A similar but easier argument establishes the following result.

THEOREM 3.2. *Let X and Y be compact metrizable* EC *spaces, let A be a retract of X, let $f\colon A \to Y$ be continuous, and let $Z = X \cup_f Y$. Then Z is* EC.

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Decomposable circle-like continua *

by

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1. Introduction. In [6] J. B. Fugate proved that a necessary and sufficient condition that the sum of two chainable continua be chainable is that the sum be atriodic and unicoherent. In this paper it is proved that a necessary and sufficient condition that the non-chainable sum of two chainable continua be circle-like is that the sum be atriodic and the common part of the two continua be not connected. The techniques used in proving this also yield a strengthened version of Fugate's theorem.

Space is assumed to be metric with metric ϱ. For definitions of terms such as chainable (snake-like) or circle-like, see [2]; the conventions used there for denoting chains (or circular chains) are employed in this paper.

The subcontinuum H of the compact continuum M is said to be a *terminal* subcontinuum of M if and only if for each two subcontinua K and L of M which intersect H either K is a subcet of $H \cup L$ or L is a subset of $H \cup K$.

A chain C is said to be *regular* (taut) if and only if the distance between non-intersecting links of C is positive. In [4], Theorem 1, p 12, H. Cook proved that if M is chainable and D is a chain covering M then there is a regular chain covering M which is a strong refinement of D

THEOREM (Fugate, [6], Lemma 1, p. 461). *If H is a terminal subcontinuum of the chainable continuum M and $\varepsilon > 0$, then there is a regular ε-chain $C(c_1, c_2, \ldots, c_n)$ covering M such that $c_1 - (c_1 \cap \bar{c}_2)$ intersects H.*

2. Terminal continua and decomposable atriodic continua. Theorem 1 is a generalization of a theorem of Bing ([1], Theorem 14, p. 661) concerning opposite end points. The argument is similar to that given by Bing.

THEOREM 1. *If H and K are mutually exclusive terminal subcontinua of the chainable continuum M and M is irreducible with respect to containing $H \cup K$ and $\varepsilon > 0$, then there is an ε-chain $C(c_1, c_2, \ldots, c_n)$ covering M such that $c_1 - (c_1 \cap \bar{c}_2)$ intersects H and $c_n - (c_n \cap \bar{c}_{n-1})$ intersects K.*

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Proof. Suppose $\varepsilon > 0$. By the theorem of Fugate there is an ε-chain $D(d_1, d_2, \ldots, d_j)$ covering M such that $d_1 - (d_1 \cap \bar{d}_2)$ intersects H. Moreover, we assume that d_j intersects $M - H$ but $d_j - (d_j \cap \bar{d}_{j-1})$ does not intersect K. Since M is irreducible with respect to containing both H and K, there is a positive number δ less than ε such that if E is a δ-chain covering M there is a link of E contained in d_j which is between the links of E which intersect H and the links of E which intersect K. Let $E(e_1, e_2, \ldots, e_k)$ be a δ-chain covering M such that $e_1 - (e_1 \cap \bar{e}_2)$ intersects K. Denote by t the least integer such that e_t is a subset of d_j and e_t is between the links of E which intersect H and the links of E which intersect K. Let $N = e_t \cup e_{t+1} \cup \ldots \cup e_k$. Then the chain $C(c_1, c_2, \ldots, c_n)$ where

$$c_1 = d_1 \cap N, \quad c_2 = d_2 \cap N, \quad \ldots, \quad c_j = d_j \cap N, \quad c_{j+1} = e_{t-1}, \quad \ldots, \quad c_n = e_1$$

is the desired ε-chain.

LEMMA 1. *Suppose that A and B are compact continua and $A \cap B$ has at least three components. Then $A \cup B$ contains a triod.*

Proof. Suppose that H, K, and L are three components of $A \cap B$. There is an open set D_1 containing $K \cup L$ such that H does not intersect $\bar{D}_1$. Let C_1 be the component of $B - (B \cap D_1)$ which contains H. Since C_1 intersects the boundary of D_1, C_1 contains a point of $B - (A \cap B)$.

