Adjunction of locally equiconnected spaces*

by

William D. McIntosh (Columbia, Missouri)

1. Introduction. The following problem has been posed by Borsuk (cf. [1] and [6]). Suppose that X is a compact ANR and that $\{A_n\}$ is a sequence of mutually disjoint compact AR subsets of X such that

$$\lim_{n\to\infty} \operatorname{diam}(A_n) = 0 .$$

For each n, let f_n be a map from A_n onto a compact AR space Y_n. Let Z be the quotient space whose members are the sets $(f_n)^{-1}(y)$ for $y \in Y_n$, and the points of $X - \bigcup_{n=1}^{\infty} A_n$. Is Z an ANR? Lelek [6] has solved this problem affirmatively in the case in which Z is finite-dimensional.

In attempting to generalize Lelek's results, it seems natural to consider a class of spaces which contains the class of all ANR's, and which, for finite-dimensional metrizable spaces, is the same as the class of ANR's. The class of locally equiconnected spaces, defined by Fox [4], has this property.

Let X be a topological space, and let V be a subset of $X^2 = X\times X$ which contains the diagonal $\Delta(X)$ of X^2. Then a map $\lambda\colon V\times I \to X$, where I is the closed unit interval, is said to have the *connecting property* if and only if

$$\lambda(x, y, 0) = x , \quad \lambda(x, y, 1) = y , \quad \text{and} \quad \lambda(x, x, t) = x \ \text{for all} \ (x, y, t) \in V\times I.$$

If $V = X^2$, then λ will be called a *connecting map for* X. If V is a neighborhood of $\Delta(X)$, then λ is a *local connecting map for* X. A topological space X is *equiconnected* (abbreviated EC) if and only if there is a connecting map for X. X is *locally equiconnected* (abbreviated LEC) if and only if X has a local connecting map. It is not difficult to show (see [3] and [5]) that

* This paper represents a portion of the author's Ph. D. dissertation, which was prepared under the supervision of Professor Charles J. Himmelberg and submitted to the University of Kansas in 1965. The author would like to express his sincere appreciation to Professor Himmelberg for his assistance and encouragement.

Part of the research for this paper was supported by NSF grant GP-209.

every ANR (resp., AR) is LEC (resp., EC), and that every LEC space is locally contractible. Thus for finite-dimensional metrizable spaces, the concepts of ANR and LEC space are equivalent.

One of the results essential to Lelek's proof is a theorem of Borsuk ([2], Theorem T), which says that if X, Y, and A are compact ANR's with $A \subset X$, and if $f\colon A \to Y$ is continuous, then the adjunction space $X \cup_f Y$ of X and Y by f is locally contractible (and hence an ANR if it is finite-dimensional). Whitehead ([7], Theorem 1) established, without the restriction of finite-dimensionality, that $X \cup_f Y$ is an ANR. The main result of this paper is a corresponding theorem for LEC spaces, without the hypothesis of finite-dimensionality, but with an added restriction on A and $f(A)$. Specifically, it will be shown (see Theorem 3.1) that if X and Y are compact metrizable LEC spaces, if A is an EC neighborhood retract of X, and if $f\colon A \to Y$ is a continuous function such that $f(A)$ is an EC neighborhood retract of Y, then $X \cup_f Y$ is LEC. Of course, in the case of Borsuk's problem, the requirement that $f(A)$ be a neighborhood retract of Y is no restriction since in that case, $f(A) = Y$.

Before proceeding to the proof of this theorem, we need a number of preliminary lemmas.

2. Preliminary lemmas.

LEMMA 2.1. *Let (X, d) be a compact metric LEC space with local connecting map $\lambda\colon V \times I \to X$, where V is a closed neighborhood of $\Delta(X)$. Let A be a closed subset of X and suppose that $r\colon U \supset A$ is a retraction of a closed neighborhood U of A onto A. Suppose further that $(x, r(x)) \in V$ for each $x \in U$. Let $g\colon U^2 - A^2 \to I$ be a continuous function such that if $\{(x_n, y_n)\}$ is a sequence in $U^2 - A^2$ converging to $(x, y) \in A^2 - \Delta(A)$, then $\lim\limits_{n\to\infty} g(x_n, y_n) = 1$. Let $c > 1$. Define $s\colon U^2 \to X$ by*

$$s(x, y) = \begin{cases} \lambda\big(x, r(x), cg(x, y)\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \leqslant 1/c; \\ r(x) & \text{if } (x, y) \in A^2, \text{ or if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \geqslant 1/c. \end{cases}$$

Then s is uniformly continuous.

Proof. It is clear that s is continuous on $U^2 - A^2$ and on A^2. It will now be shown that s is continuous on $\mathrm{Bd}(A^2)$. So choose $\varepsilon > 0$. Since λ is uniformly continuous, there exists $\delta > 0$ such that

$$d\big(\lambda(x, x, t),\ \lambda(x, y, t)\big) < \varepsilon/2,$$

or, equivalently, $d\big(x, \lambda(x, y, t)\big) < \varepsilon/2$, for all $(x, y, t) \in V \times I$ with $d(x, y) < \delta$. Furthermore, there exists $\eta > 0$ such that $d\big(r(x), r(y)\big) < \delta/2$ for all $(x, y) \in U^2$ with $d(x, y) < \eta$. We may also require that $2\eta < \delta < \varepsilon$.

Now let $(x, y) \in A^2$, and suppose that $\{(x_n, y_n)\}$ is a sequence in $U^2 - A^2$ converging to (x, y): If $x \neq y$, then $\lim\limits_{n\to\infty} g(x_n, y_n) = 1$, so that

$$\lim_{n\to\infty} s(x_n, y_n) = \lim_{n\to\infty} r(x_n) = r(x) = s(x, y).$$

On the other hand, suppose $x = y$. For sufficiently large n, $d(x, x_n) < \eta$. Choose such an n. If $g(x_n, y_n) \geqslant 1/c$, then

$$d\big(s(x, y), s(x_n, y_n)\big) = d\big(r(x), r(x_n)\big) < \delta/2 < \varepsilon.$$

If $g(x_n, y_n) \leqslant 1/c$, then, since $r(x) = x$, we have

$$d\big(x_n, r(x_n)\big) \leqslant d(x_n, x) + d\big(r(x), r(x_n)\big) < \eta + \delta/2 < \delta;$$

it follows (note that $s(x, y) = x$) that

$$d\big(s(x, y), s(x_n, y_n)\big) \leqslant d(x, x_n) + d\Big(x_n, \lambda\big(x_n, r(x_n), cg(x_n, y_n)\big)\Big) < \eta + \varepsilon/2 < \varepsilon.$$

Thus s is continuous, and therefore uniformly continuous. Q.e.d.

The following lemma is due to Himmelberg.

