Note on M. C. Gemignani's topological geometries

by

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In this note we shall show that axiom 2) for a geometry as given in chapter I in [1], while not independent of the rest, may effectively be eliminated. Further, when A1 is added to the geometry as an axiom in chapter II, axiom 2) does become dependent. In addition we shall give the relation of the theory of dependence as given in [2] and Gemignani's notion of a geometry. All notation and references will be in accordance with [1].

Let us now consider Gemignani's axiom system for a geometry with axiom 2) deleted and show how the definitions and theorems of chapter I must be modified to remain valid.

LEMMA 1. *There is a unique* (-1)*-flat.*

Proof. $\emptyset$ is contained in a unique (-1)-flat by 4). If $f, g \in F^{-1}$, then since $\emptyset \subset f$ and $\emptyset \subset g$, $f = g$.

Henceforth we shall call this unique (-1)-flat f_{-1}.

New proof of Proposition 1.1. Suppose $f = \emptyset$. Then by 4), $f \subset f_{-1}$ which contradicts 6).

LEMMA 2. *For any flat* f, $f_{-1} \subset f$.

Proof. By Proposition 1.3, $f_{-1} \cap f$ is a k-flat with $k \leqslant -1$. But, by definition, $k \geqslant -1$. So $k = -1$. Therefore by Lemma 1,

$$f_{-1} \cap f = f_{-1}, \quad \text{i.e.} \quad f_{-1} \subset f.$$

Note. According to the definition of linear independence, $\emptyset$ is an independent set and any non-empty subset of f_{-1} is dependent.

In the proof of Proposition 1.4, Lemma 1 now provides the proof for $k = -1$. Also, Lemma 2 assures us that if $y \in f - f_q(S)$ then $y \notin f_{-1}$. So the rest of the proof remains valid as given.

Other changes. In Proposition 1.11 replace the hypothesis $x \in X$ by $x \in X - f_{-1}$. In Definition 5 replace $\emptyset$ by f_{-1}. In Proposition 1.15 replace the hypothesis $Y \neq \emptyset$ by $Y \not\subset f_{-1}$. In Definition 9 replace $f \neq \emptyset$ by $f \neq f_{-1}$. In Proposition 1.16 replace $f' \neq \emptyset$ by $f' \neq f_{-1}$.

In chapter II we have two choices.

A) Replace X by $X-f_{-1}$. Then all proofs remain valid as given.

B) Add hypothesis relating connectivity to f_{-1}. We shall do this just for Proposition 2.1 in order to indicate the procedure.

Proposition 2.1. *A subset W of X is convex iff for any flat f with $\dim(f) \leqslant 1$, $f \cap W$ is connected.*

In the proof we no longer need that $\{x, y\}$ is linearly independent, but only that $\dim(f(x, y)) \leqslant 1$. This follows immediately from 4).

PROPOSITION. *Axiom 2) is derivable from A1 and axioms 1), 4), 5), 6).*

Proof. Suppose $x \in f_{-1}$. Then $\{x\} \subset f_{-1}$ and we have a 0-flat contained in the (−1)-flat, which contradicts 6). Thus $F^{-1} = \{\emptyset\}$.

The Theory of Dependence as given in [2] together with the following definition:

DD. $f \in F^k$ *iff there exist elements* $a_0, \ldots, a_k$ *such that* $\langle a_0, \ldots, a_k\rangle$ *is independent and* $f = \{a |\ a$ *depends on* $\langle a_0, \ldots, a_k\rangle\}$

is inferentially equivalent to the system consisting of Gemignani's axioms 1), 4), 5), 6) together with the following definition:

DG. *An element a depends on the sequence $\langle a_0, \ldots, a_k\rangle$ iff a is an element of the flat determined by $\{a_0, \ldots, a_k\}$.*

We omit the proof.

Note. The interplay between the Theory of Dependence and Topological Geometry is as yet unexplored.

References

[1] M. C. Gemignani, *Topological geometry and a new characterization of* R^m, Notre Dame Journal of Formal Logic 7 (1966), pp. 57-100.

[2] A. Seidenberg, *Lectures in projective geometry*, D. Van Nostrand, Princeton 1962.

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Note on a paper of Wojdysławski

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Wojdysławski [1] gives the following characterization of an absolute retract (AR).

THEOREM. *A necessary and sufficient condition for $\mathfrak{X}$ to be an AR is the existence of a barycentric function for $\mathfrak{X}$.*

(The appropriate definitions are in [1]; all spaces there are separable metric.) To establish the necessity of the condition the author states (in Section 6) that if g is a barycentric function for $\mathfrak{Y}$ and r is a function retracting $\mathfrak{Y}$ onto $\mathfrak{X}$ then rg is a barycentric function for $\mathfrak{X}$. This statement is incorrect. The theorem is true, however, and the necessity of the condition is established below by a slight modification of the author's technique. The notation here is the same as in [1].

To see that the statement in Section 6 is wrong we take $\mathfrak{Y}$ to be the closed interval $[0: 4]$ and $\mathfrak{X}$ to be the subinterval $[0: 1]$. We construct a retraction of $\mathfrak{Y}$ onto $\mathfrak{X}$ as the composite of two maps. First let f_1: $[0: 4] \to [0: 2]$ be the map which is the identity on $[0: 1]$ and sends $[2: 3]$ linearly onto $[1: 2]$ with $f_1([1: 2]) = 1$ and $f_1([3: 4]) = 2$. Then let f_2: $[0: 2] \to [0: 1]$ be the map which is the identity on $[0: 1]$ and maps both $[1: 3/2]$ and $[3/2: 2]$ linearly onto $[1/2: 1]$ so that $f_2(1) = f_2(2) = 1$ and $f_2(3/2) = 1/2$. Then $r = f_2 f_1$ is a retraction of $\mathfrak{Y}$ onto $\mathfrak{X}$. We shall use the fact that $r([1: 2]) = r([3: 4]) = 1$ and $r(5/2) = 1/2$.

Let g: $\mathfrak{L} \to \mathfrak{Y}$ be the barycentric mapping constructed in Section 5. Choose subsequences $\{\Delta_{k_n}\}$ and $\{\Delta_{h_n}\}$ from $\{\Delta_n\}$ so that for each n we have $g(\Delta_{k_n}) \in [1: 2]$ and $g(\Delta_{h_n}) \in [3: 4]$. For each integer n let $S_{p_n} = \{\Delta_{k_n}, \Delta_{h_n}\}$. Then since g is continuous on each Δ_{p_n} and maps the endpoints of each Δ_{p_n} into $[1: 2]$ and $[3: 4]$, we have $5/2 \in g(\Delta_{p_n})$ for every n. Then for each n we have $rg(S_{p_n}) = \{1\}$ and $1/2 = r(5/2) \in rg(\Delta_{p_n})$, so diameter $rg(\Delta_{p_n}) = 1/2$. Thus condition 3.3 for barycentric functions does not hold for rg.

This example can be slightly generalized to show that rg is not a barycentric mapping for $\mathfrak{X}$ even when $\mathfrak{X}$ and $\mathfrak{Y}$ are related as in Section 7—

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