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Remark on strongly additive set functions

by

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A set function μ with values in an abelian group, defined on an additive class of sets $\mathcal{A}$ is called *additive* if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for every pair A, B of disjoint sets in $\mathcal{A}$. A set function λ defined on a lattice of sets $\mathfrak{L}$ containing the empty set $\emptyset$ is called *strongly additive* if its values lie in an abelian group, $\lambda(\emptyset) = 0$ and

$$\lambda(A) - \lambda(A \cap B) = \lambda(A \cup B) - \lambda(B)$$

for every $A \in \mathfrak{L}$ and $B \in \mathfrak{L}$. We call any additive and substractive class of sets a *ring*.

If a set function λ with values in an abelian group G defined on a lattice of sets $\mathfrak{L}$ containing the empty set may be extended to an additive set function μ with values in G defined on a ring containing $\mathfrak{L}$, then $\lambda(\emptyset) = \mu(\emptyset) = 0$ and, for any $A \in \mathfrak{L}$ and $B \in \mathfrak{L}$, we have

$$\lambda(A) - \lambda(A \cap B) = \mu(A \setminus B) = \lambda(A \cup B) - \lambda(B),$$

so that λ is a strongly additive set function. The purpose of this paper is to show that the converse is also true. Namely, we shall prove the following

THEOREM. *Every strongly additive set function defined on a lattice of sets containing the empty set may be extended in a unique manner to an additive set function defined on the smallest ring of sets containing this lattice.*

In the proof of this theorem the notion of a disjoint union of sets will be used. The disjoint union of a system of sets $A_1, A_2, \ldots, A_n$ is defined if and only if these sets are mutually disjoint and in that case it is defined as the usual set-theoretic union and is denoted by $A_1 \dot{\cup} A_2 \dot{\cup} \ldots \dot{\cup} A_n$, or by $\dot{\bigcup}_{k=1,2,\ldots,n} A_k$.

LEMMA. *Let $\mathfrak{L}$ be a lattice of sets containing the empty set. Let R be the class of all sets of the form $\dot{\bigcup}_{k=1,2,\ldots,n} (A_k \setminus B_k)$, where $A_k \in \mathfrak{L}$ and $B_k \in \mathfrak{L}$ for $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ Then R is the smallest ring containing $\mathfrak{L}$.*

This lemma is known (see [1], exercise (2), p. 25 and exercise (3e), p. 26), but we shall give a proof, since it is based on some formulas which are needed also in the sequel.

Proof of the lemma. For any sets A, B, C, D we have

$$(1)\qquad (A\setminus B)\cap(C\setminus D)=(A\cap C)\setminus(B\cup D)\,,$$

$$(2)\qquad (A\setminus B)\setminus(C\setminus D)=[A\setminus(B\cup C)]\cup[(A\cap D)\setminus B] =[A\setminus(B\cup C)]\,\dot\cup\,[(A\cap C\cap D)\setminus B]\,,$$

$$(3)\qquad (A\setminus B)\cup(C\setminus D)=[A\setminus(B\cup C)]\,\dot\cup\,[(A\cap C\cap D)\setminus B]\,\dot\cup\,[C\setminus D]\,.$$

Since clearly R contains $\mathfrak{L}$ and is itself contained in the smallest ring containing $\mathfrak{L}$, we need only to prove that R is a ring. It follows from (1) and (2) that if C, D, A_k and B_k, $k=1,2,\dots,n$, belongs to $\mathfrak{L}$, then

$$(4)\qquad \bigcap_{k=1}^{n}[(C\setminus D)\setminus(A_k\setminus B_k)]\in R,$$

so that if, furthermore, the sets $(A_k\setminus B_k)$, $k=1,2,\dots,n$, are mutually disjoint, then

$$[\dot{\bigcup_{k=1,2,\dots,n}}(A_k\setminus B_k)]\cup(C\setminus D) =[\dot{\bigcup_{k=1,2,\dots,n}}(A_k\setminus B_k)]\,\dot\cup\,[\bigcap_{k=1}^{n}\{(C\setminus D)\setminus(A_k\setminus B_k)\}]\in R\,.$$

From this, by an induction in N, we deduce that, for $N=1,2,\dots$, if $A_n\in\mathfrak{L}$ and $B_n\in\mathfrak{L}$ for $n=1,2,\dots,N$, then $\bigcup_{n=1}^{N}(A_n\setminus B_n)\in R$, which means that R is an additive class of sets. If A_m, B_m, C_n and D_n are in $\mathfrak{L}$ for $m=1,2,\dots,M$ and $n=1,2,\dots,N$, then, by (4) and since R is additive, we have

$$[\bigcup_{m=1}^{M}(A_m\setminus B_m)]\setminus[\bigcup_{n=1}^{N}(C_n\setminus D_n)]=\bigcup_{m=1}^{M}\bigcap_{n=1}^{N}[(A_m\setminus B_m)\setminus(C_n\setminus D_n)]\in R\,,$$

and so R is substractive. Thus R is a ring and the lemma is proved.