LEMMA 2.2 ([5], Lemma 1). *Let X be a metrizable space, and suppose $X = A \cup B$, where A and B are closed. Let V_A be an X^2-neighborhood of $\Delta(A)$, let V_B be a B^2-neighborhood of $\Delta(B)$, and let $\lambda_A\colon V_A \times I \to X$ and $\lambda_B\colon V_B \times I \to X$ be such that:*

a. *$\lambda_B(V_B \times I) \subset B$ and λ_B is a local connecting map for B;*

b. *λ_A has the connecting property on $A^2 \cap V_A$;*

c. *$\lambda_A(x, y, 0) = x$ and $\lambda_A(x, y, 1) = y$ if $(x, y) \in V_A$;*

d. *$\lambda_A(x, y, I) \subset B$ if $(x, y) \in V_A \cap (X - A)^2$.*

Then X is LEC, and there is a local connecting map λ for X such that λ_A is defined at each point of $(A^2 \times I) \cap \mathrm{dom}(\lambda)$ and agrees with λ there.

LEMMA 2.3. *Let (X, d) be a compact metric LEC space, and let A be an EC neighborhood retract of X. Then there exist a neighborhood V of $\Delta(X) \cup A^2$ and a local connecting map $\lambda\colon V \times I \to X$ such that $\lambda(A^2 \times I) \subset A$.*

Proof. Let λ_A be a connecting map for A. Then by a theorem of Himmelberg ([5], Theorem 7) there exist a closed neighborhood V_X of $\Delta(X)$ and a local connecting map $\lambda_X\colon V_X \times I \to X$ which agrees with λ_A on $(V_X \cap A^2) \times I$. Let $\lambda_1 = \lambda_A \cup \lambda_X$. Without loss of generality, we may assume that

$$V_X = \{(x, y) \in X^2 \mid d(x, y) \leqslant \varepsilon\}$$

for some $\varepsilon > 0$. There also exist a closed neighborhood U_1 of A and a retraction $r\colon U_1 \supset A$.

Since U_1 is compact, r is uniformly continuous, so there exists $\delta > 0$ such that $d(r(x), r(y)) < \varepsilon/2$ for all $(x, y) \in (U_1)^2$ with $d(x, y) < \delta$. We may also assume that $\delta < \varepsilon/2$, and that

$$U = \{x \in X|\ d(x, A) \leqslant \delta/2\} \subset U_1 .$$

Note that if $x \in U$, then there exists $y \in A$ such that $d(x, y) < \delta$, and hence

$$d(x, r(x)) \leqslant d(x, y) + d(r(y), r(x)) < \delta + \varepsilon/2 < \varepsilon .$$

Thus $(x, r(x)) \in V_X$ for each $x \in U$.

Define $g\colon U^2 - A^2 \to I$ by

$$g(x, y) = \frac{d(x, y) + |d(x, A) - d(y, A)|}{d(x, y) + d(x, A) + d(y, A)}$$

for all $(x, y) \in U^2 - A^2$. It is clear that g is continuous. Now define $s\colon U^2 \to X$ and $s'\colon U^2 \to X$ by

$$s(x, y) = \begin{cases} \lambda_1(x, r(x), 2g(x, y)) \\ \qquad \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \leqslant 1/2; \\ r(x) \text{ if } (x, y) \in A^2, \text{ or if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \geqslant 1/2; \end{cases}$$

and $s'(x, y) = s(y, x)$ for all $(x, y) \in U^2$. Note that $s(x, x) = s'(x, x) = x$ for all $x \in U$, since if $x \notin A$, then $g(x, x) = 0$.

It is now easily seen that the hypotheses of Lemma 2.1 are satisfied. Thus s is uniformly continuous, and hence s' is also uniformly continuous. Consequently there exists $\eta > 0$ such that

$$d(x, s(x, y)) = d(s(x, x), s(x, y)) < \varepsilon/3$$

and $d(y, s'(x, y)) < \varepsilon/3$ for all $(x, y) \in U^2$ with $d(x, y) < \eta$. We may also assume that $\eta < 2\delta/3$.

Let

$$W = \{x \in X|\ d(x, A) < \eta/2\} ,$$

and suppose $(z, w) \in W^2$. We will show that $(z, s(z, w), t)$, $(s(z, w), s'(z, w), t)$, and $(s'(z, w), w, t)$ all belong to the domain of λ_1 for each $t \in I$. If $g(z, w) \geqslant 1/2$, then, since $W \subset U$, we have $d(z, s(z, w)) = d(z, r(z)) < \varepsilon$, $d(w, s'(z, w)) < \varepsilon$, and

$$(s(z, w), s'(z, w)) \in A^2 .$$

On the other hand, suppose $g(z, w) \leqslant 1/2$. Then (see the definition of g)

$$2d(z, w) + 2|d(z, A) - d(w, A)| \leqslant d(z, w) + d(z, A) + d(w, A) ,$$

so that

$$d(z, w) \leqslant d(z, A) + d(w, A) < \eta/2 + \eta/2 = \eta .$$

Therefore $d(z, s(z, w)) < \varepsilon/3$, $d(w, s'(z, w)) < \varepsilon/3$, and

$$d(s(z, w), s'(z, w)) \leqslant d(s(z, w), z) + d(z, w) + d(w, s'(z, w)) < \varepsilon/3 + \eta + \varepsilon/3 < \varepsilon .$$

We can therefore define $\lambda_2\colon W^2 \times I \to X$ by

$$\lambda_2(x, y, t) = \begin{cases} \lambda_1(x, s(x, y), 3t) & \text{if } (x, y) \in W^2 \text{ and } 0 \leqslant t \leqslant 1/3; \\ \lambda_1(s(x, y), s'(x, y), 3t-1) & \text{if } (x, y) \in W^2 \text{ and } 1/3 \leqslant t \leqslant 2/3; \\ \lambda_1(s'(x, y), y, 3t-2) & \text{if } (x, y) \in W^2 \text{ and } 2/3 \leqslant t \leqslant 1. \end{cases}$$

It is easily seen that λ_2 is continuous and has the connecting property.

Now let

$$W_1 = \{x \in X|\ d(x, A) \leqslant \eta/3\} .$$

By Lemma 2.2, with $A = W_1$, $B = X$, $V_A = W^2$, $V_B = V_X$, $\lambda_A = \lambda_2$, and $\lambda_B = \lambda_X$, there exist a closed neighborhood V' of $\Delta(X)$ and a local connecting map $\lambda'\colon V' \times I \to X$ which agrees with λ_2 on $(V' \cap (W_1)^2) \times I$. Let $V = V' \cup (W_1)^2$. Then V is a neighborhood of $\Delta(X) \cup A^2$. Let

$$\lambda = \lambda' \cup (\lambda_2| ((W_1)^2 \times I)) .$$

Then it is clear that $\lambda\colon V \times I \to X$ is a local connecting map.

