Euclidean space modulo a cell

by

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1. Introduction. In [1] Andrews and Curtis proved that if A is an arc in Euclidean n-space E^n, then the Cartesian product of the quotient space E^n/A with E^1 is topologically E^{n+1}. The significant aspects of this result appear when $n \geqslant 3$, for in this case E^n/A is not necessarily homeomorphic to E^n. The results of this paper were motivated by a conjecture in [1] and are as follows.

THEOREM 1.1. *Suppose that D is a k-cell topologically embedded in E^n. Then $E^n/D\times E^1$ is homeomorphic to E^{n+1} provided*

I. *D is flat in E^{n+1}, or*

II. *$n-k \geqslant 2$.*

For example, Gillman has shown [4] that condition I is satisfied for $k=2$ or 3 when $n=3$. Thus, we have

COROLLARY 1.2. *If D is a k-cell topologically embedded in E^3, then $E^3/D\times E^1 \approx E^4$.*

COROLLARY 1.3. *If K is a crumpled cube in E^3 (that is, the closure of the bounded complementary domain of a 2-sphere in E^3), then $E^3/K\times E^1 \approx E^4$.*

Proof. Lininger has shown [6] that $\mathrm{Cl}(E^3-K) \approx \mathrm{Cl}(E^3-D)$ for some 3-cell D in E^3. Hence, 1.3 is an immediate consequence of 1.2.

The proof of Theorem 1.1 is broken up into two parts. In Section 2, we shall give a sufficient condition that $E^n/D\times E^1$ be homeomorphic to E^{n+1}, and in Sections 3 and 4 we shall prove that this condition is implied whenever the k-cell D satisfies I or II of Theorem 1.1. We prove the main result by showing that we can satisfy the hypothesis of the following modification of Bing's criterion [2].

THEOREM 1.4 (Bing). *Let C be a compact set in E^n. Suppose that for each $\varepsilon > 0$, there exists an isotopy h_t $(t\in[0,1])$ of E^{n+1} such that*

(1) *$h_0 =$ identity,*

(2) *$h_t =$ identity outside $N_\varepsilon(C\times E^1)$ for each $t\in[0,1]$,*

(3) *each h_t changes E^1-coordinates less than ε,*

(4) h_1 *is uniformly continuous, and*

(5) $\operatorname{diam} h_1(C\times w) < \varepsilon$ *for each* $w \in E^1$.

Then $E^n/C\times E^1 \approx E^{n+1}$.

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2. A reduction of the problem. Let $I^k = \{(x_1, \dots, x_{n+1}) \in E^{n+1} | 0 \leqslant x_i \leqslant 1$ for $1 \leqslant i \leqslant k$, and $x_i = 0$ for $i > k\}$. For technical reasons, it is desirable to consider the k-cell D in E^n as the image of I^k under an embedding $f\colon I^k \to E^n\times 0 \subset E^{n+1}$.

Given integers $1 \leqslant m \leqslant k \leqslant n$, consider $I^k = I^{k-m}\times I^m$, $I^0 = \{0\}$, and let $H(n, k, m)$ denote the following statement:

If $f\colon I^k \to E^n\times 0 \subset E^n + E^1 = E^{n+1}$ is an embedding, then for each $\varepsilon > 0$, there exists an isotopy h_t $(t\in[0,1])$ of E^{n+1} such that:

(1) $h_0 =$ identity,

(2) $h_t =$ identity outside $N_\varepsilon(f(I^k)\times E^1)$, for each $t \in [0,1]$,

(3) h_1 is uniformly continuous,

(4) h_t changes E^1-coordinates less than ε, for each $t\in[0,1]$,

(5) $\operatorname{diam} h_1(f(x\times I^m)\times w) < \varepsilon$, for each $x \in I^{k-m}$ and each $w \in E^1$,

(6) for each $w \in E^1$, there exists $y \in I^m$ such that

$$h_1(f(x\times I^m)\times w) \subset N_\varepsilon(f(x, y)\times w)$$

for each $x \in I^{k-m}$.