There is an open set D_2 containing $C_1 \cup L$ such that K does not intersect $\bar{D}_2$. Let C_2 be the component of $B - (B \cap D_2)$ which contains K. Since C_2 intersects the boundary of D_2, C_2 contains a point of $B - (A \cap B)$.

There is an open set D_3 containing $C_1 \cup C_2$ such that L does not intersect $\bar{D}_3$. Let C_3 be the component of $B - (B \cap D_3)$ which contains L. Since C_3 intersects the boundary of D_3, C_3 contains a point of $B - (A \cap B)$.

Thus, $A \cup C_1$, $A \cup C_2$, and $A \cup C_3$ are three continua containing a common point and no one of them is a subset of the sum of the other two. By a theorem of Sorgenfrey ([7], Theorem 1.8, p. 443) their sum contains a triod.

LEMMA 2. *Suppose that A and B are compact continua, $A \cap B$ is the sum of two mutually exclusive continua H and K, and $A \cup B$ is atriodic. Then H and K are terminal subcontinua of A and A is irreducible with respect to containing $H \cup K$.*

Proof. Suppose that there exist subcontinua L_1 and L_2 of A each intersecting H and there is a point P of L_1 not in $H \cup L_2$ as well as a point Q of L_2 not in $H \cup L_1$. There is an open set D containing K such that no point of H is in $\bar{D}$. Denote by C the component of $B - (B \cap D)$ which contains H, and note that C intersects $B - (A \cap B)$. Then by Sorgenfrey's theorem [7], $C \cup (H \cup L_1) \cup (H \cup L_2)$ contains a triod. Thus, H is a terminal subcontinuum of A.

Suppose that A is not irreducible with respect to containing both H and K. Then there is a proper subcontinuum A_0 of A which contains $H \cup K$. There is an open set D containing K such that H does not intersect $\bar{D}$. Denote by C the component of $B - (B \cap D)$ which contains H. There is an open set R which contains C such that $\bar{R}$ contains no point of K. Denote by E the component of $B - (B \cap R)$ containing K. Then $A \cup (A_0 \cup C) \cup (A_0 \cup E)$ contains a triod.

LEMMA 3. *Suppose that A and B are compact continua, $A \cap B$ is the sum of two mutually exclusive continua K and L, and $A \cup B$ is atriodic. If H is a subcontinuum of $A \cup B$ which intersects $A - K$ but does not intersect L, then $H \cap A$ is a continuum.*

Proof. Suppose that $H \cap A$ is not connected. Since $H \cup A$ contains no triod, by Lemma 1, $H \cap A$ has only two components L_1 and L_2. Then L_1 and L_2 are mutually exclusive continua. Since $H \cap A$ is not connected, H intersects $B - K$. Further, both L_1 and L_2 intersect K. If, for example, L_1 does not intersect K, then L_1 and $L_2 \cup (B \cap H)$ are mutually exclusive closed point sets whose sum is H. Since H intersects $A - K$, one of L_1 or L_2 intersects $A - K$. Suppose that L_1 is not a subset of K.

Case 1. Assume that *K is not a subset of H*. If L_2 intersects $A - K$ and $B \cap H$ is connected, then $K \cup L_1$, $K \cup L_2$, and $B \cap H$ are three continua which contain a common point and no one of them is a subset of the sum of the other two. By Sorgenfrey's theorem [7] their sum contains a triod. If L_2 intersects $A - K$ and $B \cap H$ is not connected, then using Lemma 1, $B \cap H$ is the sum of two mutually exclusive continua M_1 and M_2. One of M_1 and M_2 intersects $B - K$. Suppose that M_1 intersects $B - K$. Since M_1 intersects K, $(K \cup L_1) \cup (K \cup L_2) \cup M_1$ contains a triod.