Finally, let $(x, y) \in A^2$. Then since $s(x, y) = x$, $s'(x, y) = y$, and $\lambda_A(A^2 \times I) \subset A$, it is clear that $\lambda(x, y, t) \in A$ for each $t \in I$. Q.e.d.

LEMMA 2.4. *Let X be a topological space, let $\{x_n\}$ be a sequence in X, let $x \in X$, and let $\{A_1, \ldots, A_k\}$ be a finite family of subsets of X such that $\{x_n|\ n \geqslant 1\} \subset \bigcup_{i=1}^{k} A_i$. Suppose further that for each A_i containing infinitely many terms of $\{x_n\}$, the subsequence consisting of those x_n's belonging to A_i, converges to x. Then $\lim_{n\to\infty} x_n = x$.*

Proof. Every neighborhood of x contains all except a finite number of terms of each of the finitely many subsequences into which $\{x_n\}$ has been decomposed, and hence all except a finite number of terms of $\{x_n\}$. Q.e.d.

LEMMA 2.5. *Let X and Y be compact metrizable spaces, let f be a function from X to Y, and let $\{x(n)\}$ be a sequence in X. Suppose further that $y \in Y$ is such that $\lim_{k\to\infty} f(x(n_k)) = y$ for every convergent subsequence $\{x(n_k)\}$ of $\{x(n)\}$. Then $\lim_{n\to\infty} f(x(n)) = y$.*

Proof. Since Y is compact, there is a subsequence $\{f(x(n_k))\}$ of $\{f(x(n))\}$ which converges to some point z. Since X is compact, $\{x(n_k)\}$ has a convergent subsequence $\{x(n_{k_j})\}$, and by hypothesis,

$$\lim_{j\to\infty} f(x(n_{k_j})) = y .$$

But we also must have

$$\lim_{j\to\infty} f\big(x(n_{k_j})\big) = z\,,$$

so that $z = y$. Thus $\{f(x(n))\}$ has exactly one cluster point, namely y, and since Y is compact, $\lim_{n\to\infty} f\big(x(n)\big) = y$. Q.e.d.

3. Main results.

THEOREM 3.1. *Let X and Y be compact metrizable* LEC *spaces, let A be an* EC *neighborhood retract of X, let $f\colon A \to Y$ be a continuous function such that $f(A)$ is an* EC *neighborhood retract of Y, and let $Z = X \cup_f Y$. Then Z is* LEC.

Proof. Without loss of generality, we may assume that $X \cap Y = \emptyset$. Let $W = X \cup Y$, and let d be a metric for W such that $\operatorname{diam}(X) < 1$, $\operatorname{diam}(Y) < 1$, and $d(X, Y) > 1$. There exist a closed neighborhood U of A and a retraction $r\colon U \supset A$. Also, by Lemma 2.3, there exist neighborhoods V_X and V_Y of $\Delta(X) \cup A^2$ and $\Delta(Y) \cup \big(f(A)\big)^2$, respectively, and local connecting maps $\lambda_X\colon V_X \times I \to X$ and $\lambda_Y\colon V_Y \times I \to Y$ such that $\lambda_X(A^2 \times I) \subset A$. Without loss of generality, we may assume that

$$U = \{x \in X \mid d(x, A) \leqslant \varepsilon\}\,,$$
$$V_X = \{(x, y) \in X^2 \mid d(x, y) \leqslant \varepsilon\} \cup U^2\,,$$

and

$$V_Y = \{(x, y) \in Y^2 \mid d(x, y) \leqslant \varepsilon\} \cup \big\{(x, y) \in Y^2 \mid d\big(x, f(A)\big) \leqslant \varepsilon;\ d\big(y, f(A)\big) \leqslant \varepsilon\big\}$$

for some $\varepsilon > 0$. Let

$$U' = \{x \in X \mid d(x, A) < d(x, X - U)\}\,.$$

Now define $g\colon U^2 - A^2 \to I$ by

$$g(x, y) = \frac{d(x, y) + |d(x, A) - d(y, A)|}{d(x, y) + d(x, A) + d(y, A)} \times \left[1 - \left(\min\left\{\frac{d(x, A)}{d(x, X - U)}, 1\right\}\right)\left(\min\left\{\frac{d(y, A)}{d(y, X - U)}, 1\right\}\right)\right]$$

for all $(x, y) \in U^2 - A^2$. It is clear that g is continuous. Next define $s\colon U^2 \to X$, $s'\colon U^2 \to X$, $q\colon U^2 - A^2 \to X$, and $q'\colon U^2 - A^2 \to X$ by

$$s(x, y) = \begin{cases} \lambda_X\big(x, r(x), \tfrac{9}{4} g(x, y)\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \leqslant 4/9;\\ r(x) & \text{if } (x, y) \in A^2\text{, or if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \geqslant 4/9; \end{cases}$$

$$s'(x, y) = s(y, x) \quad \text{for all} \quad (x, y) \in U^2;$$

$$q(x, y) = \begin{cases} s(x, y) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \leqslant 4/9;\\ \lambda_X\big(r(x), \lambda_X\big(r(x), r(y), 1/2\big), \tfrac{9}{2} g(x, y) - 2\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } 4/9 \leqslant g(x, y) \leqslant 2/3;\\ \lambda_X\big(r(x), \lambda_X\big(r(x), r(y), 1/2\big), 3 - 3g(x, y)\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \geqslant 2/3; \end{cases}$$

$$q'(x, y) = \begin{cases} s'(x, y) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \leqslant 4/9;\\ \lambda_X\big(r(y), \lambda_X\big(r(x), r(y), 1/2\big), \tfrac{9}{2} g(x, y) - 2\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } 4/9 \leqslant g(x, y) \leqslant 2/3;\\ \lambda_X\big(r(y), \lambda_X\big(r(x), r(y), 1/2\big), 3 - 3g(x, y)\big) & \text{if } (x, y) \in U^2 - A^2 \text{ and } g(x, y) \geqslant 2/3\,. \end{cases}$$

The hypotheses of Lemma 2.1 are satisfied, so s and s' are uniformly continuous. It is also clear that q and q' are continuous, and since A and U are compact, f and r are uniformly continuous. Thus there exists $\delta > 0$ such that

$$d\big(x, s(x, y)\big) = d\big(s(x, x), s(x, y)\big) < \varepsilon/3$$

and $d\big(y, s'(x, y)\big) < \varepsilon/3$ for all $(x, y) \in U^2$ with $d(x, y) < \delta$, and such that $d\big(f(x), f(y)\big) < \varepsilon/2$ for all $(x, y) \in A^2$ with $d(x, y) < \delta$. There also exists $\eta > 0$ such that $d\big(r(x), r(y)\big) < \delta$ for all $(x, y) \in U^2$ with $d(x, y) < \eta$. In addition, we may require that $\eta < \delta < \varepsilon/3 < 2$.

