Fixed points and proximate fixed points

by

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1. Introduction. In 1961 V. Klee [5] introduced the concept of an ε-continuous single-valued function on a topological space into a metric space. At the same time he defined the proximate fixed point property (p.f.p.p.) for such functions, and proved that each compact metric absolute retract has the p.f.p.p. In his concluding paragraph Klee remarked that extensions to uniform spaces and multi-valued functions were almost immediate. In particular he noted that Kakutani's [3] and Fan's [1] fixed point theorems could be generalized.

In this paper we give definitions of ε-continuous multi-valued functions in terms of uniform spaces, and we derive a general fixed point theorem which has as a corollary the generalization of Fan's theorem mentioned by Klee. In addition the main theorem of this paper and one other theorem show the close relationship between the proximate fixed point property for certain classes of multi-valued functions and the fixed point property for certain classes of upper semi-continuous multi-valued functions.

2. Preliminaries. In this section we give the basic definitions, and derive some of the fundamental results. A *multi-valued function* or *multifunction F on a space X into a space Y* is a point to set correspondence $x \to F(x)$ such that $F(x)$ is a nonempty, closed subset of Y for each $x \in X$. Henceforth, we shall use the terms function, multi-valued function, and multifunction interchangeably. The reader should refer to chapter 6 of Kelley [4] for notation and fundamental properties of uniform spaces.

DEFINITIONS. Let X and Y be topological spaces and let $F\colon X \to Y$ be a multifunction on X into Y. Then we have the following definitions.

(1) The function F is said to be *upper semicontinuous* (u.s.c.) if and only if for each closed set $A \subset Y$, the set

$$F^{-1}(A) = \{x \in X \mid F(x) \cap A \neq \square\}$$

is a closed subset of X.

(2) The function F is said to be *lower semi-continuous* (l.s.c.) if and only if for each open set $W \subset Y$, the set $F^{-1}(W)$ is an open subset of X.

(3) The function F is *continuous* if and only if it is both u.s.c. and l.s.c.

(4) A point $x \in X$ is a *fixed point of a function* $F\colon X \to X$ in case $x \in F(x)$. Then a space X is said to *have the fixed point property* (F.p.p.) if and only if each continuous function $F\colon X \to X$ has a fixed point.

The fixed point property for other classes of functions is defined in an analogous way. For example, we say that X has the F.p.p. for u.s.c., continuum valued functions in case each u.s.c., continuum valued function on X into X has a fixed point. In the preceeding the term continuum valued means that $F(x)$ is a compact, connected subset of X for each $x \in X$.

Now let X be a topological space, and let $(Y, \mathcal{V})$ be a uniform space. We shall assume that Y has the topology generated by the uniformity $\mathcal{V}$. In the following definitions assume that F is a multifunction on X into Y and $V \in \mathcal{V}$.

DEFINITIONS. (1) The function F is called *upper V-continuous* (u.V-c.) if and only if for each $x \in X$, there exists a neighborhood U of x such that

$$F(U) \subset V[F(x)] = \{y' \in Y \mid \text{for some } y \in F(x),\ (y, y') \in V\}.$$

(2) The function F is *lower V-continuous* (l.V-c.) if and only if for each $x \in X$ and for each $y \in F(x)$, there is a neighborhood U of x such that $x' \in U$ implies that $F(x') \cap V[y] \neq \square$.

(3) The function F is *V-continuous* (V-c.) if and only if it is both u.V-c. and l.V-c.

(4) The function F is *strongly lower V-continuous* (s.l. V-c.) if and only if for each $x \in X$ there exists a neighborhood U of x such that $x' \in U$ implies that $F(x') \cap V[y] \neq \square$ for all $y \in F(x)$.

It is clear that if F is s.l. V-c., then F is l.V-c. Moreover, if $F(x)$ is compact for each $x \in X$, then we can get a partial converse.

Before stating the converse we make the following convention. Let P be a property of sets (e.g. compact, closed, etc). Then we say that F *is point P* in case $F(x)$ has the property P for each $x \in X$.

