(X)$, then $A$ is a maximal $\Phi$-subalgebra of $D(\mathcal{M}(A))$. In the latter case, $A$ is said to be a maximal $\Phi$-algebra.

4. Stationary sets for maximal $\Phi$-subalgebras.

4.1. Connectedness in the next theorem depends only on the fact that $A$ is local and contains the constant functions.
**Theorem.** Let $A$ be a maximal $\Phi$-subalgebra of $D(X)$. Every stationary set of $A$ with more than one point is closed, nowhere dense, and connected.

**Proof.** Let $S$ be a stationary set of $A$ with more than one point, and let $x \in S$. Then $S = \bigcap \{ f^{-1} \{ f(x) \} : f \in A \}$, so $S$ is closed.

If $y \in \text{Int} S$ and $y \neq x$, then there is an open neighborhood $U$ of $y$ such that $x \notin U \subseteq S$. There is a closed neighborhood $V$ of $y$ such that $V \subseteq U$. Let $g \in C(X)$ with $g(y) = 0$ and $g|V = 1$. Every element of $A$ is constant on the open set $U$, and $g$ is constant on the open set $X \setminus V$, so $\sum f g^i$ is defined whenever $f x A \{ 0 \leq i \leq n \}$. Hence $g \in A$. But $g$ separates $x$ and $y$.

Suppose $S = B \cap C$ where $B$ and $C$ are closed and disjoint, and $C \neq \emptyset$. Let $g \in C(X)$ with $0 \leq g \leq 1$, $g(U) = \{ 0 \}$, and $g(V) = \{ 1 \}$ for some open neighborhoods $U$ and $V$ of $x$ and $B$ and $C$, respectively. Let $Z = \{ x \in X : g(x) \neq 1 \}$. Note that $Z$ is closed and $Z$ is disjoint from $U \cup V$. Now, $A$ separates each point of $Z$ from $S$, so $Z$ is connected by the compactness of $X$, $A$ separates $Z$ from $S$; hence there is $f x A$ such that $f$ vanishes on a neighborhood of $Z$, is 1 on a neighborhood of $S$, and is bounded. Then $f g$ is in $A$ at every point of $Z$, so $f x A$. Since $f$ separates $B$ and $C$, we must have $B = \emptyset$. Hence $S$ is connected.

It has been pointed out to the author that connectedness of stationary sets also arises in a different context. The following statement was proved by M. Katětov [10]. Lemma 18, see [3], 16. Since $A$ is a subring of $C(X)$ containing the constant functions and $f$ belongs to $A$ whenever $f x A$, then every stationary set of $A$ is connected.

4.2. Corollary. If $X$ is a totally-ordered space or is zero-dimensional, then every maximal $\Phi$-subalgebra of $D(X)$ separates points of $X$.

4.3. Lemma. Let $h \in D(X)$ and let $Y$ be a connected subset of $\mathcal{N}(h) = \{ x \in X : h(x) \in [0, 1] \}$ satisfying

(*) If $f : Y \to [-1, 1]$ is continuous and onto, then there exists a $x \in [-1, 1]$ such that $f(x)$ is nowhere dense in $Y$.

Then if $g \in D(X)$ is not constant on $Y$, $g$ does not belong to any $\Phi$-subalgebra containing $h$.

**Proof.** Since $Y$ is connected, either $Y \subseteq \text{pos} h$ or $Y \subseteq \text{neg} h$; we can assume $h \leq 0$. Suppose $x, x' \in X$ with $h(x) = -1$ and $h(x') = 1$. Let $r \in (-1, 1)$ such that $g(r) = 0$. Then $g(x)$ is nowhere dense in $Y$—we can assume that $r = 0$. Finally, we suppose that $\text{gh}$ is defined and arrive at a contradiction.

Suppose that there is an $x \in Y$ such that every $Y$-neighborhood $U$ of $x$ contains points $x(U)$ and $y(U)$ at which $g$ is, resp., strictly negative and strictly positive. Let $U$ be a $Y$-neighborhood of $x$. Clearly $\text{gh}[x(U)] = -\infty$ and $\text{gh}[y(U)] = \infty$. Hence every neighborhood, $V$, of $x$ contains points $x(V \setminus Y)$ and $y(V \setminus Y)$ at which $\text{gh}$ is $-\infty$, $+\infty$, resp., contradicting continuity of $\text{gh}$.

Hence for each $x \in Y$ there exists a $\text{Y}$-neighborhood, $U_x$, of $x$ such that either $U_x \subseteq \text{pos} g \cup \text{Z}(g)$ or $U_x \subseteq \text{neg} g \cup \text{Z}(g)$. Let $B$ be the set of all $x \in Y$ satisfying the former and $C$, the set of $x$ satisfying the latter. Since $Z(g)$ is nowhere dense in $Y$, $B \cup C = \emptyset$. If $x \in B$, then Int $U_x \subseteq C$, so $B$ is open in $Y$; similarly, $C$ is open in $Y$. Now, $Y = B \cup C$, $x \in C$, and $y \in B$. But this contradicts connectedness of $Y$.

Note that if $Y$ is a space in which every disjoint family of open sets is countable—e.g., if $Y$ is 2nd-countable—then $Y$ satisfies condition (\*) of the lemma.

4.4. Let $A$ be a maximal $\Phi$-subalgebra of $D(X)$. The sets $\mathcal{N}(g)$ for $g \in A$ are called *infinity sets* of $A$. Stationary sets of $A$ with more than one point arise as a result of certain functions being "kept out of" $A$. Since functions are kept out of $A$ by infinity sets of $A$, it is reasonable to expect a close relationship between stationary sets of $A$ and infinity sets of $A$, and this is indeed the case (see 4.5). However, there may be a stationary set on which no function of $A$ is infinite, and which is not even a zero set.

**Example.** A stationary set for a maximal $\Phi$-subalgebra $A$ of $D(X)$ which is not a zero set, which has more than one point, and which is disjoint from every infinity set of $A$.