LEMMA 2.1. *For each* n *and* k, $1 \leqslant k \leqslant n$, $H(n, k, 1)$ *implies* $H(n, k, k)$.

Proof. Assume inductively that for $1 < m \leqslant k \leqslant n$, $H(n, k, 1)$ implies $H(n, k, m-1)$. Given $H(n, k, 1)$ and $\varepsilon > 0$, let f_t $(t\in[0,1])$ be the isotopy of E^{n+1} given by $H(n, k, m-1)$ with $\varepsilon/3$ replacing ε. Choose $\delta > 0$ such that if $x \subset E^{n+1}$ and $\operatorname{diam} X < \delta$, then $\operatorname{diam} f_1(X) < \varepsilon/3$. Consider $I^k = I^{k-m}\times I\times I^{m-1}$ with respective coordinates x, y, and z. Applying $H(n, k, 1)$, we can find an isotopy g_t $(t\in[0,1])$ of E^{n+1} satisfying (1) through (4) of $H(n, k, 1)$ with δ replacing ε; (5) $\operatorname{diam} g_1(f(x\times I\times z)\times w) < \delta$, for each $x \in I^{k-m}$, $z \in I^{m-1}$, and $w \in E^1$; and (6) for each $w \in E^1$, there exists $y \in I$ such that

$$g_1(f(x\times I\times z)\times w) \subset N_\delta(f(x, y, z)\times w) \quad \text{for each } x \in I^{k-m} \text{ and } z \in I^{m-1}.$$

Let $h_t = f_t g_t$ $(t\in[0,1])$. Then clearly h_t satisfies (1) through (4) of $H(n, k, m)$ with respect to ε. Given $w \in E^1$, choose $y \in I$ such that

$$g_1(f(x\times I\times z)\times w) \subset N_\delta(f(x, y, z)\times w) \quad \text{for each } x \in I^{k-m},\ z \in I^{m-1}.$$

Then

$$f_1 g_1(f(x\times I\times z)\times w) \subset N_{\varepsilon/3}\big(f_1(f(x, y, z)\times w)\big)$$

so that

$$f_1 g_1\big(f(x\times I\times I^{m-1})\times w\big) \subset N_{\varepsilon/3}\big(f_1(f(x\times y\times I^{m-1})\times w)\big).$$

But $\operatorname{diam} f_1(f(x\times y\times I^{m-1})\times w) < \varepsilon/3$ for each $x \in I^{k-m}$, $y \in I$. Hence

$$\operatorname{diam} h_1(f(x\times I\times I^{m-1})\times w) < \varepsilon.$$

Now choose $z \in I^{m-1}$ corresponding to the given $w \in E^1$ such that

$$f_1(f(x\times y\times I^{m-1})\times w) \subset N_{\varepsilon/3}(f(x, y, z)\times w).$$

Then

$$f_1 g_1(f(x\times I\times I^{m-1})\times w) \subset N_{\varepsilon/3}\big(f_1(f(x\times y\times I^{m-1})\times w)\big) \subset N_{\varepsilon/3}(f(x, y, z)\times w)$$

for each $x \in I^{k-m}$. Thus (5) and (6) of $H(n, k, m)$ are also satisfied.

LEMMA 2.2. $H(n, k, k)$ *implies that if* D *is a* k*-cell in* E^n, *then* $E^n/D\times E^1 \approx E^{n+1}$.

Proof. Conditions (1) through (5) of $H(n, k, k)$ are clearly sufficient to guarantee that the hypotheses of Theorem 1.4 are also satisfied.