Let

$$V_W = \{(x, y) \in (W - A)^2 \mid d(x, y) < \eta/2\} \cup \{(x, y) \in X^2 \mid d(x, y) < \eta/4\}\,.$$

Then it is clear that V_W is a neighborhood of $\Delta(W)$. Note also that since $\eta/2 < 1$, we have $V_W \subset X^2 \cup Y^2$. Let $p\colon W \to Z$ be projection, and note that p is closed since W is compact and Z is Hausdorff. Thus $\{p^{-1}(a) \mid a \in Z\}$ is an upper semi-continuous decomposition of W. Let

$$V = \bigcup \left\{\big(p\big(V_W[p^{-1}(a)]\big)\big)^2 \,\middle|\, a \in Z\right\}.$$

We claim that V is a neighborhood of $\Delta(Z)$. So let $b \in Z$. Then $V_W[p^{-1}(b)]$ is an open neighborhood of $p^{-1}(b)$. Since $\{p^{-1}(a) \mid a \in Z\}$ is upper semi-continuous, there is an open union T of equivalence classes such that $p^{-1}(b) \subset T \subset V_W[p^{-1}(b)]$. Then $\big(p(T)\big)^2$ is open in Z^2, and

$$(b, b) \in \big(p(T)\big)^2 \subset \big(p\big(V_W[p^{-1}(b)]\big)\big)^2 \subset V\,.$$

Thus V is a neighborhood of (b, b), and hence of $\Delta(Z)$.

We now proceed to find a local connecting map $\lambda\colon V \times I \to Z$. For each $a \in Z$, denote by $h(a)$ the unique member of $p^{-1}(a) \cap (W - A)$. In order to insure that the definition of λ, to be given below, is meaningful, a number of preliminary observations are needed. First of all, it must be shown that

(1) *if* $(a, b) \in V \cap \left[\left(p(U-A)\right)^2 - \left(p(X-U')\right)^2\right]$, *then*

(a) $$\left(h(a), s\left(h(a), h(b)\right)\right) \in V_X;$$

(b) $$\left(s\left(h(a), h(b)\right), q\left(h(a), h(b)\right)\right) \in V_X;$$

(c) $$\left(q\left(h(a), h(b)\right), q'\left(h(a), h(b)\right)\right) \in V_X \quad \text{if} \quad g\left(h(a), h(b)\right) \leqslant 4/9;$$

(d) $$\left(q\left(h(a), h(b)\right), q'\left(h(a), h(b)\right)\right) \in A^2 \quad \text{if} \quad g\left(h(a), h(b)\right) \geqslant 4/9;$$

(e) $$\left(fq\left(h(a), h(b)\right), fq'\left(h(a), h(b)\right)\right) \in V_Y \quad \text{if} \quad g\left(h(a), h(b)\right) \geqslant 2/3;$$

(f) $$\left(q'\left(h(a), h(b)\right), s'\left(h(a), h(b)\right)\right) \in V_X;$$

(g) $$\left(s'\left(h(a), h(b)\right), h(b)\right) \in V_X .$$

We will also show that

(2) $$\text{if } (a, b) \in V \cap \left(p(X-U')\right)^2, \text{ then } \left(h(a), h(b)\right) \in V_X;$$

that

(3) $$\text{if } (a, b) \in V \cap \left(p(Y)\right)^2, \text{ then } \left(h(a), h(b)\right) \in V_Y;$$

that

(4) *if* $(a, b) \in V \cap \left(p(X-A) \times p(Y)\right)$, *then*

(a) $$h(a) \in U;$$

(b) $$\left(h(a), rh(a)\right) \in V_X;$$

(c) $$\left(frh(a), h(b)\right) \in V_Y;$$

and that

(5) *if* $(a, b) \in V \cap \left(p(Y) \times p(X-A)\right)$, *then*

(a) $$h(b) \in U;$$

(b) $$\left(h(a), frh(b)\right) \in V_Y;$$

(c) $$\left(rh(b), h(b)\right) \in V_X .$$

Finally, in order to guarantee that the domain of λ is actually $V \times I$, it will be shown that

(6) $$V \cap \left(p(X-A)\right)^2 \subset V \cap \left[\left(p(U-A)\right)^2 \cup \left(p(X-U')\right)^2\right].$$

Let $(a, b) \in V$. Then there exists $c \in Z$ such that

$$(a, b) \in \left(p\left(V_W[p^{-1}(c)]\right)\right)^2.$$

Thus

$$p^{-1}(a) \cap V_W[p^{-1}(c)] \neq \emptyset$$

and

$$p^{-1}(b) \cap V_W[p^{-1}(c)] \neq \emptyset .$$

Note first of all that (1d) is trivial, that (1d) implies (1e), that (4a) implies (4b) and that (5a) implies (5c).

Case 1. There exists $w \in W-A$ such that $d\left(h(a), w\right) < \eta/2$ and $d\left(h(b), w\right) < \eta/2$. Then $d\left(h(a), h(b)\right) < \varepsilon$, establishing (2) and (3). Now suppose

$$(a, b) \in \left(p(U-A)\right)^2 - \left(p(X-U')\right)^2 .$$

Then since $d\left(h(a), h(b)\right) < \delta$, we have

$$d\left(h(a), s\left(h(a), h(b)\right)\right) < \varepsilon/3$$

and

$$d\left(s'\left(h(a), h(b)\right), h(b)\right) < \varepsilon/3 ,$$

which establishes (1a) and (1g), If $g\left(h(a), h(b)\right) \leqslant 4/9$, then

$$q\left(h(a), h(b)\right) = s\left(h(a), h(b)\right) \quad \text{and} \quad q'\left(h(a), h(b)\right) = s'\left(h(a), h(b)\right),$$

whereas if $g\left(h(a), h(b)\right) \geqslant 4/9$, then

$$\left(s\left(h(a), h(b)\right), q\left(h(a), h(b)\right)\right) \in A^2$$

and

$$\left(q'\left(h(a), h(b)\right), s'\left(h(a), h(b)\right)\right) \in A^2 .$$

Thus (1b) and (1f) are established. Again, if $g\left(h(a), h(b)\right) \leqslant 4/9$, then by what has been said above,

$$d\left(q\left(h(a), h(b)\right), q'\left(h(a), h(b)\right)\right) \leqslant d\left(s\left(h(a), h(b)\right), h(a)\right) + d\left(h(a), h(b)\right) + \\ + d\left(h(b), s'\left(h(a), h(b)\right)\right) < \varepsilon/3 + \delta + \varepsilon/3 < \varepsilon .$$

Thus (1c) holds. Finally, we remark that under the hypothesis of Case 1, we cannot have

$$\left(h(a), h(b)\right) \in \left((X-A) \times Y\right) \cup \left(Y \times (X-A)\right),$$

so that (4) and (5) hold vacuously.