Proof of the theorem. It is clear from the lemma that if the desired extension μ of λ exists, then for any set $A\in R$ and any representation of it in the form

$$(5)\qquad A=\dot{\bigcup_{n=1,2,\dots,N}}(A_n\setminus B_n)\,,$$

where $A_n\in\mathfrak{L}$ and $B_n\in\mathfrak{L}$, we have

$$(6)\qquad \mu(A)=\sum_{n=1}^{N}\big(\lambda(A_n)-\lambda(A_n\cap B_n)\big)\,.$$

From this the uniqueness of the extension μ is obvious. It is also obvious that for the proof of the existence of such an extension we need only to prove that, for any set $A\in R$, the right side of (6) is independent of the particular representation of A in form (5).

The proof of such independence will be corned out in several steps. Everywhere in the sequel the big Latin letters denote the elements of $\mathfrak{L}$.

Step 1. If $A\setminus B=(C_1\setminus D_1)\,\dot\cup\,(C_2\setminus D_2)$, then

$$\lambda(A)-\lambda(A\cap B)=\lambda(C_1)-\lambda(C_1\cap D_1)+\lambda(C_2)-\lambda(C_2\cap D_2)\,.$$

Proof. Since $C_1\setminus D_1$ and $C_2\setminus D_2$ are disjoint, by (1) we have

$$(7)\qquad C_1\cap C_2\subset D_1\cup D_2\,.$$

Since $C_i\setminus D_i\subset A\setminus B$, by (2) we have

$$\emptyset=(C_i\setminus D_i)\setminus(A\setminus B)=[C_i\setminus(D_i\cup A)]\cup[(C_i\cap B)\setminus D_i]\,,$$

form which

$$(8)\qquad C_i\subset A\cup D_i$$

and

$$(9)\qquad B\cap C_i\subset D_i$$

or $i=1,2$. Since, by (2)

$$[A\setminus(B\cup C_1)]\cup[(A\cap D_1)\setminus B]=(A\setminus B)\setminus(C_1\setminus D_1)\subset C_2\setminus D_2\,,$$

we have, again by (2),

$$[A\setminus(B\cup C_1\cup C_2)]\cup[(A\cap D_2)\setminus(B\cup C_1)]=[A\setminus(B\cup C_1)]\setminus[C_2\setminus D_2]=\emptyset$$

and

$$[(A\cap D_1)\setminus(B\cup C_2)]\cup[(A\cap D_1\cap D_2)\setminus B]=[(A\cap D_1)\setminus B]\setminus[C_2\setminus D_2]=\emptyset\,,$$

from which

$$(10)\qquad A\subset B\cup C_1\cup C_2\,,$$

$$(11)\qquad A\cap D_2\subset B\cup C_1\,,$$

$$(12)\qquad A\cap D_1\subset B\cup C_2\,,$$

$$(13)\qquad A\cap D_1\cap D_2\subset B\,.$$

Put

$$\mathfrak{V}=\lambda(A)-\lambda(A\cap B)-\lambda(C_1)+\lambda(C_1\cap D_1)-\lambda(C_2)+\lambda(C_2\cap D_2)\,.$$

We have to prove that $\mathfrak{V}=0$. Since, by (10) and by strong additivity of λ,

$$\begin{aligned}\lambda(A)-\lambda(A\cap B)&=\lambda\big(A\cap(B\cup C_1\cup C_2)\big)-\lambda(A\cap B)\\&=\lambda\big(A\cap(C_1\cup C_2)\big)-\lambda\big(A\cap B\cap(C_1\cup C_2)\big)\\&=\lambda(A\cap C_1)+\lambda(A\cap C_2)-\lambda(A\cap C_1\cap C_2)-\\&\quad-\lambda(A\cap B\cap C_1)-\lambda(A\cap B\cap C_2)+\lambda(A\cap B\cap C_1\cap C_2)\,,\end{aligned}$$

and, by (8) and by the strong additivity of λ,

$$\lambda(C_i)-\lambda(C_i\cap D_i)=\lambda\big(C_i\cap(A\cup D_i)\big)-\lambda(C_i\cap D_i)$$
$$=\lambda(A\cap C_i)-\lambda(A\cap C_i\cap D_i)$$

for $i=1,2$, we have

$$\mho=\lambda(A\cap C_1\cap D_1)+\lambda(A\cap C_2\cap D_2)+\lambda(A\cap B\cap C_1\cap C_2)-$$
$$-\lambda(A\cap B\cap C_1)-\lambda(A\cap B\cap C_2)-\lambda(A\cap C_1\cap C_2)\,.$$