If L_2 is a subset of K and $H \cap B$ is connected, then L_1 intersects $H \cap B$. If not, L_1 and $(L_2 \cup (H \cap B))$ are mutually exclusive closed point sets whose sum is H. Thus, $K \cup L_1 \cup (H \cap B)$ contains a triod (since K is not a subset of H).

If L_2 is a subset of K and $H \cap B$ is the sum of two mutually exclusive continua M_1 and M_2, then M_1 or M_2 intersects $B - K$. Suppose that M_1 intersects $B - K$. If M_2 also intersects $B - K$, $(K \cup L_1) \cup (K \cup M_1) \cup (K \cup M_2)$ contains a triod. If M_2 is a subset of K, then L_1 intersects M_1. If not, M_2 intersects L_1 since L_1 intersects $B \cap H$ and thus M_2 does not intersect L_2 for M_2 is a subcontinuum of A. Then $(L_1 \cup M_2)$ and $(L_2 \cup M_1)$ are mutually exclusive closed point sets whose sum is H. Thus L_1 intersects M_1. Any point common to L_1 and M_1 is in K, so $L_1 \cup M_1 \cup K$ contains a triod.

We have shown that Case 1 does not occur, so we have that K is a subset of H.

Case 2. *K is a subset of H*. Then K is a subset of L_1 or of L_2. Assume that K is a subset of L_2. Then L_1 and $L_2 \cup (H \cap B)$ are mutually exclusive closed point sets whose sum is H. Thus, $A \cap H$ is connected.

3. Main theorem.

THEOREM 2. *Suppose that A and B are chainable continua which intersect. Then*

(1) *$A \cup B$ is a chainable continuum if and only if $A \cup B$ is atriodic and $A \cap B$ is connected.*

(2) *$A \cup B$ is a non-chainable, circle-like continuum if and only if $A \cup B$ is atriodic and $A \cap B$ is not connected.*

Note that (1) is a refinement of Fugate's theorem ([6], Theorem 1). The technique used below to argue (2) can be modified somewhat to argue (1).

Proof of necessity. Suppose that A and B are chainable continua such that $A \cup B$ is a non-chainable, circle-like continuum. Since $A \cup B$ is circle-like, $A \cup B$ is atriodic ([3], Theorem 1, p. 654). Using Fugate's theorem ([6], Theorem 1, p. 466), $A \cup B$ is not unicoherent. By [3], Theorem 5, p. 656, $A \cup B$ is bicoherent so $A \cap B$ is not connected.

Proof of sufficiency. Suppose that A and B are chainable continua such that $A \cap B$ is not connected and $A \cup B$ is atriodic. By Lemma 1, $A \cap B$ is the sum of two mutually exclusive continua H and K.

Suppose $\varepsilon > 0$. Let δ be the smaller of ε and $\frac{1}{4}\varrho(H, K)$. There exist mutually exclusive open sets R_1 and R_2 containing $A-(A \cap B)$ and $B-(A \cap B)$ respectively. By Lemma 2, H and K are terminal subcontinua of A (and of B). By Theorem 1 and [4], Theorem 1, p. 12, there is a regular δ-chain $C(c_1, c_2, \ldots, c_n)$ covering A such that $c_1-(c_1 \cap \bar{c}_2)$ intersects H and $c_n-(c_n \cap \bar{c}_{n-1})$ intersects K. Moreover, we may assume that if c is a link of C which does not intersect $H \cup K$ then c is a subset of R_1. There is a positive number e less than δ such that if D is an e-chain covering H or K then D is a refinement of C. There is a regular e-chain $D(d_1, d_2, \ldots, d_m)$ covering B such that $d_1-(d_1 \cap \bar{d}_2)$ intersects H and $d_m-(d_m \cap \bar{d}_{m-1})$ intersects K and if d is a link of D which intersects H or K then $\bar{d}$ is a subset of some link of C. Moreover, we may assume that if d is a link of D which does not intersect $H \cup K$ then d is a subset of R_2.