PROPOSITION 1. *Let $F\colon X \to Y$ be a point compact multifunction. Suppose that V, V', and $\hat{V}$ are members of $\mathcal{V}$ such that $V \subset V \circ \hat{V} \subset V'$. If F is l.V-c., then F is s.l.V'-c.*

Proof. First, since each member of $\mathcal{V}$ contains a symmetric member of $\mathcal{V}$, we may assume that $\hat{V}$ is symmetric. Let $x \in X$. Since $F(x)$ is compact, there exists a finite set $\{y_1, \ldots, y_k\} \subset F(x)$ such that

$$F(x) \subset \bigcup \{\hat{V}[y_i] \mid i = 1, \ldots, k\}.$$

If F is l.V-c., then there is a neighborhood U_i of x, $1 \leqslant i \leqslant k$, such that $x' \in U_i$ implies that $F(x') \cap V[y_i] \neq \square$. Let $U = \bigcap \{U_i \mid i = 1, \ldots, k\}$, and let $x' \in U$. Then

$$F(x') \cap V[y_i] \neq \square \quad \text{for all} \quad i = 1, \ldots, k.$$

Further, if $y \in F(x)$, then there exists an i such that $y \in \hat{V}[y_i]$. Since there exists a $y' \in F(x')$ such that $(y_i, y') \in V$ and since $\hat{V}$ is symmetric, we have $(y, y') \in V \circ \hat{V} \subset V'$. That is, $y' \in V'[y]$, and so $F(x') \cap V'[y] \neq \square$. Thus, F is s.l.V'-c.

DEFINITION. A uniform space $(X, \mathcal{U})$ has the *proximate fixed point property* (p.F.p.p.) if and only if for each $U \in \mathcal{U}$ there exists a $\hat{U} \in \mathcal{U}$ such that whenever $F\colon X \to X$ is a $\hat{U}$-c. function there is a point $x \in X$ such that $F(x) \cap U[x] \neq \square$.

The proximate fixed point property for other classes of functions is defined analogously.

THEOREM 2. *Let X be a compact, Hausdorff space. If X has the* p.F.p.p., *then X has the* F.p.p.

Proof. Let $F\colon X \to X$ be continuous. Since X is compact, T_2, the collection $\mathcal{U}$ of all neighborhoods of the diagonal is the unique uniformity for X. Since F is U-c. for each $U \in \mathcal{U}$, there is an $x_U \in X$ such that $F(x_U) \cap U[x_U] \neq \square$ (under the assumption that X has the p.F.p.p.). Thus, for each $U \in \mathcal{U}$, there are points x_U and y_U in X such that $y_U \in F(x_U) \cap U[x_U]$. Since $\mathcal{U}$ is directed by inclusion, we define a net $N\colon \mathcal{U} \to X \times X$ by: $N(U) = (x_U, y_U)$ for each $U \in \mathcal{U}$. Further, since X is compact, the net N has a cluster point (x_0, y_0) in $X \times X$. If $(x_0, y_0) \notin \Delta = \{(x, x) \mid x \in X\}$, then there exist disjoint, open subsets S and T of $X \times X$ such that $(x_0, y_0) \in S$ and $\Delta \subset T$ (since X is compact, Hausdorff). But this is contrary to the facts that $\mathcal{U}$ is the family of all neighborhoods of Δ and that (x_0, y_0) is a cluster point of N. Hence, $(x_0, y_0) \in \Delta$; that is, $x_0 = y_0$. Finally, since F is u.s.c. and point closed, $x_0 = y_0 \in F(x_0)$, and hence, x_0 is a fixed point of F.

Remark. Theorem 2 can be stated for u.s.c. and u.U-c functions. Though in this case it is necessary to make some assumptions about the sets $F(x)$ in order to assure the existence of spaces with the p.F.p.p. and F.p.p. The main result of the paper is a converse of this form of Theorem 2.

3. The main theorem.

The main result of this paper has two variations, each of which is a converse of Theorem 2. Before stating the first form of the result, we need the following definitions.

DEFINITION. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces and let $V \in \mathcal{V}$. Then a multifunction $F\colon X \to Y$ is *uniformly upper V-continuous* (unif. u.V-c.) if and only if there is a $U \in \mathfrak{U}$ such that $F(U[x]) \subset V[F(x)]$ for all $x \in X$.