3. Preliminary constructions. Throughout the remainder of this paper we shall assume that $D = f(I^k)$ is a k-cell in $E^n\times 0 \subset E^{n+1}$ satisfying either

I. D *is flat in* E^{n+1} or

II. $n - k \geqslant 2$.

Moreover, whenever condition II is applied we shall make the further assumption that $n \geqslant 4$ since the case $n = 3$ is covered by Andrews and Curtis [1]. What we wish to do now is prove that these assumptions imply $H(n, k, 1)$. The proof of this assertion is very similar to the proof of Theorem 1 in [1].

Given positive integers m and p, subdivide I^k into rectangular k-cells as follows:

Let $A = \{(a_1, \dots, a_k) |\ a_i$ is a positive integer, $1 \leqslant a_i \leqslant p$ for $1 \leqslant i < k$, and $1 \leqslant a_k \leqslant 2m\}$. Consider I^k as $I\times I\times \dots\times I$, k factors, and for $a = (a_1, \dots, a_k) \in A$, let

$$I_a = \left[\frac{a_1-1}{p}, \frac{a_1}{p}\right]\times \dots\times \left[\frac{a_{k-1}-1}{p}, \frac{a_{k-1}}{p}\right]\times \left[\frac{a_k-1}{2m}, \frac{a_k}{2m}\right] \quad \text{and} \quad D_a = f(I_a).$$

Given $a \in A$, we shall let a_i denote its ith coordinate.

For any subset X of $E^{n+1} = E^n\times E^1$, define

$$S(X) = \bigcup \{x\times E^1 |\ x \in E^n \text{ and } (x\times E^1) \cap X \neq \emptyset\}.$$

LEMMA 3.1. *For each* $\varepsilon > 0$ *there exists a covering* $\{P_a |\ a \in A\}$ *of* I^k *in* E^{n+1} *of* $(n+1)$*-cubes* P_a *and a homeomorphism* g *of* E^{n+1} *such that:*

(3.1.1) *Each* P_a *is a product of closed intervals,*

(3.1.2) $P = \bigcup\{P_a|\ a \in A\}$ *is an* $(n+1)$*-cell with* $I^k \subset \operatorname{int} P$,

(3.1.3) $P_a \cap P_b$ *is a face of each,*

(3.1.4) $P_a \cap I_k = I_a$,

(3.1.5) $D \subset g(\operatorname{int} P)$,

(3.1.6) $g(P_a) \subset N_\varepsilon(D_a)$, *and*

(3.1.7) $S\big(g(P_a)\big) \cap S\big(g(P_b)\big) = \emptyset$ *whenever* $I_a \cap I_b = \emptyset$.

Proof. Notice that $S(D_a) \cap S(D_b) = \emptyset$ whenever $I_a \cap I_b = \emptyset$ since $D = f(I^k) \subset E^n \times 0$. Hence, there exists $0 < \delta < \frac{1}{2}\varepsilon$ such that if $\{X_a|\ a \in A\}$ is a collection of subsets of E^{n+1} with $X_a \subset N_{2\delta}(D_a)$, then $S(X_a) \cap S(X_b) = \emptyset$ whenever $I_a \cap I_b = \emptyset$.

I. If D is flat in E^{n+1}, then $f\colon I^k \to E^{n+1}$ extends to a homeomorphism $g\colon E^{n+1} \to E^{n+1}$. Since $g(I_a) = f(I_a) = D_a$, we can easily find the desired covering $\{P_a|\ a \in A\}$ of I^k.

II. If $n-k \geqslant 2$ (and $n \geqslant 4$), we shall require the following two facts.

HOMMA'S APPROXIMATION THEOREM [5]. *Let f be an embedding of a closed combinatorial m-manifold M into a combinatorial n-manifold N, $n-m \geqslant 3$. Then for each $\varepsilon > 0$, there exists a* PL *embedding $g\colon M \to N$ such that $d\big(f(x), g(x)\big) < \varepsilon$ for all $x \in M$.*