Let $L$ be the 1-point compactification of the long line (see, for example, [3], p. 252). Let $X = L \times [0, 1]$. For convenience, if $x \in L$, we denote by $x(a)$ the "vertical line segment" $(a) \times [0, 1]$. It will be shown that $V(\infty)$ is a stationary set for some maximal $\Phi$-subalgebra $A$.

Recall that the long line is the lexicographic product $W \times [0, 1)$ with the first coordinate dominating, where $W$ is the space of countable ordinals, and the product is given the order topology. For $s \in W$, define $f_s \in D(X)$ by $f_s(x) = 0$, if $x \in [0, 1]$, and $f_s(x, y) = 1$, if $y \in [0, 1]$. Since $s \not\equiv t$ implies that $f_s$ vanishes on a neighborhood of $\mathcal{N}(f_t)$, $f_s \in W$ is contained in a subalgebra of $D(X)$. Let $A$ be a maximal $\Phi$-subalgebra containing all of the $f_s$.

Since $\mathcal{N}(f_t) = \{ x \in X : f_t \neq 0 \}$ is homeomorphic to $[0, 1]$, lemma 4.3 implies that each $\mathcal{N}(f_t)$ is a stationary set of $A$. Since $\infty$ is a limit point of $\{ (x, y) : (x, y) \in W \}$, $\mathcal{N}(\infty)$ is also a stationary set of $A$. Since $\infty$ is not a $\mathcal{G}_\delta$ set, it is not a zero set.

If $f \in A$ is infinite at any point of $\mathcal{N}(\infty)$, it is infinite on all of $\mathcal{N}(\infty)$; $\mathcal{N}(f)$ is a zero set. But every $\mathcal{G}_\delta$ set containing $\mathcal{V}(\infty)$ has non-empty interior.
4.5. Proposition. If \( S \) is a stationary set of a maximal \( \Phi \)-subalgebra \( A \) of \( D(X) \) with more than one point, and if \( V \) is any open set intersecting \( S \), then \( V \) contains points at which some element of \( A \) is infinite.

Proof. Suppose \( V \subset \mathcal{S}(A) \). Since \( S \) is connected, \( S \cap V \) has more than one point; let \( x \) and \( y \) be distinct points of \( S \cap V \). Let \( g \in \mathcal{O}(X) \) with \( g(\{x\}) = 1 \) and \( g([x] \sim W) = 0 \) for some closed neighborhood \( W \) of \( x \) contained in \( V \sim \{y\} \). Since \( V \subset \mathcal{S}(A) \), \( g \in \mathcal{A}(x) \) (see 2.2). But \( g \) is not constant on \( S \).

4.6. Theorem. If \( Y \) is a nowhere dense connected zero set of \( X \) and if \( Y \) satisfies \( (*) \) of lemma 4.3, then \( X \) is a stationary set for some maximal \( \Phi \)-subalgebra of \( D(X) \).

Proof. If \( Y = Z(f) \), let \( h = \lceil f \rceil^{-1} \) and let \( A \) be a maximal \( \Phi \)-subalgebra of \( D(X) \) containing \( h \).

This theorem is a partial converse to theorem 4.1. As example 4.4 shows, theorem 4.1 cannot be strengthened to include zero set in its conclusion.

4.7. As the next two examples show, stationary sets for maximal \( \Phi \)-subalgebras may be very large.

Example. A 1-dimensional space \( X \) with maximal \( \Phi \)-subalgebra which has a 1-dimensional stationary set.

Let \( X = \{z: \sin(\text{Im}(z)) \in (0, 1) \cup (0, 1) \times [-1, 1], x 0, 1, 1 \} \) a nowhere dense connected zero set satisfying \( (*) \) of lemma 4.3. By theorem 4.6, \( Y \) is a stationary set for some maximal \( \Phi \)-subalgebra \( A \) of \( D(X) \).

4.8. Example. A maximal \( \Phi \)-subalgebra of \( D([0, 1]) \) for any cardinal \( \alpha > 1 \) with structure space \([0, 1], \).

Index the rationals in \([0, 1], Q \cap [0, 1] = \{q: i \in \mathbb{N}\} \) so that \( q_0 = 0 \). Define \( f_i : D([0, 1]) \rightarrow [0, 1] \) by \( f_i(x) = 1 \) for \( f_i(0) = \infty \). Let \( U_n \) be a closed interval in \( [0, 1] \) containing \( q_n \) in its interior but not containing \( q_{i+1} \) for \( i < n \). Let \( U_n \) be a closed interval contained in the interior of \( U_n \) and containing \( q_n \) in its interior. Let \( h_n \in \mathcal{O}([0, 1]) \) such that \( h_n([0, 1] \sim U_n) = 0 \) and \( h_n|_{U_n} = 1 \). Let \( \theta = h_n \leq 1 \).

For \( 1 < n < \mathbb{N} \), define \( f_n \) inductively by

\[
\begin{align*}
f_n(x) &= h_n(x) \lfloor \inf \{j | f_j(x) \neq f_j(q_n)\}^{-1}: 1 \leq i < n \}, \\
f_n(q_n) &= \infty.
\end{align*}
\]

Then \( f_n \in D([0, 1]) \).

(a) Since \( h_n \) vanishes on a neighborhood of \( \{q_i: i < n\} \), so does \( f_n \).

(b) Since \( f_j(x) = f_j(q_n) \) iff \( x = \{q_i: i < n\} \), \( f_n \), and \( f_n \) is infinite only at \( q_n \).

(c) If \( j < n \), then

\[
0 < \lim_{\sigma \to \infty} \frac{\|f_j(x) - f_j(q_n)\|}{\sigma^{m_n(x)}} = \lim_{\sigma \to \infty} \frac{\|f_j(x) - f_j(q_n)\|}{\sigma^{m_n(x)}} = \left(\frac{1}{\inf_{i < n}}\right)^m
\]

(d) If \( P \) is a polynomial in \( f_1, ..., f_{n-1} \) which vanishes at \( q_n \), then \( \lim_{\sigma \to \infty} f_n(x) P(x) = 0 \).