Denote by c_i and d_j the last links of C and D respectively which intersect H. Denote by c_x and d_y the first links of C and D respectively which intersect K. We assume that $\bar{d}_{j+2} \cup \bar{d}_{j+3} \cup \ldots \cup \bar{d}_m$ does not intersect $c_1 \cup c_2 \cup \ldots \cup c_i$ and $\bar{d}_1 \cup \bar{d}_2 \cup \ldots \cup \bar{d}_{y-2}$ does not intersect $c_x \cup c_{x+1} \cup \cup \ldots \cup c_n$ since $c_1, c_2, \ldots, c_i$ and $c_x, c_{x+1}, \ldots, c_n$ can be modified so that this is true.

Consider the closed set $F = B \cap [d_{j+1}-(d_j \cup d_{j+2})]$. Since D is a regular chain, F is a point set. There is an open set R containing F such that $\bar{R}$ is a subset of d_{j+1}. By replacing d_{j+1} by R in D we obtain a chain D' which covers B and has the property that the closure of the $(j+1)$st link of D' does not intersect A. Similarly, we may replace in D' the $(y-1)$st link with one such that its closure does not intersect A, thus obtaining a chain D'' covering B such that if d is a link of D'' which does not intersect A then $\bar{d}$ does not intersect A. For notational convenience we will denote D'' by D.

Denote by d_k the last link of D which is a subset of c_1 and by d_w the first link of D which is a subset of c_n. Then, also, we assume that $\bar{d}_{k+2} \cup \ldots \cup \bar{d}_m$ does not intersect c_1 and $\bar{d}_1 \cup \ldots \cup \bar{d}_{w-2}$ does not intersect c_n, and that $\bar{d}_{j+1}$ and $\bar{d}_{y-1}$ each intersects only one link of C (respectively one which contains $\bar{d}_j$ and one which contains $\bar{d}_y$).

We consider the case that d_{j+1} does not intersect c_1 and d_{y-1} intersects c_n. We note the similarity in the other cases (e.g. d_{j+1} does not intersect c_1 and d_{y-1} does not intercset c_n) and omit the arguments.

Denote by J the point set $(A \cup B) \cap \overline{[(c_2 \cup c_3 \cup \ldots \cup c_i)-c_1]}$ and let $N_1 = J \cap \bar{d}_{j+1}$ and $N_2 = J-[J \cap (d_{k+1} \cup \ldots \cup d_m)]$. There is a point P of B in $\beta(d_j) \cap d_{j+1}$. Since P is in $\bar{d}_j$, P is in one of $c_1, c_2, \ldots, c_i$, but d_{j+1} does not intersect c_1 so P is in J. Then P is a point of N_1. There is a point Q of A in $\beta(c_i) \cap c_{i+1}$. Then Q is in J. The point Q is not in B because Q is in c_{i+1} and c_{i+1} contains no point of B. (c_{i+1} contains no point of H by the choice of i, c_{i+1} contains no point of K by choice of δ, and so c_{i+1} contains no point of $B-(A \cap B)$ by choice of R_1). Then Q is in no link of D, so Q is in N_2. Moreover, N_1 and N_2 are mutually exclusive closed subsets of J.

No subcontinuum of J intersects both N_1 and N_2 for suppose that M is a subcontinuum of J intersecting both N_1 and N_2. Since M intersects N_1, M intersects $B-(A \cap B)$. If M intersects $A-(A \cap B)$, then, by Lemma 3, $M \cap A$ and $M \cap B$ are continua. Thus $H \cup (M \cap A) \cup (M \cap B)$ contains a triod, so M does not intersect $A-(A \cap B)$. Therefore M is a subset of B, so M intersects one of $d_1, d_2, \ldots, d_k$ and, thus, M intersects d_k. However this is impossible since J does not intersect c_1. Therefore, J is the sum of two mutually exclusive closed sets J_1 and J_2 containing N_1 and N_2 respectively.

