3.2. *If M is an* i.s.s. *of topological space X between points a and b, and if H is a clopen subset of $X-M$ such that $a \in H$ but $b \notin H$, then each point p of M either is a limit point of H or is such that $\{p\}$ has a limit point in H.*

4. THEOREM. *Any irreducible separating set M of a topological space X is the union of two disjoint sets (not always non-empty) one of which is open and the other closed.*

Proof. The closed set is $\bar{H} \cap \bar{K}$. For a point p in $M-(\bar{H} \cap \bar{K})$, the lemma guarantees a limit point of $\{p\}$ in H and another in K. Then the intersection W of suitable neighborhoods of each satisfies

$$p \in W \subset M-(\bar{H} \cap \bar{K}).$$

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SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY,
MONTANA STATE UNIVERSITY

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Prime z-ideal structure of $C(\mathbf{R})$

by

Mark Mandelker* (Rochester, N. Y.)

Introduction. In the study of the ideal structure of the ring $C(X)$ of all real-valued continuous functions on a topological space X, a special role is played by the class of z-ideals. A *z-ideal* is an ideal that is maximal with respect to the sets of zeros of its members. Maximal ideals are z-ideals, and every z-ideal is an intersection of prime ideals. These and other basic facts concerning the algebraic structure of $C(X)$ are found in the Gillman and Jerison text *Rings of Continuous Functions* [GJ]. Early results concerning the *prime* ideal structure of $C(X)$ are summarized and extended in [GJ], Chapter 14. For example, a prime ideal is contained in a unique maximal ideal; in fact, the prime ideals containing a given prime ideal form a chain. Later results are found in [GJ_1], [FG], and [K].

Of special interest is the family of *prime z-ideals*. For example, minimal prime ideals are z-ideals, and prime z-ideals have interesting connections with the topology of the space. In the case of a completely regular Hausdorff space X, prime z-ideals in $C(X)$ are related to convergence problems in the Stone–Čech compactification βX.

In this work we consider the real line $\mathbf{R}$ and examine the prime z-ideal structure of $C(\mathbf{R})$. However, some basic results are obtained in Part I for a completely regular Hausdorff space. The main results for the real line are obtained in Part II. The main effort is directed toward the determination of the order types of chains of prime z-ideals in $C(\mathbf{R})$.

We find that the prime z-ideal structure contrasts greatly with the prime ideal structure. For example, although every maximal chain of prime ideals, of cardinal greater than one, has cardinal at least $2^{\aleph_1}$ and contains η_1-sets, we shall demonstrate, assuming the continuum hypothesis, the existence of maximal chains of prime z-ideals in $C(\mathbf{R})$ of all cardinals $\mathfrak{m}$ with $1 \leqslant \mathfrak{m} \leqslant \aleph_0$. In fact, we characterize all countable decreasing well-ordered maximal chains; they exist precisely for all countable ordinals

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not λ or $\lambda+1$, where λ is a limit ordinal (Theorem 13.3). The case $\mathfrak{m}=1$ yields a maximal ideal in $C(\mathbf{R})$ that contains no other prime ideal. In contrast, there is no such maximal ideal in the subring $C^*(\mathbf{R})$ of bounded continuous functions or in the ring $C([0,1])$.

Minimal prime ideals in $C(\mathbf{R})$ are characterized as those for which each member vanishes on some nonempty open set (Theorem 8.1). It is shown that the intersection of a countable chain of nonminimal prime z-ideals is never minimal (Theorem 8.4).

Concerning the lengths of chains of prime z-ideals in $C(\mathbf{R})$, we show that there exist uncountable decreasing well-ordered chains (Theorem 8.5), and there exist countable increasing well-ordered chains for every countable ordinal (Theorem 14.1). The question of uncountable increasing well-ordered chains is open (Question 14.2). Other questions are also raised.

Other results concern the predecessors, in the family of all prime z-ideals, of a given prime z-ideal in $C(\mathbf{R})$. A large class of prime z-ideals is found for which every predecessor is contained in an immediate predecessor (Theorem 10.6). In contrast, there exists a prime z-ideal that has an immediate predecessor but also a predecessor that is contained in no immediate predecessor (Corollary 14.3). A fundamental result is that every nonminimal prime z-ideal has a family of predecessors that is order-isomorphic with the family of *all* prime z-ideals (Theorem 12.8).

The structure of the prime z-ideals in $C(\mathbf{R})$ is the same as the structure of the prime z-filters on $\mathbf{R}$; it is convenient to study these directly, thus all the discussion is in terms of prime z-filters.