Proof. For fixed \( n > 1 \), the proof is by induction on the degree of \( P \). The statement is clear if degree of \( P \) is 0.

Suppose that degree of \( P \) is \( s > 0 \), and the statement holds for every polynomial in \( f_1, ..., f_{n-1} \) of degree \( < s \). For notational reasons, we assume that there are only two terms of \( P \) of highest degree: \( t_j, t_{n-1} \), and \( f_j, f_{n-1} \). We suppose that \( t_m 
eq 0, t_n = 0 \). Write \( P \) as \( f_j - f_j(q_n) + f_j - f_j(q_n) + R \), where \( Q = \{t_j, ..., t_{n-1}\} \), \( R = \{f_j, f_{n-1}, ..., f_{n-1}\} \), and \( S \) has degree \( < s \). Since \( P \), \( f_j - f_j(q_n) \), and \( f_j - f_j(q_n) \) vanish at \( q_n \), so does \( S \). By the induction hypothesis and (c) above, \( \lim_{\sigma \to \infty} f_n(x) P(x) = 0 \).

(e) \( \{f_i: i < \mathbb{N}\} \) belongs to some subalgebra of \( D([0, 1], 1) \).

Proof. Let \( S \) be a polynomial in \( f_0, ..., f_n \). We can write \( S = \sum \sum f_0 \sum f_1 \), (by induction) each \( P \) is defined as a continuous function on \([0, 1], 1 \), and which is finite except (possibly) at \( q_1, ..., q_{n-1} \).

Write \( P(x) \) for \( P(x) - P(q_n) \). Then

\[
S = \sum \sum f_0 \sum f_1 P(q_n) + P(q_n).
\]

By (d), the first term vanishes at \( q_n; P_n \) is finite at \( q_n \); the middle term is \( +\infty, -\infty, or 0 \) according as \( P(q_n) > 0, P(q_n) < 0 \) where \( j \) is the highest index for which \( P(q_n) \neq 0 \) and \( f_j > 0 \), or all \( P(q_n) \) (i.e. 0) are zero.

(f) Now, write \( \{0, 1\} \) as \( \{0, 1\} \times \{0, 1\} \) and define \( \Phi \in D([0, 1]) \) by \( \Phi(x, y) = f_n(x) \). Let \( A \) be a maximal \( \Phi \)-subalgebra of \( D([0, 1]) \) containing the \( \mathbb{N} \). By the above, \( \Phi \) is infinite only at \( \mathbb{N} \). By the above, \( \Phi \) is infinite only at \( \mathbb{N} \).
Hence $(0, 1)^f$ satisfies $(*)$ of lemma 4.3. By that lemma, each $(g_n) \times (0, 1)^f$ is a stationary set of $A$. By denseness of $Q$, each $(r) \times (0, 1)^f$ is a stationary set of $A \cap [0, 1]^f$. Hence $\mathcal{A}(A)$ is homeomorphic to $(0, 1]^f$.

5. Intersection of maximal $\Phi$-subalgebras.

5.1. Lemma. If $x \in X$ has a countable base of neighborhoods and if $f$ belongs to all maximal $\Phi$-subalgebras of $D(X)$, then $f$ is constant on some neighborhood of $x$.

Proof. Suppose that $f \in D(X)$ is not constant on any neighborhood of $x$, and that $x$ has a countable base of neighborhoods. Then $f$ takes infinitely many values on each neighborhood of $x$. Let $(U_n: n \in \mathbf{N})$ be a base of neighborhoods of $x$ such that $cl U_{n+1} \subseteq U_n$ for all $n \in \mathbf{N}$. Let $(a_n)_{n \in \mathbf{N}}$ be a sequence in $X$ such that $f(a_n) \neq f(a_j)$, $a \in U_i$, $a_j \neq x$, and $i \neq j$ implies that $f(a_n) \neq f(a_i)$. We can assume that $a_i \notin U_{i+1}$ (by choosing subsequences of $(U_i)_{i \in \mathbf{N}}$ and $(a_i)_{i \in \mathbf{N}}$). Now, if $(f(a))_{n \in \mathbf{N}}$ is a sequence in $\mathbf{R}$ converging to $f(x)$, then $(f(a))_{n \in \mathbf{N}}$ has either an increasing or a decreasing subsequence; suppose the former, and—by a change of notation—suppose that $(f(a))_{n \in \mathbf{N}}$ is increasing.

Case 1. If $f(x) = \infty$. We can assume that $f(a_n) \to \infty$ for all $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, there exists $g_n \in \mathcal{C}(X)$ satisfying

$$g_n(a_n) = (−1)^n (f(a_n))^{−1},$$

the $g_n$ are alternately positive and negative, and $g_n$ is bounded by $g_1(a_n), \ldots, g_n(a_n)$.

Since $(g_n(x): n \in \mathbf{N})$ is locally finite and $(f(a))_{n \in \mathbf{N}}$ converges to $\infty$, we can define $g = \sum \gamma_n$, continuous. However, $g(x) = \sum \gamma_n (f(x))^{−1}$, so $fg$ is not defined at $x$.

Case 2. If $f(x) < \infty$. Since $f$ belongs to a maximal $\Phi$-subalgebra $A$ iff $f = f(x) \in A$, we may assume $f(x) = 0$; also, $f \in A$ iff $−f \in A$. We can treat the case $f(x) = 0, 1 > f(a_n) > 0$ and $f(a_x) = 0$. Hence $fg$ is decreasing. We may define $g$ by

$$g(x) = \sup \{g_n(x): n \in \mathbf{N}\} \quad (g \neq 0), \quad g(x) = \infty.$$
then the set of $U(r)$ for strictly positive real numbers $r$ is a base for a
uniformity on $A$. The uniform topology is 1st-countable and conver-
gence in this topology is just uniform convergence as extended real-valued
functions on $\mathcal{K}(A)$. If $A$ is a complete uniform space with this
uniformity, then $A$ is said to be uniformly closed.