ENGULFING THEOREM (Bryant and Seebeck [3]). *Suppose that K is a compact k-dimensional* ANR *in E^n, $n-k \geqslant 3$, $n \geqslant 5$, such that $E^n - K$ is uniformly locally simply connected* (1-ULC) *and $\delta > 0$. Then there exists $\eta > 0$ such that if $f\colon K \to E^n$ is an η-homeomorphism and U is an open subset of E^n containing $f(K)$, then there is a δ-homeomorphism h of E^n such that $h(U) \supset K$.*

Since $n-k \geqslant 2$ and $D \subset E^n \times 0$, it is clear that $E^{n+1} - D$ is 1-ULC. Choose $\eta > 0$ corresponding to the δ we have chosen according to the Engulfing Theorem. Note that $f\colon I^k \to E \times 0$ extends to an embedding $\bar{f}\colon \operatorname{Bd} I^{k+1} \to E^{n+1}$, so that Homma's Approximation Theorem applies to give a piecewise linear embedding $\varphi\colon I^k \to E^{n+1}$ such that $d\big(\varphi(x), f(x)\big) < \eta$ for each $x \in I^k$. Since $(n+1)-k \geqslant 3$, we may extend φ to a homeomorphism g' of E^{n+1} onto itself [1].

Assume that $\eta < \delta$ and construct a covering $\{P_a|\ a \in A\}$ of I^k in E^{n+1} by $(n+1)$-cubes satisfying (3.1.1) through (3.1.4) and with the additional property that $g'(P_a) \subset N_\delta(D_a)$. Then, by the Engulfing Theorem, there exists a δ-homeomorphism g'' of E^{n+1} such that $g''\big(g'(\operatorname{int} P)\big) \supset D$.

Let $g = g''g'$. Then for each $a \in A$, $g(P_a) \subset N_{2\delta}(D_a) \subset N_\varepsilon(D_a)$, and $S\big(g(P_a)\big) \cap S\big(g(P_b)\big) = \emptyset$ whenever $I_a \cap I_b = \emptyset$. Q.E.D.

Let N_1 be a compact neighborhood of D in E^n and let ε_1 be a positive number. Choose $\{P_a^1|\ a \in A\}$, with $P^1 = \bigcup\{P_a^1|\ a \in A\}$, and g_1 as in Lemma 3.1 so that $g_1(P^1) \subset N_1 \times [-\varepsilon_1, \varepsilon_1]$. Let N_2 be a compact neighborhood of D in E^n and $\varepsilon_2 > 0$, $\varepsilon_2 < \varepsilon_1$, such that $N_2 \times [-\varepsilon_2, \varepsilon_2] \subset g_1(P^1)$. Choose $\{P_a^2|\ a \in A\}$, with $P^2 = \bigcup\{P_a^2|\ a \in A\}$, and g_2 as in Lemma 3.1 so that $g_2(P^2) \subset N_2 \times [-\varepsilon_2, \varepsilon_2]$.

Continue in this manner to obtain coverings $\{P_a^1\}, \{P_a^2\}, \ldots, \{P_a^{m-1}\}$ of I^k, with $P^i = \bigcup\{P_a^i|\ a \in A\}$; homeomorphisms $g_1, g_2, \ldots, g_{m-1}$ of E^{n+1}; and neighborhoods

$$M_1 = N_1 \times [-\varepsilon_1, \varepsilon_1]\,, \;\ldots,\; M_m = N_m \times [-\varepsilon_m, \varepsilon_m] \text{ of } D \text{ in } E^{n+1},$$

with $\varepsilon_1 > \ldots > \varepsilon_m$, such that

$$M_1 \supset g_1(P^1) \supset M_2 \supset g_2(P^2) \supset \ldots \supset g_{m-1}(P^{m-1}) \supset M_m\,.$$

It follows from the proof of Lemma 3.1 that g_i and $\{P_a^i|\ a \in A\}$ can be chosen so that