The methods used are primarily set-theoretic considerations involving ultrafilters on $\mathbf{R}$, the topology of $\mathbf{R}$, and the Stone–Čech compactification $\beta\mathbf{R}$.

PART I. Completely regular spaces

Many of the basic properties of prime z-ideals in $C(\mathbf{R})$ are also valid for an arbitrary topological space Y. However, since $C(Y)$ is isomorphic with $C(X)$ for some completely regular Hausdorff space X ([GJ], 3.9), it suffices to consider only completely regular Hausdorff spaces.

1. Preliminaries. In all of the following, inclusion is denoted by $\subseteq$ and $\subset$ is reserved for proper inclusion.

Throughout Part I, *X denotes a completely regular Hausdorff space.* In this and the next section, we recall some basic concepts developed in [GJ].

1.1. $C(X)$ will denote the ring of *all* real-valued continuous functions on X. For $f \in C(X)$, the *zero-set* $Z(f)$ of f is the set of points of X at which f vanishes. The family $Z(X)$ of all zero-sets in X is a lattice and forms a base for the closed subsets of X. The zero-set-neighborhoods of a point in X form a base for the neighborhoods of the point.

A *z-filter* on X is a nonempty family of nonempty zero-sets that is closed under finite intersection and supersets. The union and intersection of a chain of z-filters are also z-filters. A *z-ultrafilter* is a maximal z-filter. The z-filter generated by a z-filter $\mathcal{F}$ and a zero-set Z that meets every member of $\mathcal{F}$ is denoted by $(\mathcal{F}, Z)$; it is easily seen that

$$(\mathcal{F}, Z) = \{W \in Z(X) \ \text{ for some } F \in \mathcal{F},\ F \cap Z \subseteq W\}.$$

1.2. A *z-ideal* in $C(X)$ is a proper ideal I that contains a function f in $C(X)$ whenever $Z(f) = Z(g)$ for some g in I. A z-ideal is characterized algebraically as a proper ideal I that contains a function f in $C(X)$ whenever f belongs to the same maximal ideals as some function in I. Minimal prime ideals are z-ideals.

For every ideal I of $C(X)$, the family $Z[I] = \{Z(f)\colon f \in I\}$ is a z-filter on X, and $Z^{\leftarrow}[Z[I]] = \{g \in C(X)\colon Z(g) \in Z[I]\}$ is the smallest z-ideal containing I. The mapping $I \to Z[I]$ is an order-isomorphism from the family of all z-ideals in $C(X)$ onto the family of all z-filters on X. This mapping yields a one-to-one correspondence between the maximal ideals of $C(X)$ and the z-ultrafilters on X.

1.3. Let X be dense in a completely regular Hausdorff space T. A z-filter $\mathcal{F}$ on X *converges* to a point p in T if every neighborhood of p contains a member of $\mathcal{F}$. For every point p in T there is at least one z-ultrafilter on X converging to p. If T is compact, every z-ultrafilter on X converges to a point of T.

The Stone–Čech compactification of X, denoted by βX, is a compact Hausdorff space such that X is dense in βX and distinct z-ultrafilters on X converge to distinct points of βX. These properties characterize βX.

The z-ultrafilter on X converging to a point p in βX is denoted by $\mathcal{M}^p$, and is given by

$$\mathcal{M}^p = \{Z \in Z(X)\colon p \in \mathrm{cl}_{\beta X} Z\}.$$

For each $p \in \beta X$, the family

$$\mathcal{O}^p = \{Z \in Z(X)\colon p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z\}$$

is a z-filter on X. When necessary, we write $\mathcal{M}^p_X$, etc. In the case that p is in X, $\mathcal{M}^p$ consists of all zero-sets containing p and $\mathcal{O}^p$ is the family of all zero-set-neighborhoods of p.

A z-filter is *fixed* if its members have a point in common, otherwise it is *free*. The z-ultrafilter $\mathcal{M}^p$ is fixed if and only if p is in X.

2. Prime z-filters. A z-filter $\mathcal{F}$ is *prime* if whenever the union of two zero-sets is in $\mathcal{F}$, then one of them is in $\mathcal{F}$. Throughout the paper,

$\mathcal{P}, \mathcal{Q}, \mathcal{P}_1, \ldots$ denote prime z-filters. The family of all prime z-filters on X is denoted by $\boldsymbol{P}(X)$.

A prime z-filter is *minimal* if it is a minimal element of $\boldsymbol{P}(X)$. If $\mathcal{P} \subset \mathcal{Q}$, we say that $\mathcal{P}$ is a *predecessor* of $\mathcal{Q}$ and that $\mathcal{Q}$ is a *successor* of $\mathcal{P}$. In the case where $\mathcal{P} \subset \mathcal{Q}$ and there is no prime z-filter between them, we use the terms *immediate predecessor* and *immediate successor*.