Case 2. There does not exist $w \in W-A$ such that $d\left(h(a), w\right) < \eta/2$ and $d\left(h(b), w\right) < \eta/2$. We claim first of all that

$$\left(h(a), h(b)\right) \notin (X-U')^2 .$$

For suppose otherwise. Then $p^{-1}(a) = \{h(a)\}$, so $h(a) \in V_W[p^{-1}(c)]$. Hence $p^{-1}(c) \cap X \neq \emptyset$, which implies that $h(c) \notin Y - f(A)$. So suppose $h(c) \in f(A)$. Then there must exist $w \in f^{-1}(h(c))$ such that $(h(a), w) \in V_W$, so $d(h(a), A) \leqslant d(h(a), w) < \eta/4 < \varepsilon/2$. It is then clear that $d(h(a), X - U) \geqslant \varepsilon/2$, so that $h(a) \in U'$, contradicting the supposition that $h(a) \in X - U'$. We must therefore have $h(c) \in X - A$. But then $(h(a), h(c)) \in V_W$, and, by a similar argument, $(h(b), h(c)) \in V_W$, so that $d(h(a), h(c)) < \eta/2$ and $d(h(b), h(c)) < \eta/2$, contrary to the assumption of Case 2. Thus (2) holds vacuously.

Case 2a. $(a, b) \in (p(U - A))^2 - (p(X - U'))^2$. Then we must have $h(a) \in V_W[p^{-1}(c)]$ and $h(b) \in V_W[p^{-1}(c)]$, which, as above, implies that $h(c) \notin Y - f(A)$. Again, as above, if $h(c) \in X - A$, then the hypothesis of Case 2 is contradicted. Thus $h(c) \in f(A)$, so there are points w and v of $f^{-1}(h(c))$ such that $(h(a), w) \in V_W$ and $(h(b), v) \in V_W$. It follows that $d(h(a), A) < \eta/4$ and $d(h(b), A) < \eta/4$. Hence $d(h(a), X - U) \geqslant \varepsilon - \eta/4 > 3\eta/4$, and

$$d(h(a), A) + d(h(b), A) < \eta/2 \leqslant d(h(a), h(b))$$

by the Case 2 hypothesis. Thus

$$\frac{d(h(a), A)}{d(h(a), X - U)} < \frac{\eta/4}{3\eta/4} = \frac{1}{3}.$$

Similarly,

$$\frac{d(h(b), A)}{d(h(b), X - U)} < \frac{1}{3}.$$

Therefore

$$g(h(a), h(b)) > \frac{d(h(a), h(b))}{d(h(a), h(b)) + d(h(a), h(b))}\left(1 - \frac{1}{3}\cdot\frac{1}{3}\right) = \frac{4}{9}.$$

Thus (1c) holds vacuously. It also follows that $s(h(a), h(b)) = rh(a)$ and $s'(h(a), h(b)) = rh(b)$. Since $d(h(a), w) < \eta/4$ and $r(w) = w$, we have $d(rh(a), w) < \delta$, so that

$$d(h(a), s(h(a), h(b))) \leqslant d(h(a), w) + d(w, rh(a)) < \eta/4 + \delta < \varepsilon.$$

Similarly,

$$d(s'(h(a), h(b)), h(b)) < \varepsilon,$$

so (1a) and (1g) are established. Moreover,

$$(s(h(a), h(b)), q(h(a), h(b))) \in A^2$$

and

$$(q'(h(a), h(b)), s'(h(a), h(b))) \in A^2,$$

from which (1b) and (1f) follow.

Case 2b. $(a, b) \in (p(Y))^2$. If $h(c) \in Y - f(A)$, then $V_W[p^{-1}(c)] \subset Y$, so that $(h(a), h(c)) \in V_W$ and $(h(b), h(c)) \in V_W$, contrary to the assumption of Case 2. Next suppose $h(c) \in X - A$. Then $V_W[p^{-1}(c)] \subset X$, so we must have $h(a) \in f(A)$ and $h(b) \in f(A)$, from which (3) follows. Finally, suppose $h(c) \in f(A)$. Then we claim that $d(h(a), h(c)) < \varepsilon/2$. For suppose $d(h(a), h(c)) \geqslant \varepsilon/2 > \eta/2$. Then $(h(a), h(c)) \notin V_W$, so it is easily seen that $h(a) \in f(A)$ and that there exist $w \in f^{-1}(h(a))$ and $v \in f^{-1}(h(c))$ such that $(w, v) \in V_W$, i.e., such that $d(w, v) < \eta/4 < \delta$. It then follows that

$$d(h(a), h(c)) = d(f(w), f(v)) < \varepsilon/2,$$

which is a contradiction. Similarly we establish that $d(h(b), h(c)) < \varepsilon/2$. Thus $d(h(a), h(b)) < \varepsilon$, which verifies (3).

Case 2c. $(a, b) \in p(X - A) \times p(Y)$. Then $h(a) \in V_W[p^{-1}(c)]$, which, as has been seen above, implies that $h(c) \notin Y - f(A)$. So suppose first that $h(c) \in X - A$. Then $V_W[p^{-1}(c)] \subset X$, so it is clear that $h(b) \in f(A)$, and that there exists $w \in f^{-1}(h(b))$ such that $(w, h(c)) \in V_W$, i.e., such that $d(w, h(c)) < \eta/4$. But we also know that $d(h(a), h(c)) < \eta/2$, so that $d(h(a), w) < 3\eta/4$, which implies that $h(a) \in U$, establishing (4a) and hence (4b). Moreover, $(frh(a), h(b)) \in (f(A))^2$, so that (4c) holds.

Now suppose that $h(c) \in f(A)$. Then there exists $v \in f^{-1}(h(c))$ such that $(h(a), v) \in V_W$, i.e., such that $d(h(a), v) < \eta/4$. Thus $h(a) \in U$, establishing (4a) and (4b). Now if $h(b) \in f(A)$, then (4c) follows immediately from the fact that $(frh(a), h(b)) \in (f(A))^2$. On the other hand, suppose $h(b) \in Y - f(A)$. Then $h(b) \in V_W[p^{-1}(c)]$, so that $d(h(b), h(c)) < \eta/2$. Since $d(h(a), v) < \eta$, we have $d(rh(a), v) < \delta$, so that

$$d(frh(a), h(c)) = d(frh(a), f(v)) < \varepsilon/2.$$

Thus $d(frh(a), h(b)) < \varepsilon/2 + \eta/2 < \varepsilon$, which establishes (4c).