Throughout the remainder of this section, we assume that $(X, \mathfrak{U})$ is a uniform space and that $\mathfrak{U}_0$ is a base for $\mathfrak{U}$ which consists entirely of symmetric subsets of $X \times X$. In the following we let P denote a property of sets and let $P(X)$ be the subsets of X which have property P.

DEFINITION. A function $K\colon 2^X \to 2^X$ (where 2^X denotes the collection of all subsets of the set X) is called a *P-operator on X* if and only if the following conditions hold:

(i) if $\square \neq A \subset X$, then $\square \neq K(A) \in P(X)$;

(ii) if $A \subset B$, then $K(A) \subset K(B)$;

(iii) if $A \in P(X)$, then $K(A) = A$.

We consider the following conditions which P might satisfy:

(1) if $A \in P(X)$, then $A^* \in P(X)$, where A^* is the closure of A.

(2) If $A \in P(X)$ and if $U \in \mathfrak{U}_0$, then $U[A] \in P(X)$.

(3) There exists a P-operator on X.

THEOREM 3. *Let X be a compact, Hausdorff space, and let P be a property for which conditions* (1), (2), *and* (3) *are satisfied. If X has the* F.p.p. *for* u.s.c., *point P functions, then X has the* p.F.p.p. *for point P, uniformly upper V-continuous functions.*

Proof. Let $U \in \mathfrak{U}$. Choose $V \in \mathfrak{U}_0$ such that $V^2 = V \circ V \subset U$, and let F be a uniformly upper V-continuous function on X into X. Then there exists a $V' \in \mathfrak{U}$ such that $x' \in V'[x]$ implies that $F(x') \subset V[F(x)]$. Let $\hat{V} \in \mathfrak{U}_0$ be such that $\hat{V} \circ \hat{V} \subset V'$. Since X is compact, there is a finite set $\{x_i|\ i = 1, \ldots, k\}$ such that $X = \bigcup \{\hat{V}[x_i]^\circ|\ i = 1, \ldots, k\}$ where $^\circ$ denotes the interior. For each $x \in X$ define a nonempty, open set W_x by:

$$W_x = \bigcap \{\hat{V}[x_i]^\circ|\ x \in \hat{V}[x_i]^\circ\}.$$

Now define a function $G\colon X \to X$ by $G(x) = K\big(F(W_x)\big)^*$ where K is a P-operator whose existence is guaranteed by (3). Clearly G is a point P multifunction. Furthermore, if $G(x) \subset S = S^\circ \subset X$, then $G(W_x) \subset K\big(F(W_x)\big)^* \subset S$, and so G is upper semi-continuous. Hence, there exists an $x_0 \in X$ such that $x_0 \in G(x_0)$.

From the definition of G there is an $x_1 \in K\big(F(W_{x_0})\big)$ such that $x_1 \in V[x_0]$. Further, if $x \in W_{x_0}$, then $x \in \hat{V}[x_i]$ and $x_0 \in \hat{V}[x_i]$ for some x_i, and thus, $x \in \hat{V}^2[x_0]$. Therefore $F(x) \subset V[F(x_0)]$ since $\hat{V}^2 \subset V'$ and F is unif. u.V-c. That is, $F(W_{x_0}) \subset V[F(x_0)]$, and hence, $K\big(F(W_{x_0})\big) \subset K(V[F(x_0)]) = V[F(x_0)]$ by (2) and the facts that K is a P-operator and F is point P. Combining this result with $x_1 \in V[x_0]$ gives $F(x_0) \cap \hat{V}^2[x_0] \neq \square$. Therefore, $F(x_0) \cap U[x_0] \neq \square$, and the theorem follows.

The proof of Theorem 4 uses Proposition 1 and a construction similar to the one in the proof of Theorem 3.