2.1. The order-isomorphism $I \to \boldsymbol{Z}[I]$ maps the prime z-ideals of $C(X)$ onto the prime z-filters on X, thus the prime z-ideal structure of $C(X)$ is the same as the structure of $\boldsymbol{P}(X)$. *For each result stated in terms of prime z-filters, the corresponding statement concerning prime z-ideals is also valid.*

Every z-ultrafilter is prime. The union and intersection of a chain of prime z-filters are also prime. A prime z-filter is contained in a unique z-ultrafilter; when $\mathcal{Q} \subseteq \mathcal{M}^p$, $\mathcal{Q}$ is fixed if and only if $p \in X$.

For any z-filter $\mathcal{F}$, the following are equivalent: (a) $\mathcal{F}$ is prime. (b) $\mathcal{F}$ contains a prime z-filter. (c) If the union of two zero-sets is all of X, then one of them is in $\mathcal{F}$. (d) The prime z-filters containing $\mathcal{F}$ form a chain. (e) The z-filters containing $\mathcal{F}$ form a chain.

When X is dense in a completely regular space T and $p \in T$, a prime z-filter $\mathcal{P}$ on X converges to p if and only if $p \in \operatorname{cl}_T Z$ for every $Z \in \mathcal{P}$.

For $\mathcal{Q} \in \boldsymbol{P}(X)$ and $p \in \beta X$, the following are equivalent: (a) $\mathcal{Q}$ converges to p. (b) $\mathcal{Q} \subseteq \mathcal{M}^p$. (c) $\mathcal{O}^p \subseteq \mathcal{Q}$.

Let $p \in \beta X$. The prime z-filters contained in $\mathcal{M}^p$ form a chain if and only if $\mathcal{O}^p$ is prime. The z-ultrafilter $\mathcal{M}^p$ is a minimal prime z-filter if and only if $\mathcal{M}^p = \mathcal{O}^p$.

2.2. It is clear that a prime z-filter $\mathcal{P}$ has an immediate successor if and only if it is not the intersection of its successors. In this case, *the immediate successor is unique* and will be denoted by $\mathcal{P}^+$.

2.3. *Let $\mathcal{P} \subset \mathcal{Q}$. Then $\mathcal{Q} = \mathcal{P}^+$ if and only if $\mathcal{Q} = (\mathcal{P}, F)$ for every $F \in \mathcal{Q} - \mathcal{P}$.*

Proof. The necessity is clear. Now let $\mathcal{Q} = (\mathcal{P}, F)$ whenever $F \in \mathcal{Q} - \mathcal{P}$. If $\mathcal{P} \subset \mathcal{P}' \subseteq \mathcal{Q}$, then for any $F \in \mathcal{P}' - \mathcal{P}$ we have $\mathcal{Q} = (\mathcal{P}, F) \subseteq \mathcal{P}'$. Hence $\mathcal{Q} = \mathcal{P}^+$.

2.4. *$\mathcal{Q}$ has an immediate predecessor if and only if there exist $\mathcal{P} \subset \mathcal{Q}$ and $F \in \mathcal{Q}$ such that $\mathcal{Q} = (\mathcal{P}, F)$.*

Proof. The necessity is clear. Conversely, let $\mathcal{P} \subset \mathcal{Q}$, $F \in \mathcal{Q}$, and $\mathcal{Q} = (\mathcal{P}, F)$. Following [GJ₁], 3.7, let $\mathcal{P}_0$ be the union of the chain of all prime z-filters containing $\mathcal{P}$ but not F; it is clear that $\mathcal{Q} = \mathcal{P}_0^+$.

2.5. *The following are equivalent:* (a) *Every predecessor of $\mathcal{Q}$ is contained in an immediate predecessor.* (b) *$\mathcal{Q}$ is not the union of a chain of predecessors.* (c) *For every predecessor $\mathcal{P}$ of $\mathcal{Q}$ there is an $F \in \mathcal{Q}$ such that $\mathcal{Q} = (\mathcal{P}, F)$.*

Proof. (b) implies (a). If $\mathcal{P}$ is a predecessor of $\mathcal{Q}$, and $\mathcal{P}_0$ is the union of the chain of all predecessors of $\mathcal{Q}$ containing $\mathcal{P}$, then clearly $\mathcal{Q} = \mathcal{P}_0^+$.