Case 2d. $(a, b) \in p(Y) \times p(X - A)$. Then the same argument as in Case 2c shows that (5) holds.

We have thus verified (1)-(5). To establish (6), note first that $X - A = (X - U') \cup (U - A)$, that $(X - U') - (U - A) = X - U$, and that $(U - A) - (X - U') = U' - A$. It then follows from a trivial set-theoretic identity that

$$(X - A)^2 = (X - U')^2 \cup (U - A)^2 \cup ((X - U) \times (U' - A)) \cup ((U' - A) \times (X - U)).$$

Now suppose $(a, b) \in V \cap (p(X - A))^2$ but that

$$(a, b) \notin (p(U - A))^2 \cup (p(X - U'))^2.$$

Then

$$(h(a),h(b))\in\big((X-U)\times(U'-A)\big)\cup\big((U'-A)\times(X-U)\big).$$

Suppose, to be specific, that $(h(a),h(b))\in(X-U)\times(U'-A)$. Then $h(a)\in V_W[p^{-1}(c)]$, which, as before, implies that $h(c)\notin Y-f(A)$. If $h(c)\in f(A)$, then there exists $w\in f^{-1}(h(c))$ such that $(h(a),w)\in V_W$, and hence $d(h(a),A)<\eta/4<\varepsilon$, contradicting the fact that $h(a)\in X-U$. We therefore must have $h(c)\in X-A$, and hence $(h(a),h(c))\in V_W$ and $(h(b),h(c))\in V_W$. It follows that $d(h(a),h(b))<\eta$. But since $h(b)\in U'$, we also know that

$$d(h(b),A)<d(h(b),X-U)\leqslant d(h(b),h(a))<\eta,$$

so that

$$d(h(a),A)\leqslant d(h(a),h(b))+d(h(b),A)<2\eta<\varepsilon,$$

again contradicting the fact that $h(a)\in X-U$. Similarly, we show that $(h(a),h(b))\notin(U'-A)\times(X-U)$. Formula (6) is thus established.

Finally, note that (1c) and (1d) together imply that

$$\big(q(h(a),h(b)),\,q'(h(a),h(b))\big)\in V_X$$

for all

$$(a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big].$$

Because of formulas (1)-(6), we can now define $\lambda\colon V\times I\to Z$ as follows.

$$\lambda(a,b,t)=\begin{cases} p\lambda_X\big(h(a),s(h(a),h(b)),6t\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big] \text{ and } 0\leqslant t\leqslant 1/6; \\ p\lambda_X\big(s(h(a),h(b)),q(h(a),h(b)),6t-1\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big] \text{ and } 1/6\leqslant t\leqslant 1/3; \\ p\lambda_X\big(q(h(a),h(b)),q'(h(a),h(b)),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big],\ 1/3\leqslant t\leqslant 2/3, \\ \qquad\qquad\text{and } g(h(a),h(b))\leqslant 2/3; \\ p\lambda_Y\big(fq(h(a),h(b)),fq'(h(a),h(b)),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big],\ 1/3\leqslant t\leqslant 2/3, \\ \qquad\qquad\text{and } g(h(a),h(b))\geqslant 2/3; \end{cases}$$

$$\lambda(a,b,t)=\begin{cases} p\lambda_X\big(q'(h(a),h(b)),s'(h(a),h(b)),6t-4\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big] \text{ and } 2/3\leqslant t\leqslant 5/6; \\ p\lambda_X\big(s'(h(a),h(b)),h(b),6t-5\big) \\ \quad\text{if } (a,b)\in V\cap\big[(p(U-A))^2-(p(X-U'))^2\big] \text{ and } 5/6\leqslant t\leqslant 1; \\ ph(a) \\ \quad\text{if } (a,b)\in V\cap(p(X-U'))^2 \text{ and } 0\leqslant t\leqslant 1/3; \\ p\lambda_X\big(h(a),h(b),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap(p(X-U'))^2 \text{ and } 1/3\leqslant t\leqslant 2/3; \\ ph(b) \\ \quad\text{if } (a,b)\in V\cap(p(X-U'))^2 \text{ and } 2/3\leqslant t\leqslant 1; \\ ph(a) \\ \quad\text{if } (a,b)\in V\cap(p(Y))^2 \text{ and } 0\leqslant t\leqslant 1/3; \\ p\lambda_Y\big(h(a),h(b),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap(p(Y))^2 \text{ and } 1/3\leqslant t\leqslant 2/3; \\ ph(b) \\ \quad\text{if } (a,b)\in V\cap(p(Y))^2 \text{ and } 2/3\leqslant t\leqslant 1; \\ p\lambda_X\big(h(a),rh(a),6t\big) \\ \quad\text{if } (a,b)\in V\cap\big(p(X-A)\times p(Y)\big) \text{ and } 0\leqslant t\leqslant 1/6; \\ prh(a) \\ \quad\text{if } (a,b)\in V\cap\big(p(X-A)\times p(Y)\big) \text{ and } 1/6\leqslant t\leqslant 1/3; \\ p\lambda_Y\big(frh(a),h(b),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap\big(p(X-A)\times p(Y)\big) \text{ and } 1/3\leqslant t\leqslant 2/3; \\ ph(b) \\ \quad\text{if } (a,b)\in V\cap\big(p(X-A)\times p(Y)\big) \text{ and } 2/3\leqslant t\leqslant 1; \\ ph(a) \\ \quad\text{if } (a,b)\in V\cap\big(p(Y)\times p(X-A)\big) \text{ and } 0\leqslant t\leqslant 1/3; \\ p\lambda_Y\big(h(a),frh(b),3t-1\big) \\ \quad\text{if } (a,b)\in V\cap\big(p(Y)\times p(X-A)\big) \text{ and } 1/3\leqslant t\leqslant 2/3; \end{cases}$$

$$\lambda(a,b,t)=\begin{cases} prh(b) \\ \quad \text{if } (a,b)\in V\cap\big(p(Y)\times p(X-A)\big) \text{ and } 2/3\leqslant t\leqslant 5/6; \\ p\lambda_X\big(rh(b),h(b),6t-5\big) \\ \quad \text{if } (a,b)\in V\cap\big(p(Y)\times p(X-A)\big) \text{ and } 5/6\leqslant t\leqslant 1. \end{cases}$$

It is easily seen that λ has the connecting property. It must now be shown that λ is continuous. First of all, it is clear that λ is continuous on each of the sets

$$\Big[V\cap\Big(\big(p(U-A)\big)^2-\big(p(X-U')\big)^2\Big)\Big]\times I, \qquad \Big[V\cap\big(p(X-U')\big)^2\Big]\times I,$$

$$\Big[V\cap\big(p(Y)\big)^2\Big]\times I, \qquad \Big[V\cap\big(p(X-A)\times p(Y)\big)\Big]\times I,$$

and

$$\Big[V\cap\big(p(Y)\times p(X-A)\big)\Big]\times I,$$

since $p|(X-A)$ and $p|Y$ are homeomorphisms, and

$$h(e)=\begin{cases}\big(p|(X-A)\big)^{-1}(e) & \text{if } e\in p(X-A); \\ (p|Y)^{-1}(e) & \text{if } e\in p(Y). \end{cases}$$

Now let $\{(a_n,b_n,t_n)\}$ be a sequence in $V\times I$ converging to $(a,b,t)\in V\times I$. By Lemma 2.4, we may assume that $\{t_n|\ n\geqslant 1\}$ is contained in one of the intervals $[0,1/6]$, $[1/6,1/3]$, $[1/3,2/3]$, $[2/3,5/6]$, or $[5/6,1]$.

