Compactifications as closures of graphs

by

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In this paper we will discuss a procedure for the construction of compactifications of locally compact spaces. By a *compactification of X* we will mean a compact Hausdorff space in which $\hat{X}$ can be densely embedded. Thus, all spaces considered will be Hausdorff. If $\hat{X}$ is a compactification of X, we will call $\hat{X}-X$ the *remainder of X in $\hat{X}$*.

Methods for providing compactifications with given remainders for certain classes of spaces are given in the works of Magill [3] and Aarts and Van Emde Boas [1]. We wish to show that their results are special cases of a general theorem which can be abstracted from [1] and also has other applications.

Let X be a locally compact non-compact space and $X^* = X \cup \{\omega\}$ denote its one-point compactification. By $N(\omega)$ we will mean a neighborhood of ω in X^*

THEOREM. *Let X be locally compact and non-compact and K be compact. If there is a continuous map f of X into K such that $f[N(\omega) \cap X]$ is dense in K for all $N(\omega)$, then X has a compactification $\hat{X}$ with K as remainder. Indeed, such an $\hat{X}$ is the closure of the graph of f in $X^* \times K$.*

Proof. It is easy to see that the mapping $h(x) = (x, f(x))$ is a homeomorphism of X onto the graph G of f in $X^* \times K$. Since K is Hausdorff and f is continuous, it follows that no point of the form (x, k), $k \neq f(x)$ is in $\overline{G}$, the closure of G. That all points of the form (ω, k) are in $\overline{G}$ is a result of the density condition on f. Thus $\overline{G}$ is the desired compactification with remainder homeomorphic to K.

A simple example which may be easily visualized is the following. Let X be the half open interval $(0, 1]$ and K be the compact interval $[-1, 1]$. The desired mapping is $f(x) = \sin(1/x)$.

COROLLARY 1 (Magill). *Let X be a locally compact normal space which contains an infinite discrete closed subset, D. Then for any Peano space K, there exists a compactification with remainder K.*

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Proof. Let F be a countably infinite subset of D and f a map of F onto the rationals in the interval $[0,1]$. Now F is closed in X and f is continuous. Thus by the Tietze Extension Theorem, f may be extended to a map f' on X. Now, let g be a continuous mapping of $[0,1]$ onto K. The mapping $g\circ f'$ is the desired mapping of X into K since every $N(\omega)$ must contain all but a finite number of the elements of F.

It is a curious fact that Magill used exactly these kinds of mappings to map $\beta X-X$ onto K in his proof.

COROLLARY 2 (Aarts and Van Emde Boas). *If X is a locally compact, non-compact separable metric space, then each separable metric continuum K is a remainder of X in some compactification.*

Proof. In [1] Aarts and Van Emde Boas give an easy construction of a continuous map from X into K which maps a sequence of points in X converging to ω in X^* onto a dense subset of K. The theorem can now be applied to this mapping.

COROLLARY 3. *Let X be an infinite discrete space and K be a compact space with a dense subset of cardinality less than or equal to that of X. Then X has a compactification with K as remainder.*

Proof. Let $\aleph$ be the cardinality of X. Now express X as the union of disjoint subsets X_i, $i=1,2,\ldots$, of cardinality $\aleph$. Let D be a dense subset of K with cardinality less than or equal to $\aleph$. We can construct a mapping f of X into K which maps each X_i onto D. Since every $N(\omega)$ is cofinite in X and every mapping on X is continuous, the theorem may be applied.

We should point out that not all compactifications of a locally compact space may be obtained by this procedure. For example, let N be a countable discrete space. Then $\beta N-N$ contains uncountably many disjoint open sets ([2], p. 97), and cannot have a countable dense subset.

Also, the theorem will not allow disconnected remainders to be attached to connected spaces. A small extension may be obtained in the following way. Our method essentially replaces a remainder of one point by another compact remainder. Suppose that X is a locally compact space allowing an n-point compactification (i.e. a finite remainder consisting of n points). We may now replace each of the n points by compact sets. It follows from a theorem of Magill [4] that X may be written as a union of disjoint sets $X=U\cup\bigcup_{i=1}^{n}S_i$, where U is open and has a compact closure in X and each S_i is closed and non-compact. Now, if the one-point remainder in S_i^* may be replaced by a compact set K_i, it is not hard to see that X has a compactification with remainder $\bigcup_{i=1}^{n}K_i$.

References

[1] J. M. Aarts and P. Van Emde Boas, *Continua as remainders in compact extensions*, Nieuw Archief voor Wiskunde 15 (1967), pp. 34-37.

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[3] K. D. Magill, Jr., *A note on compactifications*, Math. Zeitschr. 94 (1966), pp. 322-325.

[4] — *N-point compactifications*, Amer. Math. Monthly 72 (1965), pp. 1075-1081.

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ERRATA

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201^{17}	O_z, что $g^{-1}Oz\subseteq Oz$.	Oz, что $g^{-1}Oz\subseteq O_z$.
204_2	$3'=3$	$3'\Rightarrow 3$