(a) implies (c). Let $\mathcal{P} \subset \mathcal{Q}$, and choose $\mathcal{P}_0$ so that $\mathcal{P} \subseteq \mathcal{P}_0$ and $\mathcal{Q} = \mathcal{P}_0^+$. Choose $F \in \mathcal{Q} - \mathcal{P}_0$. Since $\mathcal{P}_0$ and $(\mathcal{P}, F)$ are comparable, but $F \notin \mathcal{P}_0$, we have $\mathcal{P}_0 \subset (\mathcal{P}, F) \subseteq \mathcal{Q}$, and thus $(\mathcal{P}, F) = \mathcal{Q}$.

(c) implies (b). Suppose that $\mathcal{Q}$ is the union of a chain $\boldsymbol{Q}$ of predecessors. Choose any $\mathcal{P}_0 \in \boldsymbol{Q}$ and choose $F \in \mathcal{Q}$ such that $\mathcal{Q} = (\mathcal{P}_0, F)$. Choose $\mathcal{P}_1 \in \boldsymbol{Q}$ such that $F \in \mathcal{P}_1$. Since $F \notin \mathcal{P}_0$, we have $\mathcal{P}_0 \subset \mathcal{P}_1$. Thus $\mathcal{Q} = (\mathcal{P}_0, F) \subseteq \mathcal{P}_1$, contradicting the choice of $\mathcal{P}_1$.

Remark. There is a prime z-filter that satisfies the conditions of 2.4 but not those of 2.5; see Corollary 14.3.

3. Co-ideals.

A proper dual ideal in a lattice L is *prime* if and only if it is the complement of an ideal. Thus the proper prime dual ideals (e.g., the prime z-filters in the lattice $\boldsymbol{Z}(X)$) correspond to the decompositions $L = D \cup I$, where $D \cap I = \emptyset$, D is a dual ideal, and I is an ideal. M. H. Stone has studied these decompositions in the case where L is distributive; in this section his results are applied to $\boldsymbol{Z}(X)$.

The main fact about co-ideals (see definition below) in $\boldsymbol{Z}(X)$ is that every co-ideal contains a prime z-filter. This will be applied in Section 4: if a prime z-filter properly contains a co-ideal, then we know that it has a predecessor. Among other applications of co-ideals is the construction of increasing sequences in Section 14.

3.1. A nonempty subset of a lattice L will be called a *co-ideal* in L if its complement is an ideal. Thus a co-ideal in $\boldsymbol{Z}(X)$ is a nonempty family of nonempty zero-sets that is closed under supersets and that contains at least one of a pair of zero-sets whenever it contains their union.

3.2. *Let $\mathcal{D}$ be a co-ideal in $\boldsymbol{Z}(X)$.* (a) *$\mathcal{D}$ contains a prime z-filter.* (b) *If Z is a member of $\mathcal{D}$, then $\mathcal{D}$ contains a prime z-filter containing Z.* (c) *If $\mathcal{F}$ is a z-filter contained in $\mathcal{D}$, then $\mathcal{D}$ contains a prime z-filter containing $\mathcal{F}$.* (d) *If $\mathcal{F}$ is a z-filter that is maximal among the z-filters contained in $\mathcal{D}$, then $\mathcal{F}$ is prime.* ([St], Theorem 6.)

Proof. We first prove (d). Let $Z_1, Z_2 \in \boldsymbol{Z}(X)$ with $Z_1 \cup Z_2 \in \mathcal{F}$. Suppose $Z_i \notin \mathcal{F}$, $i = 1, 2$. For each i, there is $W_i \in \mathcal{F}$ such that $W_i \cap Z_i \notin \mathcal{D}$, for otherwise $\mathcal{F} \subset (\mathcal{F}, Z_i) \subseteq \mathcal{D}$, contradicting the maximality of $\mathcal{F}$. Define $W = W_1 \cap W_2$. Thus $W \cap Z_i \notin \mathcal{D}$, $i = 1, 2$, and hence $W \cap (Z_1 \cup Z_2) \notin \mathcal{D}$. Thus $W \cap (Z_1 \cup Z_2) \notin \mathcal{F}$ so $W \notin \mathcal{F}$, contradicting the choice of W_1 and W_2. This completes the proof of (d). Now (c) follows from (d) and Hausdorff's maximal principle. For (b), we apply (c) to the z-filter generated by Z. Finally, (a) follows immediately from (b).

4. Nonminimal prime z-filters. The construction of decreasing sequences requires a supply of nonminimal prime z-filters. We give a characterization of these in Corollary 4.4; it will be used in Section 8 to obtain a simpler characterization in the case of the real line.