The major result of this section is Theorem 6.3: There is a unique
maximal $\Phi$-subalgebra, $U(X)$, of $D(X)$ containing $C(X)$. $U(X)$ is
the set of all functions in $D(X)$ whose coinequality sets are $C^*$-embedded.
Since every uniformly closed $\Phi$-algebra $A$ contains $C(\mathcal{K}(A))$, this
theorem yields a very useful characterization of uniformly closed maximal $\Phi$-algebras:

6.1. If $A$ is a $\Phi$-subalgebra of $D(X)$ containing the constant functions
and separating points of $X$, then by the Stone-Weierstrass theorem, $A$ is
uniformly closed iff $C(X) \subseteq A$ iff $A^* = C(X)$ (by [6], 3.7, $A$ is uni-
formly closed iff $A^*$ is). In view of the first part of 3.3, the following
proposition is clear.

**Proposition.** Let $A$ be a $\Phi$-subalgebra of $D(X)$ containing the constant
functions. Then $A$ is uniformly closed iff $A$ contains $\{f \in C(X) : f$ is
constant on all stationary sets of $A\}$.

6.2. **Lemma.** Let $Y$ be a completely regular space, and let $f \in D(Y)$.
If $fg$ is defined whenever $g \in (C^*(Y))^+$ with $Z(g) = N(f)$, then $\mathcal{K}(f)$ is
$C^*$-embedded in $Y$.

**Proof.** Suppose $1 \leq f \in D(Y)$ such that $fg$ is defined whenever
$g \in (C^*(Y))^+$ with $Z(g) = N(f)$. Let $1 \leq h \in C(\mathcal{K}(f))$. Define $g : Y \to \mathbb{R}$ by $g(x) = h(x)/f(x)$ for $x \in \mathcal{K}(f)$, and $g(x) = 0$ for $x \in N(f)$. Then $g \in (C^*(Y))^+$
and $Z(g) = N(f)$. Now, $fg$ extends $h$ over $Y$. If $k \in C(\mathcal{K}(f))$, then $k + n \geq 1$ for some $n \in N$, so $k + n$, and hence $k$, has an extension over $Y$.
Hence, in this case, $\mathcal{K}(f)$ is $C^*$-embedded.

Let $f \in D(Y)$ such that $fg$ is defined whenever $g$ belongs to $(C^*(Y))^+$
with $Z(g) = N(f)$. Let $f_1 = f^+ + 1$ and $f_2 = f^- + 1$. Let $\lambda \in (C^*(Y))^+$
with $Z(\lambda) = N(f)$. Then $Z(\lambda) = N(f^+)$. Let $k \in (C^*(Y))^+$ with $Z(k) = N(f^-)$
and $k|Z(f^-) = 1$. Then $Z(k) = N(f^-)$ and $kk$ agrees with $h$ on $f$. $k$, $kk$, $kk'$, and $kk''$, are $C^*$-embedded. $kk'$ is defined and $kk''$ is $C^*$-embedded on $f$; $f^+$ is bounded
on $f + x$. If $f(x) < 1$, then $kk''$ is defined. Then $kk'' = h^* + k$ is defined since
a bounded function can always be added to any element of $D(Y)$. By
the first part, $\mathcal{K}(f)$ is $C^*$-embedded in $Y$. Similarly, $\mathcal{K}(f)$ is $C^*$-embedded
in $Y$. Since the intersection of two open dense $C^*$-embedded sets is always
$C^*$-embedded (see [3], 9 N), $\mathcal{K}(f) = \mathcal{K}(f) \cap \mathcal{K}(f)$ is $C^*$-embedded.

6.3. **Theorem.** Let $Y$ be a completely regular space and let $U(Y) = (f \in D(Y) : fg$ is defined for all $g \in C(X))$. Then

(i) $f \in U(Y)$ iff $\mathcal{K}(f)$ is $C^*$-embedded in $Y$, and
(ii) $U(Y)$ is the unique maximal $\Phi$-subalgebra of $D(Y)$ containing $C(Y)$.

**Proof.** (i) follows from the lemma.

Since $C(Y)$ is an algebra, $C(Y) \subseteq U(Y)$; moreover, any subalgebra
of $D(Y)$ containing $C(Y)$ lies in $U(Y)$. To show that $U(Y)$ is an algebra,
we must observe that if $f$ and $g$ belong to $U(Y)$, then $\mathcal{K}(f) \cap \mathcal{K}(g)$
is $C^*$-embedded.

While the last statement in the proof of 6.3 is known (see [3], 9 N),
I cannot resist giving the following proof (a portent of things to come):
If $h \in C(\mathcal{K}(f))$, then $h$ has an extension $h' \in C(Y)$; since $\mathcal{K}(f)$
is defined, $(g|\mathcal{K}(f))$ is defined. This implies, by the lemma, that $\mathcal{K}(f) \cap \mathcal{K}(g)$
is $C^*$-embedded in $\mathcal{K}(f)$, hence in $Y$.

Hence $D(X)$ contains a unique maximal uniformly closed $\Phi$-subalgebra
which separates points of $X$.

6.4. A space $Y$ is said to be a CM-space iff $C(Y)$ is a maximal $\Phi$-subalgebra of $D(Y)$, or equivalently, iff $C(Y) = U(Y)$.

**Corollary.** A space $Y$ is a CM-space iff $Y$ has no proper dense
$C^*$-embedded nonzero subset.

**Proof.** If $X = Z(f)$ is dense and $C^*$-embedded, then $f$ is defined
and if, in addition $Z(f) \neq \emptyset$, then $f$ belongs to $U(Y) \sim C(Y)$.

6.5. The next corollary was pointed out to the author by J. R. Isbell.

**Corollary.** If $X$ is a compact space of power $< \aleph$, then $X$ is
a CM-space.

**Proof.** If $X$ had a proper dense $C^*$-embedded nonzero subset, $S$, then,
by [3], 9.5, $X \sim S$ would have power at least $2^\omega$.