Suppose first that

$$(a_n,b_n)\in\big(p(U-A)\big)^2-\big(p(X-U')\big)^2$$

for each n, and that $(a,b)\in\big(p(X-U')\big)^2$. Then $\{\big(h(a_n),h(b_n)\big)\}$ is a sequence in $(U-A)^2-(X-U')^2$ converging to $\big(h(a),h(b)\big)\in(X-U')^2$. Since $g\big(h(a),h(b)\big)=0$, we have

$$\lim_{n\to\infty} g\big(h(a_n),h(b_n)\big)=0,$$

so that

$$\lim_{n\to\infty} q\big(h(a_n),h(b_n)\big)=\lim_{n\to\infty} s\big(h(a_n),h(b_n)\big)=h(a)$$

and

$$\lim_{n\to\infty} q'\big(h(a_n),h(b_n)\big)=\lim_{n\to\infty} s'\big(h(a_n),h(b_n)\big)=h(b).$$

Thus if each $t_n\in[0,1/3]$, then

$$\lim_{n\to\infty}\lambda(a_n,b_n,t_n)=ph(a)=\lambda(a,b,t).$$

Similarly, if each $t_n\in[2/3,1]$, then

$$\lim_{n\to\infty}\lambda(a_n,b_n,t_n)=\lambda(a,b,t).$$

If each $t_n\in[1/3,2/3]$, then

$$\lim_{n\to\infty}\lambda(a_n,b_n,t_n)=p\lambda_X\big(h(a),h(b),3t-1\big)=\lambda(a,b,t).$$

We have thus shown that λ is continuous on $\big[V\cap\big(p(X-A)\big)^2\big]\times I$.

Now in order to show that λ is continuous on $V\times I$, it is clearly sufficient to consider only the cases in which the sequence $\{(a_n,b_n)\}$ is contained in one of the sets $\big(p(X-A)\big)^2$, $\big(p(Y)\big)^2$, $p(X-A)\times p(Y)$, or $p(Y)\times p(X-A)$, and in which (a,b) belongs to a different one of these sets. It is also clear that no sequence in $p(Y)$ can converge to a point of $p(X-A)$. Moreover, if $\{e_n\}$ is a sequence in $p(X-A)$ converging to $e\in p(Y)$, then $e\in p(A)$; for otherwise $Z-p(X)$ is a neighborhood of e which does not intersect $p(X-A)$. Furthermore, $p(X-U')$ is closed, since $X-U'$ is closed, and hence we need not consider the case in which $(a_n,b_n)\in\big(p(X-U')\big)^2$ for infinitely many n. Since λ may be thought of as the composition of two functions, the first of which takes (a,b,t) into $\big(h(a),h(b),t\big)$, it follows from Lemma 2.5 that in order to show that

$$\lim_{n\to\infty}\lambda(a_n,b_n,t_n)=\lambda(a,b,t),$$

it is sufficient to show that

$$\lim_{k\to\infty}\lambda(a_{n_k},b_{n_k},t_{n_k})=\lambda(a,b,t)$$

for every convergent subsequence $\{\big(h(a_{n_k}),h(b_{n_k}),t_{n_k}\big)\}$ of $\{\big(h(a_n),h(b_n),t_n\big)\}$. Without loss of generality, then, we may assume that $\{\big(h(a_n),h(b_n),t_n\big)\}$ converges, say to (x,y,t).

Suppose first that

(7) $\qquad (a_n,b_n)\in\big(p(X-A)\big)^2$ for each n and $(a,b)\in\big(p(Y)\big)^2$.

Then $\big(h(a_n),h(b_n)\big)\in(X-A)^2$ for each n, and $\big(h(a),h(b)\big)\in\big(f(A)\big)^2$. We claim that

$$\lim_{n\to\infty} d\Big(h(a_n),f^{-1}\big(h(a)\big)\Big)=0.$$

For let N be an X-neighborhood of $f^{-1}\big(h(a)\big)$. Then $N\cup Y$ is a W-neighborhood of $p^{-1}(a)$. Hence there is an open union M of equivalence classes such that $p^{-1}(a)\subset M\subset N\cup Y$. Then $p(M)$ is a neighborhood of a, and therefore contains all except a finite number of the a_n's. Since each $h(a_n)\in X$, it follows that N contains all except a finite number of the $h(a_n)$'s. Similarly,

$$\lim_{n\to\infty} d\Big(h(b_n),f^{-1}\big(h(b)\big)\Big)=0.$$

Thus

$$(x, y) \in f^{-1}(h(a)) \times f^{-1}(h(b)) .$$

We will show next that

$$\lim_{n\to\infty} s(h(a_n), h(b_n)) = \lim_{n\to\infty} q(h(a_n), h(b_n)) = x .$$

By Lemma 2.4, we may assume, without loss of generality, that $\{g(h(a_n), h(b_n)) \mid n \geqslant 1\}$ is contained in one of the intervals $[0, 4/9]$, $[4/9, 2/3]$, or $[2/3, 1]$. If $g(h(a_n), h(b_n)) \leqslant 4/9$ for each n, then

$$\lim_{n\to\infty} q(h(a_n), h(b_n)) = \lim_{n\to\infty} s(h(a_n), h(b_n))$$
$$= \lim_{n\to\infty} \lambda_X\big(h(a_n), rh(a_n), \tfrac{9}{4} g(h(a_n), h(b_n))\big) = x ,$$

since $\lim_{n\to\infty} rh(a_n) = r(x) = x$. Note that if $x = y$, then

$$\lim_{n\to\infty} rh(a_n) = \lim_{n\to\infty} rh(b_n) = x ,$$

and if $x \neq y$, then

$$\lim_{n\to\infty} g(h(a_n), h(b_n)) = 1 .$$

It is then easy to see that if $g(h(a_n), h(b_n)) \geqslant 4/9$ for each n, then

$$\lim_{n\to\infty} s(h(a_n), h(b_n)) = \lim_{n\to\infty} q(h(a_n), h(b_n)) = x .$$