4.1. For any $E \subseteq X$, we define

$$z(E) = \{Z \in \boldsymbol{Z}(X)\colon\ E \subseteq Z\}\,.$$

This z-filter replaces, for arbitrary X, the single zero-set $\mathrm{cl}\,E$ that we associate with E in the case of the line.

4.2. THEOREM. *Let F be a member of $\mathcal{Q}$, and let Z be any zero-set. Then Z is a member of some predecessor of $\mathcal{Q}$ not containing F if and only if $z(Z-F) \subseteq \mathcal{Q}$.*

Proof. Let $\mathcal{S}$ be a predecessor of $\mathcal{Q}$ not containing F, with $Z \in \mathcal{S}$. If $W \in z(Z-F)$, then $Z \subseteq F \cup W$, so that $F \cup W \in \mathcal{S}$; hence $W \in \mathcal{S}$. Thus $z(Z-F) \subseteq \mathcal{Q}$.

Conversely, let $z(Z-F) \subseteq \mathcal{Q}$. We define

$$\mathcal{E} = \{W \in \boldsymbol{Z}(X)\colon\ z(W-F) \subseteq \mathcal{Q}\}\,.$$

It is clear that $\mathcal{E}$ is a nonempty family of nonempty zero-sets that contains Z but not F. Let W be a member of $\mathcal{E}$ and T a zero-set with $W \subseteq T$. Clearly $z(T-F) \subseteq z(W-F)$ and hence T belongs to $\mathcal{E}$. Now let W_1, W_2 be zero-sets not in $\mathcal{E}$. Choose $T_i \in z(W_i-F)$ with $T_i \notin \mathcal{Q}$, for $i = 1, 2$. Clearly $T_1 \cup T_2 \in z\big((W_1 \cup W_2)-F\big)$ with $T_1 \cup T_2 \notin \mathcal{Q}$ so that $W_1 \cup W_2 \notin \mathcal{E}$. Thus $\mathcal{E}$ is a co-ideal in $\boldsymbol{Z}(X)$ that contains Z but not F. By 3.2(b), $\mathcal{E}$ contains a prime z-filter $\mathcal{S}$ that contains Z but not F. If $W \in \mathcal{E}$, then since $W \in z(W-F)$, we have $W \in \mathcal{Q}$; thus $\mathcal{E} \subseteq \mathcal{Q}$. It follows that $\mathcal{S}$ is a predecessor of $\mathcal{Q}$.

4.3. COROLLARY. *A zero-set Z is a member of some predecessor of $\mathcal{Q}$ if and only if there exists $F \in \mathcal{Q}$ such that $z(Z-F) \subseteq \mathcal{Q}$.*

4.4. COROLLARY. *$\mathcal{Q}$ is nonminimal if and only if there exists $F \in \mathcal{Q}$ such that $z(X-F) \subseteq \mathcal{Q}$.*

4.5. COROLLARY. *If a prime z-filter has a nowhere dense member, it is nonminimal.*

Remark 1. A special case of Corollary 4.5, for a z-ultrafilter or a prime z-filter with an immediate successor, is found in [K], p. 238.

Remark 2. The converse to Corollary 4.5 does not hold generally. An example may be given as follows. Let X be the one-point compactification of an uncountable discrete space S. It is easy to see that $\mathcal{M}^\infty$ consists of the subsets of X that contain ∞ and have countable complements, while $\mathcal{O}^\infty$ consists of those with finite complements. Thus $\mathcal{M}^\infty$ is nonminimal, but every member of $\mathcal{M}^\infty$ meets S, and hence has nonempty interior. (This example is used in [K], p. 238.) However, the converse is true for real line, see Theorem 8.1.

5. Traces and induced z-filters. If $Y \subseteq X$ and $\mathcal{F}$ is a z-filter on Y, it is clear that

$$\mathcal{F}^{\#} = \{Z \in \boldsymbol{Z}(X)\colon\ Z \cap Y \in \mathcal{F}\}$$

is a z-filter on X; it is called the z-filter *induced* on X by $\mathcal{F}$. If $\mathcal{F}$ is prime, then it is easy to see that $\mathcal{F}^{\#}$ is also prime.

If $Y \subseteq X$ and $\mathcal{F}$ is a z-filter on X, then

$$\mathcal{F}\,|\,Y = \{Z \cap Y\colon\ Z \in \mathcal{F}\}$$

is called the *trace* of $\mathcal{F}$ on Y.

We say that a subspace Y of a space T is *z-embedded* in T if for every $Z \in \boldsymbol{Z}(Y)$, there exists $W \in \boldsymbol{Z}(T)$ such that $Z = W \cap Y$. Any C^*-embedded subspace is z-embedded; thus X is z-embedded in βX.