By (11), (12) and (9),

$$\lambda(A\cap C_i\cap D_i)-\lambda(A\cap B\cap C_i)$$
$$=\lambda\big(A\cap C_i\cap D_i\cap(B\cup C_{3-i})\big)-\lambda(A\cap B\cap C_i\cap D_i)$$
$$=\lambda(A\cap C_1\cap C_2\cap D_i)-\lambda(A\cap B\cap C_1\cap C_2\cap D_i)\,,$$

by (7)

$$\lambda(A\cap C_1\cap C_2)=\lambda\big(A\cap C_1\cap C_2\cap(D_1\cup D_2)\big)$$
$$=\lambda(A\cap C_1\cap C_2\cap D_1)+\lambda(A\cap C_1\cap C_2\cap D_2)-\lambda(A\cap C_1\cap C_2\cap D_1\cap D_2)$$

and, by (9) and (7)

$$A\cap B\cap C_1\cap C_2=A\cap B\cap C_1\cap C_2\cap D_i=A\cap C_1\cap C_2\cap D_1\cap D_2\,.$$

Thus $\mho=0$.

Step 2. If $C\backslash D\subset A\backslash B$, then there are E_i and F_i, $i=1,2$, such that

(14) $$A\backslash B=(C\backslash D)\,\dot\cup\,(E_1\backslash F_1)\,\dot\cup\,(E_2\backslash F_2)$$

and

(15) $$\lambda(A)-\lambda(A\cap B)$$
$$=\lambda(C)-\lambda(C\cap D)+\lambda(E_1)-\lambda(E_1\cap F_1)+\lambda(E_2)-\lambda(E_2\cap F_2).$$

Indeed, put $E_1=A$, $F_1=B\cup C$, $F_2=A\cap C\cap D$, $F_2=B$. Then by (3), (14) holds and in order to prove that (15) holds we must prove that

$$\lambda(A)-\lambda(A\cap B)=\lambda(C)-\lambda(C\cap D)+\lambda(A)-\lambda\big(A\cap(B\cup C)\big)+$$
$$+\lambda(A\cap C\cap D)-\lambda(A\cap B\cap C\cap D)\,.$$

Since, by the strong additivity of λ,

$$\lambda\big(A\cap(B\cup C)\big)=\lambda(A\cap B)+\lambda(A\cap C))-\lambda(A\cap B\cap C)\,,$$

this is equivalent to proving that

(16) $$\lambda(C)-\lambda(C\cap D)-\lambda(A\cap C)+\lambda(A\cap C\cap D)+$$
$$+\lambda(A\cap B\cap C)-\lambda(A\cap B\cap C\cap D)=0\,.$$

Since $C\backslash D\subset A\backslash B$, by (2) we have

$$\emptyset=(C\backslash D)\backslash(A\backslash B)=[C\backslash(A\cup D)]\cup[(C\cap B)\backslash D]\,,$$

so that $C\subset A\cup D$ and $C\cap B\subset D$, which implies that

$$B\cap C\cap D=B\cap C$$

and

$$\lambda(C)=\lambda\big(C\cap(A\cup D)\big)=\lambda(C\cap D)+\lambda(A\cap C)-\lambda(A\cap C\cap D)\,,$$

and so (16) follows.

