Now let $\mathfrak{Q}$ have a countable member F. For each $x \in F$, choose a free z-ultrafilter $\mathfrak{Q}_x$ on $T_x = B_0(x, 1)$, converging to x, such that every countable subset of T_x is disjoint from some member of $\mathfrak{Q}_x$. For each $x \in F$, choose $S_x \in \mathfrak{Q}_x$ such that $S_x \cap F = \emptyset$, in such a way that $\operatorname{diam} S_x \to 0$. Define $\mathfrak{F}_x = \mathfrak{Q}_x|S_x$. We now apply Theorem 12.1 with $E = F$, and proceed as in the first case.

Remark 1. By 12.4 and Corollary 8.2, we can find a nonminimal nonclosed prime z-filter with no countable member. By Remark 1 following Theorem 13.2, we can find a closed nonminimal prime z-filter with a countable member. Hence the two alternative conditions used are independent.

Remark 2. Theorem 15.6 yields an example of a closed nonminimal prime z-filter with an immediate successor, and provides a counterexample to the converse of Corollary 10.4(a).

15.7. QUESTION. *When a prime z-filter is the union of all its predecessors, does it follow that it is the union of a chain of predecessors?*

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A lower bound for transfinite dimension

by

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1. Introduction. In this paper, essential mappings are used to give a lower bound for the (large, strong) transfinite inductive dimension of a space. Transfinite inductive dimension (Ind) is defined by transfinite induction as follows: (See [3], p. 161).

DEFINITION 1. (a) $\operatorname{Ind}(R) = -1$, if $R = \emptyset$. (b) $\operatorname{Ind}(R) \leqslant \alpha$ (an ordinal number) if every pair of disjoint closed subsets of R can be separated by a closed subset S such that $\operatorname{Ind}(S) < \alpha$. ($S$ *separates* A and B in R if $R - S$ is the union of disjoint open (in R) sets U, V such that $A \subset U$ and $B \subset V$.) (c) $\operatorname{Ind}(R) = \alpha$ if $\operatorname{Ind}(R) \leqslant \alpha$ and it is not true that $\operatorname{Ind}(R) < \alpha$. (d) R is said to have *transfinite dimension* (Ind) if $\operatorname{Ind}(R)$ exists.

It is known ([3], p. 209) that if a normal space R has an essential mapping onto the n-cell, I^n, then $\operatorname{Ind}(R) \geqslant n$ or $\operatorname{Ind}(R)$ does not exist. (A mapping $f\colon R \to I^n$ is *essential* if there does not exist a mapping $g\colon R \to \operatorname{Bd}(I^n)$ (Bd = Boundary) such that $f|f^{-1}(\operatorname{Bd}(I^n)) = g|f^{-1}(\operatorname{Bd}(I^n))$.) We shall construct, for each countable ordinal α, a space J^α such that (with "essential" suitably defined), if a normal space R has an essential mapping onto J^α, then $\operatorname{Ind}(R) \geqslant \alpha$ or $\operatorname{Ind}(R)$ does not exist.

Some of the ideas behind the definition of J^α and the proofs below can be found in [1], § 3, by Yu. M. Smirnov (Ю. М. Смирнов).

2. Results and questions.

DEFINITION 2 (J^α, T^α, and p^α). For each ordinal number α, greater than or equal to 0 and less than Ω (the first uncountable ordinal), we shall define a compact metric set J^α with a compact subset T^α and a point $p^\alpha \in T^\alpha$. Let $J^0 = T^0 = p^0 =$ (a point). If α is positive and finite, then let J^α be the α-dimensional cube, T^α be the $(\alpha-1)$-dimensional sphere which is the combinatorial boundary of J^α, and p^α be any point of T^α. If α is not a limit ordinal, then define

$$J^\alpha = J^{\alpha-1} \times J^1, \quad T^\alpha = (T^{\alpha-1} \times J^1) \cup (J^{\alpha-1} \times T^1), \quad \text{and} \quad p^\alpha = p^{\alpha-1} \times p^1.$$

If α is a limit ordinal, then let K^β, for $\beta < \alpha$, be the union of J^β and a half-open arc A^β, such that $A^\beta \cap J^\beta = p^\beta =$ (the end point of A^β); and define J^α

to be the one-point compactification of the (locally compact) *disjoint* union $\bigcup_{\beta<\alpha} K^\beta$, $T^\alpha = J^\alpha - \bigcup_{\beta<\alpha}(J^\beta - T^\beta)$, and p^β be the compactifying point. (I.e., neighborhoods of p^α are complements (in J^α) of compact subsets of $\bigcup_{\beta<\alpha} K^\beta$. Note that each J^β, $\beta<\alpha$, is joined (in J^α) to p^α by an arc.)