Similarly we establish that

$$\lim_{n\to\infty} s'(h(a_n), h(b_n)) = \lim_{n\to\infty} q'(h(a_n), h(b_n)) = y .$$

Therefore, if each $t_n \in [0, 1/3]$, then

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = p(x) = ph(a) = \lambda(a, b, t) .$$

Similarly, if each $t_n \in [2/3, 1]$, then

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = \lambda(a, b, t) .$$

So now suppose that each $t_n \in [1/3, 2/3]$. As in the preceding paragraph, we may assume, without loss of generality, that either $g(h(a_n), h(b_n)) \leqslant 2/3$ for each n or $g(h(a_n), h(b_n)) \geqslant 2/3$ for each n. In the first case, observe that $x = y$ (since otherwise

$$\lim_{n\to\infty} g(h(a_n), h(b_n)) = 1) ,$$

and hence $h(a) = h(b)$. It follows that if $g(h(a_n), h(b_n)) \leqslant 2/3$ for each n, then

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = p(x) = p\lambda_Y(h(a), h(b), 3t-1) = \lambda(a, b, t) .$$

On the other hand, if $g(h(a_n), h(b_n)) \geqslant 2/3$ for each n, then

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = p\lambda_Y(f(x), f(y), 3t-1) = p\lambda_Y(h(a), h(b), 3t-1) = \lambda(a, b, t).$$

Next suppose that

(8) $\quad (a_n, b_n) \in (p(X-A))^2$ for each n and $(a, b) \in p(X-A) \times p(Y)$.

Then it is clear that $x = h(a)$ and, as above, we see that $y \in f^{-1}(h(b))$. Note also that

$$\lim_{n\to\infty} g(h(a_n), h(b_n)) = 1 ,$$

so that

$$\lim_{n\to\infty} s(h(a_n), h(b_n)) = \lim_{n\to\infty} q(h(a_n), h(b_n)) = rh(a) .$$

Similarly,

$$\lim_{n\to\infty} s'(h(a_n), h(b_n)) = \lim_{n\to\infty} q'(h(a_n), h(b_n)) = r(y) = y .$$

It now requires only a straightforward argument to show that

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = \lambda(a, b, t) .$$

Similarly, we show that if

(9) $\quad (a_n, b_n) \in (p(X-A))^2$ for each n and $(a, b) \in p(Y) \times p(X-A)$,

then

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = \lambda(a, b, t) .$$

Now suppose that

(10) $\quad (a_n, b_n) \in p(X-A) \times p(Y)$ for each n and $(a, b) \in (p(Y))^2$.

Then $(h(a), h(b)) \in f(A) \times Y$, $y = h(b)$, and $x \in f^{-1}(h(a))$. Again, a straightforward argument shows that

$$\lim_{n\to\infty} \lambda(a_n, b_n, t_n) = \lambda(a, b, t) .$$

The desired result is obtained similarly if

(11) $\quad (a_n, b_n) \in p(Y) \times p(X-A)$ for each n and $(a, b) \in (p(Y))^2$.

Note that since a sequence in $p(Y)$ cannot converge to a point of $p(X-A)$, (7)–(11) are the only possibilities which must be considered. Hence λ is continuous, and so is a local connecting map for Z. Thus Z is LEC. Q.e.d.

A similar but easier argument establishes the following result.

THEOREM 3.2. *Let X and Y be compact metrizable* EC *spaces, let A be a retract of X, let $f\colon A\to Y$ be continuous, and let $Z = X\cup_f Y$. Then Z is* EC.

References

[1] K. Borsuk, *Bericht über einige neue Ergebnisse und Probleme aus dem Gebiet der anschaulichen Topologie*, J.-Ber Deutsch. Math.-Verein. 66 (1964), pp. 94-105.

[2] — *Quelques rétractes singuliers*, Fund. Math. 24 (1935), pp. 249-258.

[3] J. Dugundji, *Locally equiconnected spaces and absolute neighborhood retracts*, Fund. Math. 57 (1965), pp. 187-193.

[4] R. H. Fox, *On fibre spaces. II*. Bull. Amer. Math. Soc. 49 (1943), pp. 733-735.

[5] C. J. Himmelberg, *Some theorems on equiconnected and locally equiconnected spaces*, Trans. Amer. Math. Soc. 115 (1965), pp. 43-53.

[6] A. Lelek, *Some mappings of* ANR-*sets*, Fund. Math. 52 (1963), pp. 225-233.

[7] J. H. C. Whitehead, *Note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. 54 (1948), pp. 1125-1132.

UNIVERSITY OF MISSOURI

Reçu par la Rédaction le 20. 3. 1967

Decomposable circle-like continua*

by

W. T. Ingram (Houston, Texas)

1. Introduction. In [6] J. B. Fugate proved that a necessary and sufficient condition that the sum of two chainable continua be chainable is that the sum be atriodic and unicoherent. In this paper it is proved that a necessary and sufficient condition that the non-chainable sum of two chainable continua be circle-like is that the sum be atriodic and the common part of the two continua be not connected. The techniques used in proving this also yield a strengthened version of Fugate's theorem.

Space is assumed to be metric with metric ϱ. For definitions of terms such as chainable (snake-like) or circle-like, see [2]; the conventions used there for denoting chains (or circular chains) are employed in this paper.

The subcontinuum H of the compact continuum M is said to be a *terminal* subcontinuum of M if and only if for each two subcontinua K and L of M which intersect H either K is a subcet of $H\cup L$ or L is a subset of $H\cup K$.

A chain C is said to be *regular* (taut) if and only if the distance between non-intersecting links of C is positive. In [4], Theorem 1, p 12, H. Cook proved that if M is chainable and D is a chain covering M then there is a regular chain covering M which is a strong refinement of D

THEOREM (Fugate, [6], Lemma 1, p. 461). *If H is a terminal subcontinuum of the chainable continuum M and $\varepsilon>0$, then there is a regular ε-chain $C(c_1, c_2, \ldots, c_n)$ covering M such that $c_1-(c_1\cap\bar{c}_2)$ intersects H.*

2. Terminal continua and decomposable atriodic continua. Theorem 1 is a generalization of a theorem of Bing ([1], Theorem 14, p. 661) concerning opposite end points. The argument is similar to that given by Bing.

THEOREM 1. *If H and K are mutually exclusive terminal subcontinua of the chainable continuum M and M is irreducible with respect to containing $H\cup K$ and $\varepsilon>0$, then there is an ε-chain $C(c_1, c_2, \ldots, c_n)$ covering M such that $c_1-(c_1\cap\bar{c}_2)$ intersects H and $c_n-(c_n\cap\bar{c}_{n-1})$ intersects K.*

* Work on this paper was supported in part by NASA Grant NGR 44-005-037.