Step 3. If

$$A\backslash B=\dot{\bigcup_{j=1,2,\ldots,n}}(C_j\backslash D_j)\,,$$

then

$$\lambda(A)-\lambda(A\cap B)=\sum_{j=1}^{n}\big(\lambda(C_j)-\lambda(C_j\cap D_j)\big)\,.$$

Suppose that this statement is true for an $n=k$ and let

$$A\backslash B=\dot{\bigcup_{j=1,2,\ldots,k+1}}(C_j\backslash D_j)\,.$$

By step 2 there are E_i and F_i, $i=1,2$, such that

$$A\backslash B=(C_{k+1}\backslash D_{k+1})\,\dot\cup\,(E_1\backslash F_1)\,\dot\cup\,(E_2\backslash F_2)$$

and

(17) $$\lambda(A)-\lambda(A\cap B)=\lambda(C_{k+1})-\lambda(C_{k+1}\cap D_{k+1})+\lambda(E_1)-\lambda(E_1\cap F_1)+$$
$$+\lambda(E_2)-\lambda(E_2\cap F_2)\,.$$

We then have by (1)

$$E_i\backslash F_i=\dot{\bigcup_{j=1,2,\ldots,k}}[(C_j\backslash D_j)\cap(E_i\backslash F_i)]=\dot{\bigcup_{j=1,2,\ldots,k}}[(C_j\cap E_i)\backslash(D_j\cup F_i)]$$

for $i=1,2$ and so, since the statement of step 3 is assumed to be true for $n=k$,

(18) $$\lambda(E_i)-\lambda(E_i\cap F_i)=\sum_{j=1}^{k}\Big(\lambda(C_j\cap E_i)-\lambda\big(C_j\cap E_i\cap(D_j\cup F_i)\big)\Big)\,.$$

Since, for $j=1,2,\ldots,k$ by (1),

$$C_j\backslash D_j=\dot{\bigcup_{i=1,2}}[(C_j\backslash D_j)\cap(E_i\backslash F_i)]=\dot{\bigcup_{i=1,2}}[(C_j\cap E_i)\backslash(D_j\cup F_i)]$$

by the step 1 we have

(19) $$\lambda(C_j)-\lambda(C_j\cap D_j)=\sum_{i=1}^{2}\Big(\lambda(C_j\cap E_i)-\lambda\big(C_j\cap E_i\cap(D_j\cup F_i)\big)\Big)$$

for $j=1,2,\ldots,k$. It follows from (17), (18) and (19) that

$$\lambda(A)-\lambda(A\cap B)=\sum_{j=1}^{k+1}\big(\lambda(C_j)-\lambda(C_j\cap D_j)\big)\,.$$

Thus we see that if the statement of the step 3 is true for $n = k$, then it is also true for $n = k+1$. By step 1, this statement is true for $n = 1$ and $n = 2$. Hence, by the principle of mathematical induction, it is true for every $n = 1, 2, \ldots$

Step 4. If

$$\dot{\bigcup_{i=1,2,\ldots,m}} (A_i \setminus B_i) = \dot{\bigcup_{j=1,2,\ldots,n}} (C_j \setminus D_j),$$

then

$$\sum_{i=1}^{m} \big(\lambda(A_i) - \lambda(A_i \cap B_i)\big) = \sum_{j=1}^{n} \big(\lambda(C_j) - \lambda(C_j \cap D_j)\big).$$

We have

$$A_i \setminus B_i = \dot{\bigcup_{j=1,2,\ldots,n}} [(A_i \setminus B_i) \cap (C_j \setminus D_j)] = \dot{\bigcup_{j=1,2,\ldots,n}} [(A_i \cap C_j) \setminus (B_i \cup D_j)]$$

and

$$C_j \setminus D_j = \dot{\bigcup_{i=1,2,\ldots,m}} [(A_i \setminus B_i) \cap (C_j \setminus D_j)] = \dot{\bigcup_{i=1,2,\ldots,m}} [(A_i \cap C_j) \setminus (B_i \cup D_j)],$$

so that, by step 3,

$$\lambda(A_i) - \lambda(A_i \cap B_i) = \sum_{j=1}^{n} \Big[\lambda(A_i \cap C_j) - \lambda\big(A_i \cap C_j \cap (B_i \cup D_j)\big)\Big]$$

and

$$\lambda(C_j) - \lambda(C_j \cap D_j) = \sum_{i=1}^{m} \Big[\lambda(A_i \cap C_j) - \lambda\big(A_i \cap C_j \cap (B_i \cup D_j)\big)\Big].$$

Hence

$$\sum_{i=1}^{m} \big(\lambda(A_i) - \lambda(A_i \cap B_i)\big) = \sum_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} \Big(\lambda(A_i \cap C_j) - \lambda\big(A_i \cap C_j \cap (B_i \cup D_j)\big)\Big)$$

$$= \sum_{j=1}^{n} \big(\lambda(C_j) - \lambda(C_j \cap D_j)\big).$$

Thus the statement of step 4 is proved and so the proof of our theorem is completed.

Reference

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