The main advantage of the J^α as compared to the spaces of Smirnov ([1], § 3, Def. 2) is that Smirnov's spaces do not satisfy Theorem 1.

THEOREM 1. *Each J^α is a retract of the Hilbert cube and therefore each J^α is an absolute retract* (AR).

THEOREM 2. $\mathrm{Ind}(J^\alpha) = \alpha$.

DEFINITION 3 (Essential mappings). A continuous function (mapping or map) $f\colon X\to J^\alpha$ of a space X onto J^α is called *inessential* if there exists another map $g\colon X\to J^\alpha$ such that $g|f^{-1}(T^\alpha) = f|f^{-1}(T^\alpha)$ and $g(X)\neq J^\alpha$. Otherwise, f will be called *essential.*

THEOREM 3. *If there is an essential mapping of a normal space X onto J^α, then* $\mathrm{Ind}(X)\geqslant\alpha$ *or* $\mathrm{Ind}(X)$ *does not exist.*

It will become clear later that the identity mapping of J^α onto itself is essential; therefore, the inequality in Theorem 3 is the best possible. It would be very interesting to know if the converse to Theorem 3 is true, namely:

QUESTION. *If* $\mathrm{Ind}(X)\geqslant\alpha$, *does X have an essential mapping onto J^α?*

The answer is "yes" if X is normal and α is finite; but nothing else is known even for compact metric spaces. The next question is related to Alexandroff's unsolved problem which asks whether each weakly infinite-dimensional compact metric space is countable-dimensional. (See [2], Chapter III, § 1.)

QUESTION. *If a compact metric space X has an essential mapping onto J^α, for each $\alpha<\Omega$, then is X strongly infinite-dimensional?* (X is *strongly infinite-dimensional* if there is a map f of X onto the Hilbert Cube I^ω, such that, for each finite-dimensional face F of I^ω, $f|f^{-1}(F)\colon f^{-1}(F)\to F$ is essential. Such a map, f, is called an *essential mapping onto* I^ω.)

In proving Theorem 2 the following Proposition is used which may be of interest in its own right.

PROPOSITION 1. *Let X be a hereditarily paracompact space which is the union of two closed subsets A and B. If* $\mathrm{Ind}(A\cap B)$ *is finite and both* $\mathrm{Ind}(A)$ *and* $\mathrm{Ind}(B)$ *exist, then* $\mathrm{Ind}(X)$ *equals the larger of* $\mathrm{Ind}(A)$ *and* $\mathrm{Ind}(B)$.

3. Proof of Theorem 1. In order to prove Theorem 1, we shall prove (by transfinite induction) the following. (We consider I^ω as the Cartesian product, $I_1\times I_2\times I_3\times\ldots\times I_i\times\ldots$, where I_i is the interval $[0, 2^{-i}]$.)

THEOREM 1'. *For each $\alpha\geqslant 0$, there is an embedding $h^\alpha\colon J^\alpha\to 1^\omega$ of J^α into I^ω and a map $r^\alpha\colon I^\omega\to h^\alpha(J^\alpha)$ such that $h^\alpha(p^\alpha)$ equals the point all of whose coordinates are zero and such that $r^\alpha(p) = p$, for all $p\in h^\alpha(J^\alpha)$.*

Proof. The theorem is trivialy true for $\alpha = 0$. Assume inductively that the theorem is true for all $\beta<\alpha$. In the case that $\alpha=\gamma+1$, we know by the inductive hypothesis that J^γ can be embedded (by a map h) in I^ω so that $h(p^\gamma) = (0,0,\ldots,0,\ldots)$ and so that there is a map $r\colon I^\omega\to h(J^\gamma)$ such that $r(p) = p$, for all $p\in h(J^\gamma)$. Now, $I\times I^\omega$ is homeomorphic to I^ω under a homeomorphism k such that

$$k(0;0,0,\ldots,0,\ldots) = (0,0,0,\ldots,0,\ldots);$$

and the map $s\colon I\times I^\omega\to I\times h(J^\gamma)$, defined by $s(t,p) = t\times r(p)$, is a retraction onto $I\times h(J^\gamma)$ which is homeomorphic to $J^{\gamma+1} = J^\alpha$ under a homeomorphism which takes

$$k^{-1}(0,0,\ldots,0,\ldots) = 0\times h(p^\gamma)$$

onto the point p^α. We now only have to check the theorem in the case that α is a limit ordinal.