(3.2) $S\big(g_i(P_a^i)\big) \cap S\big(g_j(P_b^j)\big) = \emptyset$ and

(3.3) $S\big(g_i(P_a^i)\big) \cap D_b \times E^1 = \emptyset$

whenever $I_a \cap I_b = \emptyset$. By the construction of $M_1, \ldots, M_m$, there exists a homeomorphism ψ of $E^{n+1} = E^n \times E^1$ that changes only E^1-coordinates such that

$$\psi(M_1) = N_1 \times [0, 2m-1]\,,$$
$$\psi(M_2) = N_2 \times [1, 2m-2]\,,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\psi(M_m) = N_m \times [m-1, m]\,.$$

Let $Q_a^i = \psi g_i(P_a^i)$ and $Q^i = \psi g_i(P^i)$, and consider the following subsets of Q^i, $i = 1, 2, \ldots, m-1$:

(a) $Q_r^i = \bigcup\{Q_a^i|\ a_k \geqslant r\}$, $r = 1, \ldots, 2m$;

(b) for $a' \in A' = \{(a'_1, a'_2, \ldots, a'_{k-1})|\ 1 \leqslant a'_i \leqslant p\}$,

$$R_{a'}^i = \bigcup\{Q_a^i|\ a_i = a'_i, \text{ for } 1 \leqslant i < k\}\,.$$

Then, as an immediate consequence of our construction, we have:

(3.4) $S(Q_a^i) \cap S(Q_b^j) = \emptyset$ if $I_a \cap I_b = \emptyset$;

(3.5) $S(R_{a'}^i) \cap S(R_{b'}^j) = \emptyset$ if $\max\{|a'_l - b'_l|\colon l = 1, \ldots, k-1\} > 1$;

(3.6) $S(Q_r^j) \cap Q^i \subset Q_{r-1}^i$ for $r = 2, 3, \ldots, 2m$; and

(3.7) if we let $I_{a'} = \bigcup\{I_a|\ a_i = a'_i, \text{ for } 1 \leqslant i < k\}$ for $a' \in A'$, then $S(R_{a'}^i) \cap f(I_{b'}) \times E^1 = \emptyset$ whenever $\max\{|a'_l - b'_l|\colon l = 1, \ldots, k-1\} > 1$.

4. The shrinking isotopies.

LEMMA 4.1. *Given that $1 \leqslant i \leqslant m-1$, let X be a compact set in the interior of Q^i. Then for each $r = 2, 3, \ldots, 2m$, there exists an isotopy λ_t $(t \in [0,1])$ of E^{n+1} such that*

(1) $\lambda_0 = $ *identity*,

(2) $\lambda_t \,|Q_r^i \cup (E^{n+1} - Q^i) = $ *identity for each* $t \in [0,1]$,

(3) $\lambda_1(X) \subset Q_{r-1}^i$, *and*

(4) $\lambda_t(R_{a'}^i) = R_{a'}^i$ *for each* $t \in [0,1]$, $a' \in A'$.

Proof. Let $P_s^i = (\psi g_i)^{-1}(Q_s^i)$ for $1 \leqslant s \leqslant 2m$, and let $Y = (\psi g_i)^{-1}(X)$. We can easily obtain an isotopy λ_t' $(t \in [0,1])$ that changes only kth coordinates of points of E^{n+1} with the properties:

(a) $\lambda_0' = $ identity,

(b) $\lambda_t'|\ P_r^i \cup (E^{n+1} - P^i) = $ identity, and

(c) $\lambda_1'(Y) \subset P_{r-1}^i$.

Then $(\psi g_i)\lambda_t'(\psi g_i)^{-1} = \lambda_t$ $(t \in [0,1])$ has the required properties.

Remark. Let I^{k-1} denote the face of I^k determined by $x_k = 0$. Notice that for each $i = 1, 2, \ldots, m-1$, and each $x \in I^{k-1}$, (3.7) gives us that $f(x \times I) \times w$ lies in the sum of 2^{k+1} of the sets $S(R_{a'}^i)$, for each $w \in E^1$.