6.6. **Theorem.** $U(X)$ is the intersection of all of the maximal $l$-sub-
spaces of $D(X)$.

**Proof.** Suppose that $f$ belongs to the intersection. Since every
element of $D(X)$ belongs to some maximal $l$-subspace; $f + h$ is defined
for all $h \in D(X)$. Let $g \in C(\mathcal{K}(f))$. Define $h : X \to \mathbb{R}$ by $h(x) = g(x) - f(x)$
for $x \in \mathcal{K}(f)$ and $h(x) = -f(x)$ for $x \in N(f)$. Since $g$ is bounded, $h$ is
continuous; $N(h) = N(f)$ so $h \in D(X)$. Clearly $f + h$ extends $g$ over $X$. By
theorem 6.3, $f \in U(X)$.

Suppose $f \in U(Y)$ and let $h \in D(Y)$. Define $h : X \to \mathbb{R}$ by $h(x) = f(x) + h(x)$ for $x \in \mathcal{K}(f) \cap \mathcal{K}(h)$ and $h(x) = h(x)$ for $x \in \mathcal{K}(f) \cap \mathcal{K}(h)$.
Since every point of $X$ has a neighborhood on which $f$ is bounded, $h$ is
continuous. Since $f \in U(Y)$, $\mathcal{K}(f)$ is $C^*$-embedded, so $h$ has a continuous
extension $h'$ over $X$. Since $N(h') \subseteq N(f) \cap N(h)$, $h' \in D(X)$. Hence $f + h'$
is defined for all $h \in D(X)$, this implies that $f$ belongs to every maximal
subspace of $D(X)$, hence to every maximal $l$-subspace of $D(X)$ (see 2.1).
6.7. Corollary. The following are equivalent:
(a) \( D(X) \) is a vector space.
(b) \( D(X) \) is a \( \Phi \)-algebra.
(c) Every dense cozero subset of \( X \) is \( C^* \)-embedded.

Proof. (The equivalence of (b) and (c) is a result of [2]—see 1.3.)
Each of the three statements is equivalent to \( D(X) = U(X) \).

6.8. By a construction similar to the proof (case 2) of 5.1, one easily obtains the following lemma.

Lemma. Let \( Y \) be a completely regular space and let \( x \) be a non-isolated point of \( Y \) with countable base of neighborhoods. Then \( Y \sim \{x\} \) is not \( C^* \)-embedded in \( Y \).

6.9. Proposition. If \( x \) is a 1-countable point of \( Y \) and if \( f \in D(Y) \) is infinite at \( x \), then there is some \( g \in C(Y) \) for which \( fg \) is not defined.

Proof. By the lemma, \( \mathcal{A}(f) \) is not \( C^* \)-embedded, since \( \mathcal{A}(f) \sim \{x\} \) is not and \( \mathcal{A}(f) \) is dense in \( Y \sim \{x\} \). By theorem 6.3, there is a \( g \) with the desired property.

6.10. Corollary. Every 1-countable space is a CM-space.

6.11. Proposition. If \( X = \beta Y \) for some non-pseudocompact space \( Y \), then \( X \) is not a CM-space.

Proof. In this case, \( C(Y) \) properly contains \( C(X) \) (in \( D(X) \)).

6.12. Theorem. \( U(X) \) is isomorphic to \( C(Z) \) for some \( Z \) if and only if \( X = \beta Y \) for some CM-space \( Y \), and in this case, \( U(X) \) is isomorphic to \( C(Y) \).

Proof. If \( U(X) \) is isomorphic to \( C(Z) \), then 6.6 implies that \( U(X) = C(\mathcal{A}(U(X))) \) and that \( \mathcal{A}(U(X)) \) is dense and \( C^* \)-embedded in \( Y \); i.e., \( X = \beta \mathcal{A}(U(X)) \). Let \( Y = \mathcal{A}(U(X)) \). If \( S = Y \sim Z(f) \) is a dense \( C^* \)-embedded cozero set of \( Y \), then \( S \) is a \( C^* \)-embedded cozero set of \( X \), so \( S \) has an extension \( g \in D(X) \). Now, \( Y \subseteq \mathcal{A}(g) \cap Y \subseteq S \), so by 6.4, \( Y \) is a CM-space.

For the converse, if \( f \in U(X) \), then by 6.3 and 3.9, \( \mathcal{A}(f) \cap Y \) is \( C^* \)-embedded in \( Y \); since \( \mathcal{A}(f) \) is open, \( \mathcal{A}(f) \cap Y \) is dense in \( Y \); clearly \( \mathcal{A}(f) \cap Y \) is a cozero set of \( Y \); hence, since \( Y \) is a CM-space, \( \mathcal{A}(f) \) is a CM-space.

6.13. Corollary. If \( X \) has a minimal \( C^* \)-embedded dense cozero subset \( Y \), then \( C(Y) = U(X) \).

For example, \( U(\beta R) = C(R) \).

6.14. Proposition. If \( U(X) \) is isomorphic to \( C(Y) \) for a pseudocompact space \( Y \), then \( X \) is a CM-space.

Proof. If \( \varphi : C(Y) \rightarrow U(X) \) is an isomorphism, then \( \varphi(1) = \varphi(n) = n \varphi(1) \)
= \( n \varphi(n) = n \). Hence every element of \( U(X) \) is bounded.

6.15. Suppose that \( U(X) \) is \( C(Z) \) for some space \( Z \), and let \( B \) be a \( \Phi \)-algebra with structure space \( X \). Then \( B = C(Y) \) for some \( Y \) iff \( B \) is uniformly closed and closed under inversion. For, in this case, since \( B \) is uniformly closed, separates points of \( X \) and contains the constant functions, \( C(X) \subseteq B \). Since \( U(X) \) is the unique maximal \( \Phi \)-algebra containing \( C(X) \), \( B \subseteq U(X) \). Hence \( \mathcal{A}(U(X)) \subseteq \mathcal{A}(B) \). By 1.6, \( \mathcal{A}(U(X)) \) is dense and \( C^* \)-embedded in \( X \), so \( \mathcal{A}(B) \) is dense and \( C^* \)-embedded. Hence, by 1.6, \( B \) is isomorphic to \( C(\mathcal{A}(B)) \). Necessity is immediate from 1.6.