THEOREM 4. *Let X be a compact, Hausdorff space, and let P be a property which satisfies conditions* (1), (2), *and* (3). *If X has the* F.p.p. *for point P, usc functions, then X has the* p.F.p.p. *for point P functions.*

4. Applications. For the first application let X be a nonempty, compact, convex subspace of a locally convex, Hausdorff, topological vector space. Let $\mathfrak{B}_0$ be a base for the neighborhood system of 0 which consists entirely of convex, symmetric neighborhoods of 0, and let sets of the form $\{(x, y)|\ y - x \in B\}$ for $B \in \mathfrak{B}_0$ be the members of $\mathfrak{U}_0$. Then $\mathfrak{U}_0$ is a symmetric base for the left uniformity for X when considered as a topological group. For P we take the property of being convex, and then the function K such that $K(A)$ is the convex hull of A, where $A \subset X$, is a P-operator. We then have the necessary conditions satisfied, and so, Theorem 3 together with Ky Fan's [1] and Glicksberg's [2] theorem gives:

THEOREM 5. *Each nonempty compact, convex subspace of a locally convex, Hausdorff, topological vector space has the* p.F.p.p. *for point convex, uniformly upper V-continuous functions.*

Also, by using Theorem 4, we get a result of Muenzenberger [6].

THEOREM 6. *Each nonempty, compact, convex subspace of a locally convex, Hausdorff, topological vector space has the* p.F.p.p. *for point convex functions.*

For the second application let X be a hereditarily unicoherent, arcwise connected, locally connected continuum. (Here an arc is a continuum with exactly two non-cut points.) In this case a subset A of X has property P if and only if A is connected, and we define the P-operator K by setting $K(A)$ equal to the union of the arcs with distinct endpoints in A if $\mathrm{Card}(A) \geqslant 2$ and $K(A)$ equal to A if $\mathrm{Card}(A) < 2$. Let $\mathfrak{U}_0$ be the collection of all neighborhoods of the diagonal which are of the form $\bigcup \{N_x \times N_x|\ x \in X\}$ where N_x is a connected neighborhood of x. Since X is locally connected, $\mathfrak{U}_0$ is a base for the uniformity for X. Thus we have the following result by theorems of Ward [9], [10] and Wallace [8].

THEOREM 7. *Each tree has the* p.F.p.p. *for point connected, uniformly upper V-continuous functions.*

We could also use Theorem 4, but in this case a stronger result has been proved by Smithson [7]. In [7] it was shown that each dendrite has the p.F.p.p. for δ-continuous functions. This result could be extended to non-metric spaces. The concepts of δ-continuity and U-continuity (for metric spaces) are not identical, but they are closely related [6], [7].

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Remark on strongly additive set functions

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A set function μ with values in an abelian group, defined on an additive class of sets $\mathcal{A}$ is called *additive* if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for every pair A, B of disjoint sets in $\mathcal{A}$. A set function λ defined on a lattice of sets $\mathfrak{L}$ containing the empty set $\emptyset$ is called *strongly additive* if its values lie in an abelian group, $\lambda(\emptyset) = 0$ and

$$\lambda(A) - \lambda(A \cap B) = \lambda(A \cup B) - \lambda(B)$$

for every $A \in \mathfrak{L}$ and $B \in \mathfrak{L}$. We call any additive and substractive class of sets a *ring*.

If a set function λ with values in an abelian group G defined on a lattice of sets $\mathfrak{L}$ containing the empty set may be extended to an additive set function μ with values in G defined on a ring containing $\mathfrak{L}$, then $\lambda(\emptyset) = \mu(\emptyset) = 0$ and, for any $A \in \mathfrak{L}$ and $B \in \mathfrak{L}$, we have

$$\lambda(A) - \lambda(A \cap B) = \mu(A \setminus B) = \lambda(A \cup B) - \lambda(B),$$

so that λ is a strongly additive set function. The purpose of this paper is to show that the converse is also true. Namely, we shall prove the following

THEOREM. *Every strongly additive set function defined on a lattice of sets containing the empty set may be extended in a unique manner to an additive set function defined on the smallest ring of sets containing this lattice.*

In the proof of this theorem the notion of a disjoint union of sets will be used. The disjoint union of a system of sets $A_1, A_2, \ldots, A_n$ is defined if and only if these sets are mutually disjoint and in that case it is defined as the usual set-theoretic union and is denoted by $A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_n$, or by $\dot{\bigcup}_{k=1,2,\ldots,n} A_k$.

LEMMA. *Let $\mathfrak{L}$ be a lattice of sets containing the empty set. Let R be the class of all sets of the form $\dot{\bigcup}_{k=1,2,\ldots,n} (A_k \setminus B_k)$, where $A_k \in \mathfrak{L}$ and $B_k \in \mathfrak{L}$ for $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ Then R is the smallest ring containing $\mathfrak{L}$.*