THEOREM 4.2. *There exists an isotopy h_t $(t \in [0,1])$ of E^{n+1} such that*

$h_0 = $ *identity*,

$h_t|\ E^{n+1} - N_1 \times [0, 2m-1] = $ *identity*,

$h_t|\ Q_{2m}^1 = $ *identity*,

$h_t|\ Q_{2m-2}^1 \cap (E^{n+1} - N_{m-1} \times [m-2, m+1]) = $ *identity*,

$h_t|\ Q_{2m-4}^1 \cap (E^{n+1} - N_{m-2} \times [m-3, m+2]) = $ *identity*,

. . . .

$h_t|\ Q_4^1 \cap (E^{n+1} - N_2 \times [1, 2m-2]) = $ *identity*,

$h_1(D \times [m-1, m]) \subset Q_{2m-3}^1$,

$h_1(D \times [m-2, m+1]) \subset Q_{2m-5}^1$,

. . . .

$h_1(D \times [2, 2m-3]) \subset Q_3^1$,

and for each $x \in I^{k-1}$ and each $w \in E^1$, there exist 2^{k+2m-1} of the sets $S(R_{a'}^1)$ whose union is connected and contains

$$\big(f(x \times I) \times w\big) \cup h_1\big(f(x \times I) \times w\big).$$

Proof. We shall construct h_t as a composition of $m-1$ isotopies $h_t^1, h_t^2, \ldots, h_t^{m-1}$ of E^{n+1} just as in [1], Theorem 3.

Let h_t^{m-1} $(t \in [0,1])$ be the isotopy of E^{n+1} given by Lemma 4.1 with $i = m-1$, $X = D \times [m-1, m]$, and $r = 2m-1$. Notice that, by (3.6), $S(Q_{2m}^1) \cap Q^{m-1} \subset Q_{2m-1}^{m-1}$ so that $h_t^{m-1}|\ Q_{2m}^1 = $ identity. By (3.7), the Remark, and the fact that $h_1^{m-1}(R_{a'}^{m-1}) = R_{a'}^{m-1}$ for each $a' \in A'$, we have that $\big(f(x \times I) \times w\big) \cup h_1^{m-1}\big(f(x \times I) \times w\big)$ lies in the sum of $2^{(k+1)+2}$ of the sets $S(R_{a'}^1)$ that is connected.

We let h_t^{m-2} $(t \in [0,1])$ be the isotopy given by Lemma 4.1 with $i = m-2$, $X = h_1^{m-1}(D \times [m-2, m+1])$, and $r = 2m-3$. Since $Q_{2m-2}^{m-1} \subset Q_{2m-3}^{m-2}$, $h_1^{m-2}h_1^{m-1}(D \times [m-1, m]) \subset Q_{2m-2}^{m-1}$; and since $S(Q_{2m-2}^1) \cap Q^{m-2} \subset Q_{2m-3}^{m-2}$, by (3.6), we have

$$h_t^{m-2}h_t^{m-1}|\ Q_{2m-2}^1 \cap (E^{n+1} - N_{m-1} \times [m-2, m+1]) = \text{identity}.$$

Furthermore, by condition (4) of Lemma 4.1, we have that, for each $x \in I^{k-1}$ and $w \in E^1$,

$$\big(f(x \times I) \times w\big) \cup \big(h_1^{m-2}h_1^{m-1}\big(f(x \times I) \times w\big)\big)$$

lies in the sum of $2^{(k+1)+4}$ of the sets $S(R_{a'}^1)$ that is connected.

Continuing in this manner, we obtain $h_t = h_t^1 h_t^2 \ldots h_t^{m-1}$ $(t \in I)$ with the desired properties.