For each $i\in N$, let $N(i)$ be some infinite subset of the set of natural numbers, N, such that, for $i\neq j$, $N(i)$ is disjoint from $N(j)$ and such that $N = N(1)\cup N(2)\cup\ldots\cup N(i)\cup\ldots$ Note that, for each i,

$$I_i^\omega \equiv \{(p_1, p_2, p_3, \ldots, p_j, \ldots)\in I^\omega | p_j = 0\ ,\ \text{for } j\in N-N(i)\}$$

is homeomorphic to I^ω. Let $\{\gamma(i) | i\in N\}$ be the set of all ordinals less than α, where we have indexed this set by the natural numbers. From the inductive hypothesis we know that there is an embedding $h^{\gamma(i)}\colon J^{\gamma(i)}\to I^\omega$ and a retraction $r^{\gamma(i)}\colon I^\omega\to h(J^{\gamma(i)})$. Combining this retraction with Lemma 1 below, we conclude that there is a retraction

$$r_i\colon I\times I^\omega\to[I\times(0,0,\ldots,0,\ldots)]\cup[\{1\}\times h(J^{\gamma(i)})],$$

where the image space is homeomorphic to $K^{\gamma(i)}\cup p^\alpha$. (See Definition 1.) Let $h_i\colon K^{\gamma(i)}\cup p^\alpha\to I_i^\omega$ be an embedding and $r_i\colon I_i^\omega\to K^{\gamma(i)}\cup p^\alpha$ a retraction such that $h_i(p^\alpha) = (0,0,\ldots,0,\ldots)$. Define $h^\alpha\colon J^\alpha\to I^\omega$ to be the map such that $h^\alpha|K^{\gamma(i)}\cup p^\alpha = h_i$. By Lemma 1, there is a retract of I^ω onto $\bigcup_i I_i^\omega$. But, for each $i\neq j$, $I_i^\omega\cap I_j^\omega = (0,0,\ldots,0,\ldots)$ and therefore by following the retraction of I_i^ω onto $\bigcup_i I_i^\omega$ by a retraction which is, for each i, equal to r_i on I_i^ω, we obtain a retraction $r^\alpha\colon I^\omega\to J^\alpha$. This finishes the proof of Theorem 1' and therefore Theorem 1, except for the proof of Lemma 1.

LEMMA 1. *Let X be a metric space which is equal to the Cartesian product $\prod_{i\in M} A_i$, where M is finite or countable, and where, for each $i\in M$, there is a deformation retraction h_i of A_i onto $p_i\in A_i$. Then there is a deformation retraction h of X onto $\bigcup_{i\in M} A_i'$, where $A_i' = A_i\times\prod_{j\neq i} p_j$.*

Proof. A deformation retraction of X onto Y is a map $g\colon X\times[0,1]\to X$ such that $g(X\times\{1\})\subset Y$, $g(p,\{0\})=p$, for all $p\in X$, and $g(y\times[0,1])=y$, for each $y\in Y$. Let δ be the distance function for X, and define

$$B_i = \{p\in X|\ \delta(p, A_i')\leqslant \delta(p, \bigcup_{j\neq i} A_j')\}\,.$$

The reader can check that we obtain the desired deformation retraction of X by setting, for $p\in B_i$,

$$h(p,t) = h_i\left(p, \frac{t\cdot\delta(p, A_i')}{\delta(p, \bigcup_{j\neq i} A_j')}\right)\times\prod_{j\neq i} h_j(p,t)\,.$$

4. Proof of Proposition 1. The proof will be by transfinite induction on the maximum of $\mathrm{Ind}(A)$ and $\mathrm{Ind}(B)$. The proposition is known for finite-dimensional spaces (see [3], page 199). Let F and G be any two disjoint closed subsets of X and let U and V be disjoint open subsets such that $F\subset U$ and $G\subset V$. Let U' and V' be open subsets of X such that $F\subset U'\subset U$, $G\subset V'\subset V$, $\mathrm{Ind}(\mathrm{Bd}_A(U'\cap A))<\mathrm{Ind}(A)$, and $\mathrm{Ind}(\mathrm{Bd}_B(V'\cap B))<\mathrm{Ind}(B)$, where Bd_A and Bd_B denote the boundaries in the subspaces A and B, respectively. Let

$$W = \mathrm{Bd}_A(U'\cap A)\cup\mathrm{Bd}_B(V'\cap B)\cup[(A\cap B)-(U'\cup V')]\,.$$