5.1. Let Y be z-embedded in X. If $\mathcal{F}$ is a z-filter on Y, it is easily seen that $\mathcal{F}^{\#}|\,Y = \mathcal{F}$. It follows that $\boldsymbol{P}(Y)$ is order-isomorphic with a subfamily of $\boldsymbol{P}(X)$.

5.2. THEOREM. *If Y is z-embedded in X and $\mathcal{F}$ is a z-filter on X every member of which meets Y, then $\mathcal{F}|Y$ is a z-filter on Y; if $\mathcal{F}$ is prime, so is $\mathcal{F}|Y$.*

Proof. It is easy to verify the first statement. Now let $\mathcal{F}$ be prime and let Z, $W \in \boldsymbol{Z}(Y)$ with $Z \cup W = Y$. Choose S, $T \in \boldsymbol{Z}(X)$ such that $Z = S \cap Y$, $W = T \cap Y$. Clearly $\mathcal{F} \subseteq (\mathcal{F}|Y)^{\#}$; it follows that $(\mathcal{F}|Y)^{\#}$ is prime. Since $(S \cup T) \cap Y = Y$, we have $S \cup T \in (\mathcal{F}|Y)^{\#}$. If, say, $S \in (\mathcal{F}|Y)^{\#}$, then $Z \in \mathcal{F}|Y$. Hence $\mathcal{F}|Y$ is prime.

5.3. *Let $p \in \beta X$.* (a) $\mathcal{O}^p = \mathcal{O}^p_{\beta X}|X$. (b) $\mathcal{O}^p_{\beta X} = (\mathcal{O}^p)^{\#}$. (c) *$\mathcal{O}^p$ is prime if and only if $\mathcal{O}^p_{\beta X}$ is prime.* ([GJ], 7.12(a), 2B.1.)

Proof. (b) follows from [GJ], 7.12(a); (a) from (b) and 5.1; and (c) from Theorem 5.2.

6. Associated ultrafilters. An important tool used in the construction of immediate predecessors in Section 12 is a result of [GJ] relating prime z-filters with ultrafilters, it is restated below in 6.4. An extension that will be needed in sections 13 and 15 is given in Theorem 6.2.

For any ultrafilter $\mathcal{U}$ on X, we define

$$z(\mathcal{U}) = \mathcal{U} \cap \boldsymbol{Z}(X);$$

it is easy to verify that $z(\mathcal{U})$ is a prime z-filter on X. If $\mathcal{Q} = z(\mathcal{U})$, we say that $\mathcal{U}$ is *associated* with $\mathcal{Q}$.

6.1. *If $\mathfrak{D}$ is a co-ideal in 2^X and E is a member of $\mathfrak{D}$, then $\mathfrak{D}$ contains an ultrafilter containing E; and if a filter $\mathcal{F}$ is contained in $\mathfrak{D}$, then $\mathfrak{D}$ contains an ultrafilter that contains $\mathcal{F}$.* (See 3.2).

6.2. THEOREM. *$\mathfrak{Q}$ has an associated ultrafilter containing a given set E if and only if $z(Z \cap E) \subseteq \mathfrak{Q}$, for all $Z \in \mathfrak{Q}$.*

Proof. The necessity is clear. Conversely, let $z(Z \cap E) \subseteq \mathfrak{Q}$, for all $Z \in \mathfrak{Q}$. Define $\mathfrak{D} = \{S \subseteq X: z(S \cap E) \subseteq \mathfrak{Q}\}$. It is easy to verify that $\mathfrak{D}$ is a co-ideal in 2^X, and that the filter $\mathcal{F}$ on X generated by $\mathfrak{Q}$ and E is contained in $\mathfrak{D}$. By 6.1 there is an ultrafilter $\mathcal{U}$ on X with $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathfrak{D}$; it follows that $E \in \mathcal{U}$ and $\mathfrak{Q} = z(\mathcal{U})$.

6.3. COROLLARY. *Let $\mathfrak{Q} = \bigcap_{i \in J} \mathfrak{Q}_i$, where J is any index set. If E_i is a member of some ultrafilter associated with $\mathfrak{Q}_i$, for each i, then $E = \bigcup_{i \in J} E_i$ is a member of some ultrafilter associated with $\mathfrak{Q}$.*

Proof. Let $Z \in \mathfrak{Q}$. For all i, $z(Z \cap E_i) \subseteq \mathfrak{Q}_i$, and thus also $z(Z \cap E) \subseteq \mathfrak{Q}_i$. Hence $z(Z \cap E) \subseteq \mathfrak{Q}$.