Now let $U_r = g_1\big(\mathrm{Cl}(P_r^1 - P_{r-1}^1)\big)$ for $r = 1, 2, \ldots, 2m$, $P_0^1 = \emptyset$, $U = g_1(P^1)$, and $V_{a'} = g_1(R_{a'}^1)$ for $a' \in A'$. Notice that $U_r = \bigcup \{g_1(P_a^1)|\ a_k = r\}$.

THEOREM 4.3. *There exists an isotopy h_t $(t \in I)$ of E^{n+1} such that:*

(1) $h_0 = $ *identity*,

(2) $h_t\ |E^n - S(U) = $ *identity*,

(3) h_1 *is uniformly continuous*,

(4) *for each $w \in E^1$, there exists i such that*

$$h_1\big(f(I^k) \times w\big) \subset S(U_i \cup \ldots \cup U_{i+7}) \cap E^n \times [w - 2m + 1, w + 2m - 1],$$

and

(5) *for each $x \in I^{k-1}$ and $w \in E^1$, there exist 2^{k+4m-3} of the sets $S(V_{a'})$ whose union is connected and contains $\big(f(x \times I) \times w\big) \cup h_1\big(f(x \times I) \times w\big)$.*

Proof. The proof is similar to that of Lemma 2 in [2].

COROLLARY 4.4. *$H(n, k, 1)$ is true whenever either of the conditions* I *or* II *is satisfied.*

Proof. Suppose that $\varepsilon > 0$ is given. For $a' = (a'_1, \ldots, a'_{k-1}) \in A'$, $1 \leqslant a'_i \leqslant p$, let

$$J_{a'} = \left[\frac{a'_1-1}{p}, \frac{a'_1}{p}\right] \times \ldots \times \left[\frac{a'_{k-1}-1}{p}, \frac{a'_{k-1}}{p}\right].$$

Then $\{J_{a'}\}_{a' \in A'}$ is a subdivision of I^{k-1} and for $a = (a_1, a_2, \ldots, a_k) \in A$,

$$I_{a'} = J_{a'} \times \left[\frac{a_k-1}{2m}, \frac{a_k}{2m}\right],$$

where $a'_i = a_i$ for $1 \leqslant i < k$. Choose integers m and p so that for each $i = 1, \ldots, 2m-7$, the union of any 2^{k+4m-3} of the sets

$$f\left(J_{a'} \times \left[\frac{i-1}{2m}, \frac{i+7}{2m}\right]\right)$$

that is connected has diameter less than $\varepsilon/6$. Now choose the covering $\{P_a^1\}_{a \in A}$ of I^k and g_1 so that $g_1(P_a^1) \subset N_{\varepsilon/6}(D_a)$.

Change the scale on the E^1-coordinates of points in E^{n+1} so that the isotopy of Theorem 4.3 changes $(n+1)$st-coordinates less than $\varepsilon/2$. Then (1) through (4) of $H(n, k, 1)$ are clearly satisfied. Given $w \in E^1$, choose i as in (4) of Theorem 4.3 and $y \in \left[\frac{i-1}{2m}, \frac{i+7}{2m}\right]$. Then for each $x \in I^{k-1}$,

$$\big(f(x \times I) \times w\big) \subset N_\varepsilon\big(f(x, y) \times w\big) \quad \text{and} \quad \operatorname{diam}\big(f(x \times I) \times w\big) < \varepsilon .$$

Hence, (5) and (6) of $H(n, k, 1)$ are also satisfied. This completes the proof of Theorem 1.1.

It was conjectured in [1] that if D is a k-cell in E^n, then $E^n/D \times E^k \approx E^{n+k}$. If $n = 3$ or if $n-k \geqslant 2$ for $n > 3$, Theorem 1.1 gives an affirmative solution to this problem with E^1 being required instead of E^k in order to obtain a Euclidean space. In light of the results of this paper it seems reasonable to ask the following question concerning the two cases that remain unsolved.

QUESTION. *If D is an $(n-i)$-cell in E^n, $i = 0, 1$, is $E^n/D \times E^{3-i} \approx E^{n+3-i}$?*

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