7. Stationary sets for uniformly closed maximal \( \Phi \)-subalgebras. It is still an open question whether uniformly closed maximal \( \Phi \)-subalgebras of \( D(X) \) must separate points of \( X \). This section provides some partial answers and gives a condition which stationary sets for such a \( \Phi \)-algebra must satisfy.

7.1. Proposition. Let \( A \) be a uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \). If \( \mathcal{A}(A) \) is a CM-space, then \( A \) separates points of \( X \), and \( X \) is a CM-space.

Proof. Since \( \mathcal{A}(\mathcal{A}(A)) \) is maximal and \( \{\mathcal{A}(A)\} \subseteq A \), we have \( A = \mathcal{A}(\mathcal{A}(A)) \), so \( A \) contains only bounded functions. Hence \( A \subseteq C(X) \), and since \( A \) is maximal, \( U(X) = A = C(X) \).

7.2. Corollary. If \( A \) is a uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \) and \( \mathcal{A}(A) \) is 1-countable, then \( \mathcal{A}(A) = X \).

Proof. This follows from 6.10 and 3.2.

7.3. Theorem. If \( X \) is a metric space, then every uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \) separates points of \( X \). Hence \( C(X) \) is the unique uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \).

Proof. Let \( A \) be a uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \). Let \( \psi : X \rightarrow \beta X \) be the natural projection. By [11], 3.12, \( \psi \) induces an upper semicontinuous decomposition of \( X \); by [11], 5.20 ff., \( \mathcal{A}(A) \) is a metric space. By corollary 7.2, \( \mathcal{A}(A) = X \).

7.4. Proposition. If \( X \) is 1-countable and if \( A \) is a uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \), then whenever \( f \in A \) with \( N(f) \) non-empty, \( N(f) \) is a union of stationary sets each with more than one point.

Proof. Clearly, if any stationary set meets \( N(f) \), then it is contained in \( N(f) \).

Suppose that there is some \( x \in N(f) \) such that \( x \) is a stationary set of \( A \). Let \( \psi : X \rightarrow \mathcal{A}(A) \) be the projection; then \( \psi(x) \) is a 1-countable point of \( \mathcal{A}(A) \). By 6.9, \( A \) cannot contain \( C(\mathcal{A}(A)) \). But \( A \) is uniformly closed and separates points of \( \mathcal{A}(A) \).

7.5. Proposition. Let \( A \) be a uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \). If \( S \) is a non-trivial stationary set of \( A \) and if \( U \) is any open set containing \( S \), then \( U \sim S \) contains points at which some \( f \in A \) is infinite.
8. Non-uniformly closed maximal \( \Phi \)-subalgebras. The existence of uniformly closed \( \Phi \)-algebras is immediate from section 6. If \( D(X) \) is a \( \Phi \)-algebra, then \( D(X) \) contains no non-uniformly closed maximal \( \Phi \)-subalgebras. Otherwise, \( D(X) \) contains many such \( \Phi \)-subalgebras.

8.1. Theorem. If \( f \in D(X) \rightarrow U(X) \), then \( f \) belongs to a maximal \( \Phi \)-subalgebra \( M \) of \( D(X) \) which is not uniformly closed and whose only non-trivial stationary sets are contained in \( N(f) \).

Proof. Since \( S(f) \) is not \( C^* \)-embedded, lemma 6.2 implies that there is \( g \in (C^0(U))^+ \) with \( S(g) = (f) \) for which \( fg \) is not defined. Let \( \bar{S} = \{ h \in C(X) \mid h \text{ vanishes on a neighborhood of } N(f) \} \) and let \( M \) be a maximal subalgebra of \( D(X) \) containing \( \bar{S} \). Clearly, the only non-trivial stationary sets of \( M \) lie in \( N(f) \). Now, if \( g_n = (g - 1) \alpha_n \), then \( g_n \in \bar{S} \subseteq M \), but \( g_n \not\in M \), whereas \( f \not\in M \). Hence \( M \) is not uniformly closed.

8.2. Corollary. If \( D(X) \) is not a \( \Phi \)-algebra, then \( D(X) \) contains a non-uniformly closed maximal \( \Phi \)-subalgebra.

8.3. Example. \( D([0, 1]) \) contains at least \( \varepsilon \) distinct (but perhaps isomorphic) non-uniformly closed maximal \( \Phi \)-subalgebras. [By starting with one of the \( M \) of this example and a family of \( \varepsilon \) homeomorphisms of \([0, 1]\) onto itself, one obtains a family of \( \varepsilon \) isomorphic but distinct maximal \( \Phi \)-subalgebras.]

For each \( x \in [0, 1] \), let \( f_x \in D([0, 1]) \) with \( N(f_x) = \{ x \} \). Then if \( x \neq y \), there is \( g \in C([0, 1]) \) vanishing on a neighborhood of \( x \) and having the property that \( g f_x \) is not defined (see 6.9). Then (by construction of \( M_x \)) \( g \not\in M_x \), \( M_x \not\subseteq M_y \),

8.4. A \( \Phi \)-subalgebra \( A \) of \( D(X) \) is closed under composition if whenever \( f \in A \) and \( g \in C(R) \), then \( g \circ (f \circ R) \) has an extension belonging to \( A \). The extension is denoted \( g(f) \).

Proposition. Let \( A \) be a non-uniformly closed maximal \( \Phi \)-subalgebra of \( D(X) \). If \( X \) is 1-st-countable or locally connected, then \( A \) is not closed under composition.