Then W separates F and G in X, and by the induction hypothesis $\mathrm{Ind}(W)$ equals the maximum of

$$\mathrm{Ind}\{\mathrm{Bd}_A(U'\cap A)\cup[(A\cap B)-(U'\cup V')]\}\equiv\alpha$$

and

$$\mathrm{Ind}\{\mathrm{Bd}_B(V'\cap B)\cup[(A\cap B)-(U'\cup V')]\}\equiv\beta\,.$$

Making use of the proposition in the finite-dimensional case and the induction hypothesis, we can conclude that α and β are either finite or are $<\mathrm{Ind}(A)$ and $<\mathrm{Ind}(B)$, respectively. But since the proposition is already true for finite-dimensional spaces, we may assume that either $\mathrm{Ind}(A)$ or $\mathrm{Ind}(B)$ are infinite and thus conclude that $\mathrm{Ind}(W)$ is less than the maximum of $\mathrm{Ind}(A)$ and $\mathrm{Ind}(B)$. Because F and G were arbitrary, we have reached the desired conclusion.

5. Proof of Theorem 2. The theorem is obvious if α is finite. If α is a limit ordinal, then $J^\alpha = R^\alpha\cup B^\alpha$, where R^α is the compactum $J^\alpha - \bigcup_{\beta<\alpha}(K^\beta - J^\beta)$ and

$$B^\alpha = p^\alpha\cup\bigcup_{\beta<\alpha} A^\beta = \mathrm{Closure}\,\big(\bigcup_{\beta<\alpha}(K^\beta - J^\beta)\big)\,.$$

If α is equal to $\gamma+k$, where γ is a limit ordinal and k is finite, then $J^\alpha = R^\alpha\cup B^\alpha$, where $R^\alpha = R^\gamma\times J^k$ and $B^\alpha = B^\gamma\times J^k$. Clearly, $\mathrm{Ind}B^\alpha = k+1$. Thus the theorem will follow from Proposition 1 if we can establish

LEMMA 2. $\mathrm{Ind}R^\alpha = \alpha$.

Proof of Lemma 2. A comparison of definitions will show that for finite and positive k, $R^{\omega+k} = Q^{\omega+k}$ (ω = the first infinite ordinal) where Q^α is the compactum defined by Smirnov ([1], Definition 2). A straightforward, transfinite induction gives us the conclusion that Q^α is a closed subset of R^α, for all α between ω and Ω. Therefore, by Lemma 8 of [1], we can conclude that $\mathrm{Ind}R^\alpha\geqslant\alpha$. That $\mathrm{Ind}R^\alpha\leqslant\alpha$ follows from Lemma 6 of [1], since R^α is easily shown to be one of the compacta K^α ([1], Definition 3). [Note: there is a misprint in the English translation of [1], Lemma 6; the conclusion of the Lemma should read "$\mathrm{Ind}K^\beta\leqslant\beta$".]

6. Three propositions. These results are needed for the proof of Theorem 3.

PROPOSITION 2. *For each $\alpha\geqslant 0$, every component of $J^\alpha - T^\alpha$ is an open set which is homeomorphic to some finite-dimensional Euclidean space and has as closure (in J^α) a cell, C, of the same dimension. In addition, $C\cap$ (closure $(J^\alpha - C)$) is a cell.*

The proof follows from a straight-forward (transfinite) induction and is left to the reader.

PROPOSITION 3. *For each $\alpha\geqslant 1$, a map $f\colon X\to J^\alpha$ is essential if and only if f is essential when restricted to the inverse images of the closure each of the components of $J^\alpha - T^\alpha$.* (See Proposition 2.)

Proof. We shall prove the countra-positive of the proposition. Firstly, suppose that f is inessential when restricted to the inverse image, $A = f^{-1}(C)$, of one of the cells, C, which is the closure of a component of $J^\alpha - T^\alpha$. Then there is a map $g\colon A\to C$ such that g equals f on $f^{-1}(\mathrm{Bd}\,C)$ and such that $g(A)\neq C$. The map $G\colon X\to J^\alpha$, defined by setting $G|A = g$ and $G|\mathrm{closure}(X-A) = f|\mathrm{closure}(X-A)$, shows that f is inessential.