6.4. *Every prime z-filter has at least one associated ultrafilter.* ([GJ], 14F.1). (Apply Theorem 6.2 with $E = X$.)

PART II. The real line

Attention is now restricted to the space of major interest. However, each result obtained in Part II for the real line may be extended to metric spaces with various restrictions. In particular, each result is valid in any euclidean space.

7. Preliminaries. For $p \in \mathbf{R}$ and $r > 0$, we use the notation

$$B(p, r) = \{x \in \mathbf{R}: \varrho(p, x) \leqslant r\},$$

$$B_0(p, r) = B(p, r) - \{p\}.$$

The lattice $Z(\mathbf{R})$ is now the lattice of all closed subsets of $\mathbf{R}$. For applications to the line, we may now replace the condition $z(E) \subseteq \mathfrak{Q}$, used in Part I, by the simpler condition $\mathrm{cl}\,E \in \mathfrak{Q}$. For any ultrafilter $\mathcal{U}$ on $\mathbf{R}$, the prime z-filter $z(\mathcal{U})$ is now given by $z(\mathcal{U}) = \{\mathrm{cl}\,E: E \in \mathcal{U}\}$. Every subspace of $\mathbf{R}$ is z-embedded.

The *derived set* of E is denoted by E'. If $Z' \in \mathfrak{Q}$ for every $Z \in \mathfrak{Q}$, we say that $\mathfrak{Q}$ is *closed.*

The cardinal of $\mathbf{R}$ is denoted by $\mathfrak{c}$; thus the cardinal of $\beta\mathbf{R}$ is $2^{\mathfrak{c}}$, there are $2^{\mathfrak{c}}$ prime z-filters on $\mathbf{R}$, and $2^{2^{\mathfrak{c}}}$ ultrafilters. The continuum hypothesis is designated by [CH].

A chain of prime z-filters is said to be *full* if it contains with each pair of its elements, all prime z-filters between them. Thus a chain is *maximal* if it is full and has a z-ultrafilter as greatest element and a minimal prime z-filter as least.

We denote by $\mathbf{P}^*(Y)$ the family of all free prime z-filters on a space Y.

7.1. *A prime z-filter is free if and only if each of its members is unbounded.*

Proof. If p is a point belonging to each member of $\mathfrak{Q}$, then the bounded set $B(p, 1)$ is a member of $\mathfrak{Q}$. Conversely, if $\mathfrak{Q}$ has a bounded member F, then F is compact, and it follows that $\mathfrak{Q}$ is fixed.

7.2. Let $\mathfrak{u}(\mathfrak{Q})$ denote the number of ultrafilters associated with $\mathfrak{Q}$. Clearly $\mathfrak{u}(\mathcal{M}^p) = 1$ when $p \in \mathbf{R}$. On the other hand, since there are $2^{2^{\mathfrak{c}}}$ ultrafilters, but only $2^{\mathfrak{c}}$ prime z-filters, it is easily shown that there is a prime z-filter $\mathfrak{Q}$ with $\mathfrak{u}(\mathfrak{Q}) = 2^{2^{\mathfrak{c}}}$.

7.3. QUESTION. *What other cardinals occur as $\mathfrak{u}(\mathfrak{Q})$?*

7.4. *For any $p \in \mathbf{R}$ and $r > 0$, $\mathbf{P}^*(B_0(p, r))$ is order-isomorphic with $\mathbf{P}^*(\mathbf{R})$.*

Proof. Since $B_0(p, r)$ is homeomorphic with the subspace $Y = \{x \in \mathbf{R}: \varrho(p, x) \geqslant r\}$, it follows that $\mathbf{P}^*(B_0(p, r))$ is order-isomorphic with $\mathbf{P}^*(Y)$. Finally, it is easy to verify that $\mathfrak{I} \to \mathfrak{I}^{\#}$ is an order-isomorphism from $\mathbf{P}^*(Y)$ onto $\mathbf{P}^*(\mathbf{R})$.

7.5. *For any $p \in \mathbf{R}$, the family of predecessors of $\mathcal{M}^p$ is order-isomorphic with $\mathbf{P}^*(\mathbf{R})$.*

Proof. It is easy to verify that $\mathfrak{I} \to \mathfrak{I}|B_0(p, 1)$, where $\mathfrak{I} \subset \mathcal{M}^p$, is an order-isomorphism into $\mathbf{P}^*(B_0(p, 1))$; by 5.1, it is onto. The conclusion now follows from 7.4.