There exist mutually exlusive open sets U and V containing J_1 and J_2 respectively. Then $E(d_j \cap U, d_{j-1} \cap U, \ldots, d_{k+1} \cap U, c_1, c_2 \cap V, \ldots, c_i \cap V, c_{i+1}, \ldots, c_n, d_{y-1}, \ldots, d_{j+1})$ is a circular chain irreducibly covering $A \cup B$.

THEOREM 3. *If M is a compact continuum and every proper subcontinuum of M is chainable, then M is atriodic.*

Proof. The only possibility is that M is a triod, so suppose that M is the sum of three continua M_1, M_2, and M_3 such that $M_1 \cap M_2 = M_1 \cap \cap M_3 = M_2 \cap M_3 = M_1 \cap M_2 \cap M_3$ is a proper subcontinuum of each of M_1, M_2, and M_3. Let $H = M_1 \cap M_2$ and suppose that P is a point of M_1 which is not in $M_2 \cup M_3$. Then P is not in H so there is a region R containing P such that $\bar{R}$ does not intersect H. Let N be the component of $M_1 - R$ containing H. Then $N \cup M_2 \cup M_3$ is a triod which is a proper subcontinuum of M.

THEOREM 4. *If M is a compact continuum such that every proper subcontinuum of M is chainable and M is neither chainable nor circle-like, then M is indecomposable.*

In a recent conversation with the author Professor H. Cook raised the question: If M is a compact continuum such that every proper subcontinuum of M is chainable, is M chainable or circle-like? From Theorem 4 it follows that the answer is "yes" if M is decomposable. However, in [5] Cook presents an example showing that, in general, the answer is "no".

Using a theorem of Bing ([1], Theorem 11, p. 660) and the results of this paper we have the following theorem.

THEOREM 5. *Suppose that M is a compact, hereditarily decomposable, non-chainable continuum. Then M is circle-like if and only if M is atriodic and M is not unicoherent but every proper subcontinuum of M is unicoherent.*

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Уплотнения на бикомпакты и связь с бикомпактными расширениями и с ретракцией *

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Уплотнениями называют взаимно-однозначные непрерывные (schlichte) отображения „на". Эти отображения являют собой достаточно „хороший" класс отображений, удобный для многих целей, как в топологии так и в анализе.

Повидимому впервые П. С. Александровым был поставлен вопрос *о нахождении необходимых и достаточных условий для того, чтобы топологическое пространство* (¹) (хотя бы со счётной базой) *могло быть уплотнено на компакт* (бикомпакт). Он же задал следующий более конкретный вопрос: *можно ли уплотнить на компакт связное, локально-связное, полное метрическое пространство со счётной базой, являющееся суммой счётного числа компактов?* В связи с этим А. С. Пархоменко спросил меня: *можно ли в условии этого вопроса дополнительно потребовать, чтобы пространство было одномерным?*

Здесь мы даем ответы на все эти вопросы при тех или иных, как нам кажется, достаточно широких предположениях.

Предварительно заметим, что до сих пор искались лишь *внутренние* необходимые и достаточные условия. Мне кажется, что ныне сильно развитая теория бикомпактных расширений должна здесь играть важную роль. Именно с помощью этой теории я даю здесь *внешние* необходимые и достаточные условия для любых вполне регулярных пространств (теорема 1). На следующий „конкретный" вопрос П. С. Александрова ответ отрицателен: *я строю пример множества A, лежащего в трёхмерном пространстве, которое обладает всеми требуемыми свойствами, но, тем не менее, A не уплотняется ни на один бикомпакт* (пример 1). Решение этой задачи тесно связано с вопросами ретракции: *я доказываю, что множество A нельзя ретрагировать на некоторое подмножество A_N* (²) (оно является окружностью), *но в то же*

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(¹) Под пространством мы всюду понимаем здесь вполне регулярное T_1-пространство.

(²) Через X_N мы обозначаем множество всех тех точек пространства X, в которых X не локально бикомпактно, а через X_B — дополнительное множество $X \setminus X_N$ (где X локально-бикомпактно).