Secondly, suppose that $f\colon X\to J^\alpha$ is inessential and thus that there exists a map $g\colon X\to J^\alpha$ such that $g(X)\neq J^\alpha$ and $g|f^{-1}(T^\alpha) = f|f^{-1}(T^\alpha)$. For the closure C of some component of $J^\alpha - T^\alpha$, it must be true that

$C \not\subset g(X)$. Since (Proposition 2) $C \cap (\text{closure } (J^\alpha - C))$ is a cell, the map $g|(g^{-1}(C) \cap f^{-1}(C))$ can be extended to a map $h\colon f^{-1}(C) \to C$ such that

$$h|f^{-1}(\mathrm{Bd}\, C) = g|f^{-1}(\mathrm{Bd}\, C) = f|f^{-1}(\mathrm{Bd}\, C)$$

and $h(f^{-1}(C)) \neq C$.

PROPOSITION 4. *If $f\colon X \to (J^\alpha \times J^1) = J^{\alpha+1}$ is essential, X is normal, and Y is a closed subset of X such that $X - Y$ is the union of two disjoint open sets, U, V, each of which contains the inverse image of one of the two components of $J^\alpha \times T^1$, then $\pi f|Y\colon Y \to J^\alpha$ is essential where $\pi\colon J^\alpha \times J^1 \to J^\alpha$ is the standard projection.*

Proof. We first assume that α is finite (and positive, in order to exclude a trivial case). If $\pi f|Y$ is inessential, then there is easily seen to be a map $g\colon Y \to T^\alpha$ such that g agrees with $\pi f|Y$ on $(\pi f|Y)^{-1}(T^\alpha)$. We now define a new map $h\colon (f^{-1}(T^{\alpha+1}) \cup Y) \to T^{\alpha+1}$, by setting h equal to f on $f^{-1}(T^{\alpha+1})$ and, for $y \in Y$, letting $h(y)$ be that point of $T^\alpha \times J^1$ whose first α coordinates are equal to $g(y)$ and whose last coordinate is the same as the last coordinate of $f(y)$. Call the two points of T^1, u and v, so that $U \supset f^{-1}(J^\alpha \times u)$ and $V \supset f^{-1}(J^\alpha \times v)$. Now

$$h\big((f^{-1}(T^{\alpha+1}) \cup Y) \cap \mathrm{Cl}(U)\big)$$

is contained in $(T^\alpha \times J^1) \cup (J^\alpha \times u)$, a α-cell, and so we can extend h so that it takes all of U into $T^{\alpha+1}$. (A closed subset of a normal space in normal.) We can do the same for V and thus extend h to a mapping which takes all of X into $T^{\alpha+1}$ and which agrees with f on $f^{-1}(T^{\alpha+1})$. This is a contradiction, because f is essential. Thus the proposition is true if α is finite. If α is infinite and $\pi f|Y$ is inessential, then there is a map $g\colon Y \to J^\alpha$ which agrees with $\pi f|Y$ on $(\pi f|Y)^{-1}(T^\alpha)$ and such that there is a point $p \in J^\alpha - g(Y) \subset J^\alpha - T^\alpha$. But then p belongs to the interior of one the finite-dimensional cells mentioned in Proposition 2. Restricting f to the inverse image of the closure of this cell, we reach (using Proposition 3) a contradiction of Proposition 4 in the case that α is finite. Thus Proposition 4 is true for all α.

7. Proof of Theorem 3. (By transfinite induction). Any mapping of the empty set into J^0 is inessential, therefore X is non-empty and thus $\mathrm{Ind}(X) \geqslant 0$ if $\alpha = 0$. Now assume that the theorem has been proven true for all ordinals less than α. If α is a limit ordinal, then J^α is the union of $\bigcup_{\beta<\alpha} J^\beta$ and a certain countable collection of arcs. Proposition 3 shows that $f|f^{-1}(J^\beta)\colon f^{-1}(J^\beta) \to J^\beta$ is essential for each $\beta < \alpha$. Therefore X contains closed subsets of dimension $\geqslant \beta$ for each $\beta < \alpha$. Thus, since Ind is monotone on closed subsets, we conclude that $\mathrm{Ind}(X) \geqslant \alpha$, if α is a limit ordinal. If α equal $\beta+1$, then by Proposition 4 any closed set which separates the inverse image (under f) of one component of $J^\beta \times T^1$ from the inverse image of the other component has an essential mapping onto J^β and therefore that separating set has dimension (Ind) $\geqslant \beta$. Therefore, by definition, $\mathrm{Ind}(X) \geqslant \beta+1 = \alpha$.

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