Proof. First, suppose that \( X \) is locally connected. Let \( 1 < f \in A \) with \( f(\varepsilon) = \infty \) and let \( g \in C(R) \) be given by \( g(\varepsilon) = \sin \varepsilon \). Let \( U \) be a connected neighborhood of \( x \); \( f(U) \) is connected and contains \( \varepsilon \), so it contains \( x = f^{-1}[1/2] \) for two successive integers \( n \). Hence every neighborhood of \( x \) contains points at which \( g(f(x)) = 1 \) and \( -1 \).

Suppose that \( X \) is 1-st-countable. Let \( f \in A \rightarrow U(X) \) and let \( \varepsilon \in N(f) \).

Let \( (\alpha_n) \) be a sequence in \( S(f) \) converging to \( \varepsilon \); \( (\alpha_n) \cdot N \) converges to \( \infty \). We suppose that all \( f(\alpha_n) \) are distinct. Let \( g \in C(R) \) with \( g(f(\alpha_n)) = (-1)^n \). Then \( g \circ (f \circ R) \) has no continuous extension over \( X \).

9. Quotient fields. A totally ordered set \( S \) is an \( \eta \)-set iff whenever \( Q \) and \( R \) are (perhaps empty) countable subsets of \( S \) with every element of \( Q \) less than every element of \( R \) (denoted \( Q < R \)), then there exists \( s \in S \) greater than every element of \( Q \) and less than every element of \( R \). A totally-ordered field \( K \) is an \( \eta \)-field if every \( \eta \)-set \( N(f) \) is a \( \eta \)-field.

It is proved in [7], 15.8 that every prime ideal in a uniformly closed \( \Phi \)-algebra is an \( \eta \)-ideal.

Let \( A \) be a uniformly closed \( \Phi \)-algebra. A maximal ideal \( I \) of \( A \) is said to be real iff \( A/I \) is isomorphic to the real field; otherwise, it is hyperreal. In either case, \( A/I \) is a totally-ordered field containing a canonical copy of \( \mathbb{R} \)—i.e., the set of \( \{ r + i \mid r \in \mathbb{R} \} \).

It is known ([3], 13.8.13.4) that if \( I \) is a hyper-real maximal ideal of \( C(X) \), then \( C(X)/I \) is a real closed \( \eta \)-field.

This statement is true for arbitrary uniformly closed \( \Phi \)-algebras. M. Henriksen, J. R. Isbell, and D. G. Johnson give an example ([7], 1.9) of a uniformly closed \( \Phi \)-algebra \( A \), closed under composition, with a hyper-real maximal ideal \( M \) such that \( A/M \) has a countable cofinal subset.

It is the purpose of this section to show that the above-mentioned theorem does hold for uniformly closed maximal \( \Phi \)-algebras.

9.1. If \( A \) is any uniformly closed maximal \( \Phi \)-algebra, then \( A = U(\mathcal{A}(A)) \). The next proposition is clear from 6.3.

Proposition. Every uniformly closed maximal \( \Phi \)-algebra is closed under composition.

9.2. The next proposition is proved in the same way as the special case for \( C(\mathbb{F}) \). See [3], 7.15.
Proposition. Let $A$ be a uniformly closed $\Phi$-algebra. Every prime ideal $P$ of $A$ contains

$$O_2 = \{ f \in A : Z(f) \text{ is a neighborhood of } x \}$$

for a unique $x \in M(A)$. Hence $M_A$ is the unique maximal ideal of $A$ containing $P$.

9.3. Theorem. ([7, 1.7]) Let $P$ be a prime ideal of the uniformly closed $\Phi$-algebra $A$. If $S$ and $T$ are non-empty countable subsets of $A/P$ such that $S \leq T$, there is then an $a \in A/P$ such that $S \leq a \leq T$.

9.4. A $\Phi$-algebra $A$ is closed under countable composition if and only if whenever $(f_n)_{n \in N}$ is a sequence in $A$ and $g \in C(R^n)$, then there exists $h \in A$ such that $h(x) = g(f_1(x), f_2(x), ...)$ for all $x \in \sum_{n \in N} A(f_n)$. If it is known that every uniformly closed $\Phi$-algebra is closed under countable composition, the next theorem would be an easy consequence of the following result ([7], 2.6.6): Every $\Phi$-algebra closed under countable composition is a homomorphic image of $C(Y)$ for some space $Y$. However, it is not known whether this is true, so a direct approach must be taken.

Theorem. Let $M_A$ be a hyper-real maximal ideal of $U(X)$ containing a prime ideal $P$. Then $U(X)/P$ has no countable cofinal subset.

Proof. (This proof is a modification of the proof of 12.6.6 for $C(Y)$.) Let $a_1 < a_2 < ...$ be a sequence of elements of $U(X)/P$. We can suppose that $a_1 > 0$. By ([3], 13.5), there exist $f_1 < f_2 < ... \in U(X)$ with $P(f_n) = a_n$.

Since $M_A$ is hyper-real, $U(X)/M_A$ is non-Archimedean—let $1 \leq g \in U(X)$ such that $M_A(g) \geq \infty$ in $U(X)/M_A$ for all $n \in N$. Note that $g(0) = \infty$. We can assume that $N(g)$ contains $N^+(g)$ (since we can replace $f_n$ with $f_n + g$). For $i \in N$, define $\Phi_i : C(R) \to \Phi_i(0)$ by $\Phi_i(x) = x - i$ and $\Phi_i(x) = x - a_i$, and $\Phi_i$ is continuous on $[-i, i]$ and $[i, i+1]$.

Since $U(X)$ is closed under composition, $\Phi_i(g) \in U(X)$ for all $i \in N$. Let $x \in \Phi_i(g)$; let $V_x = \{ x \in X : g(x) < a \}$. For $x \in V_x$,

$$\sum_{i \in N} \Phi_i(g(x)) = \lim_{i \to \infty} \Phi_i(g(x))$$

where $a$ is the greatest integer in $g(x)$. Hence the function $h'$ given by

$$h'(x) = \sum_{i \in N} \Phi_i(g(x))$$

is defined and continuous on $\Phi_i(g)$. Since $\Phi_i(g)$ is $C^*$-embedded in $X$, $h'$ has a continuous extension $h$ by the Baire category theorem; $\cap \Phi_i(f) \in M(A)$ implies $h \in D(X)$.

Let $h \in C^*(X)$. Since $\Phi_i(g) = \Phi_i(f)$ is defined for all $i \in N$, $\sum_{i \in N} \Phi_i(g(x)) = \lim_{i \to \infty} \Phi_i(g(x))$ is defined and continuous. But $\Phi_i(g)$ is $C^*$-embedded in $X$, so this function has a continuous extension which must be $h$. Hence $h$ is defined for all $x \in C^*(X)$, so $h \in U(X)$.

Let $U_n = \{ x \in X : g(x) < n \}$. For $x \in U_n \cap \Phi_i(g)$,

$$h(x) = \lim_{i \to \infty} \Phi_i(g(x))$$

and

$$h(x) \geq \lim_{i \to \infty} \Phi_i(g(x))$$

Here $h(x) \geq \lim_{i \to \infty} \Phi_i(g(x))$ is a neighborhood of $y$. Hence $Z(h - f_n^{-1})$ is a neighborhood of $y$, so $(h - f_n^{-1}) \in U_n$. This implies $P(h) \supseteq P(f_n)$. For all $n \in N$.

It is interesting to note that among all uniformly closed $\Phi$-algebras with structure space $X$, the largest, $U(X)$ and the smallest, $C(X)$ share an algebraic property which need not be enjoyed by $\Phi$-algebras between them. It seems that in going from $C(X)$ to a larger $\Phi$-algebra $A \neq U(X)$, one can add enough functions to create problems without adding enough to solve them.

9.5. This theorem together with the Henricksen, Isbell, Johnson result (9.3) yields the following, just as in the $C(Y)$ case (see [3], 13.8).

Theorem. If $M$ is a hyper-real maximal ideal of a uniformly closed maximal $\Phi$-algebra $A$, then $A/M$ is an $\mathfrak{H}$-field.

10. Completion.

10.1. Let $C$ be a $\Phi$-subalgebra of $B$. Then $C$ is said to be order-dense in $B$ iff $b = \sup \{ c \in C : c < b \}$ for all $b \in B$.

Proposition. Let $B$ be a $\Phi$-algebra and let $C$ be a $\Phi$-subalgebra of $B$ containing the identity element of $B$. Then there is a continuous onto function $\phi : \mathcal{M}(B) \to \mathcal{M}(C)$ which induces the embedding of $C$ in $B$. Moreover, if $C$ is order-dense in $B$, then $\phi$ is tight.

Proof. We have $C \subseteq D(C(B))$, $C$ contains the constant functions, and $\mathcal{M}(B)$ is compact. Identify $\mathcal{M}(C)$ with $\mathcal{M}(B)/\mathcal{M}(C)$ by theorem 3.2. Theorem 3.3 implies that $\mathcal{M}(C)$ is a $\mathcal{M}(B)$-field onto $\mathcal{M}(C)$ induces the embedding of $C$ into $B$ (via $f \mapsto f * \phi$).

We will consider $B$ and $C$ as $\Phi$-subalgebras of $D(C(B))$. For $x \in \mathcal{M}(B)$, $M_B$ denotes the maximal $1$-ideal $\{ f \in C : f(g) = 0 \}$ for all $g \in B$, and $M_C$ denotes $\{ f \in C : f(g) = 0 \}$ for all $g \in C$. We next prove that $C$ is dense in $B$, then $M_B$ in $M_C$, and that $\mathcal{M}(C)$ is dense in $\mathcal{M}(B)$. Suppose $M_C > 0$ for some $f \in C$, $g \in B$. Denseness of $C$ in $B$ implies that there exists $h \in C$ with $h \geq |g|$, so $|fh|(x) > 0$.

Let $\{ M \in \mathcal{M}(B) : a \in M \}$ be a non-empty basic open subset of $\mathcal{M}(B)$.
Let $b \in C$ such that $0 < b < |a|$. If $M \not\subset \{I \in \mathcal{M}(C); b \not\in I\}$, then $b \not\in M$, so a $\not\in M$. Since $b \not\in M$, the set $\{I \in \mathcal{M}(C); b \not\in I\}$ is non-empty. Hence $\varphi$ is tight.

10.2. D. G. Johnson has proved [8] the following theorem:

If $A$ is a $\Phi$-algebra, there is a complete isomorphism $\vartheta$ of $A$ onto an order-dense $\Phi$-subalgebra of a complete $\Phi$-algebra $A'$. Moreover, $A'$ is unique in the following sense: if $\Phi$ is an isomorphism of $A$ onto a dense $\Phi$-subalgebra of a complete $\Phi$-algebra $B$, then there is an isomorphism $\Psi$ of $B$ onto $A'$ such that $\Psi \circ \Phi = \vartheta$.

The next proposition characterizes the structure space of $A'$.

If $B$ is a complete $\Phi$-algebra, then $B$ is uniformly closed ([6], page 94), so $B^*$, the set of bounded elements of $B$, is isomorphic to $C(\mathfrak{M}(B))$. Since $B^*$ is complete, $\mathfrak{M}(B)$ is extremally disconnected ([15], [13]).

**Proposition.** Let $A$ be a $\Phi$-algebra. Then the structure space of the completion of $A$ is the minimal projective extension of the structure space of $A$: $\mathfrak{M}(A') = (\mathfrak{M}(A))_m$.

**Proof.** As remarked above, $\mathfrak{M}(A')$ is extremally disconnected. By proposition 10.1, there is a tight map of $\mathfrak{M}(A')$ onto $\mathfrak{M}(A)$. By the uniqueness statement in 1.1, the proof is complete.

It is remarked in [13] that the above proposition holds when $A = C(Y)$ for some $Y$.

**References**