8. Nonminimal prime z-filters. In this section we characterize a *nonminimal* prime z-filter as one that has a nowhere dense member. A fundamental result is that the intersection of a countable chain of nonminimal prime z-filters is also nonminimal. We use this to demonstrate the existence of uncountable decreasing sequences.

8.1. THEOREM. *A prime z-filter is nonminimal if and only if it has a nowhere dense member.*

Proof. Let $\mathfrak{Q}$ be nonminimal. Choose $\mathfrak{I} \subset \mathfrak{Q}$ and $Z \in \mathfrak{Q} - \mathfrak{I}$. Since $\mathbf{R} = Z \cup \mathrm{cl}(\mathbf{R} - Z)$ with $Z \notin \mathfrak{I}$, we have $\mathrm{cl}(\mathbf{R} - Z) \in \mathfrak{I}$. Thus the nowhere dense set $\mathrm{bdry}\,Z = Z \cap \mathrm{cl}(\mathbf{R} - Z)$ is in $\mathfrak{Q}$. The sufficiency is a special case of Corollary 4.5.

8.2. COROLLARY. *Every nonminimal prime z-filter has a nonclosed, nonminimal predecessor.*

Proof. Let $\mathfrak{Q}$ be nonminimal and choose a nowhere dense set $F \in \mathfrak{Q}$. Choose a discrete set D such that $D' = F$ and define $G = \mathrm{cl} D$. It is clear that $\mathrm{cl}(G-F) \in \mathfrak{Q}$. By Theorem 4.2, $\mathfrak{Q}$ has a predecessor $\mathfrak{F}$ containing G but not F. Since $G' = F$, $\mathfrak{F}$ is nonclosed, and since G is nowhere dense, $\mathfrak{F}$ is nonminimal.

Remark. For any nowhere dense $F \in \mathfrak{Q}$, there is a predecessor $\mathfrak{F}$ of $\mathfrak{Q}$ with a member G such that $G' = F$.

8.3. Corollary. *Every nonminimal prime z-filter has noncomparable predecessors.*

Proof. Let $\mathfrak{Q}$ be nonminimal and choose a nowhere dense set $F \in \mathfrak{Q}$. Choose disjoint discrete sets D_1 and D_2 such that $D_i' = F$, for $i = 1, 2$, and define $G_i = \mathrm{cl} D_i$. As in 8.2, $\mathfrak{Q}$ has a predecessor $\mathfrak{F}_i$ containing G_i but not F, for $i = 1, 2$. Since $G_1 \cap G_2 = F$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are noncomparable.

8.4. Theorem. *The intersection of a countable chain of nonminimal prime z-filters is also nonminimal.*

Proof. We index the chain by $\mathbf{N}$: $\{\mathfrak{Q}_n\}_{n\in\mathbf{N}}$, and define $\mathfrak{Q} = \bigcap_{n\in\mathbf{N}} \mathfrak{Q}_n$. For each n, choose a nowhere dense set $F_n \in \mathfrak{Q}_n$. By 7.5, it suffices to consider the case in which $\mathfrak{Q}$ is free. Define $Z_n = \mathbf{R} - (-n, n)$. By 7.1, $Z_n \in \mathfrak{Q}_n$; hence we may assume that $F_n \subseteq Z_n$, for all n. Since the family $\{Z_n\}_{n\in\mathbf{N}}$ is locally finite, $F = \bigcup_{n\in\mathbf{N}} F_n$ is closed and nowhere dense; clearly $F \in \mathfrak{Q}$.

Remark 1. If each element of the chain has a countable member, then so does the intersection of the chain. If each element of the chain contains a subset of a fixed set K, then the intersection also contains a subset of K.

Remark 2. On a different space, there is an example of a decreasing ω-sequence with minimal intersection ([K], Example 2); this shows that Theorem 8.4 does not extend to arbitrary spaces.

8.5. Theorem. *Every nonminimal prime z-filter has a decreasing ω_1-sequence of predecessors.*

Proof. The construction proceeds by transfinite induction, using Corollary 8.2 and Theorem 8.4.

9. Minimal prime z-filters.

In this section we discuss some basic properties of a minimal prime z-filter, its relation to its successors, and its associated ultrafilters.

9.1. Theorem. *Let $\mathfrak{Q}$ be minimal. Then* (i) *$\mathfrak{Q}$ is closed;* (ii) *for each $Z \in \mathfrak{Q}$, we have* $\mathrm{cl}(\mathrm{int} Z) \in \mathfrak{Q}$; *and* (iii) *$\mathfrak{Q}$ is generated by its regular members.*

Proof. Let $Z \in \mathfrak{Q}$. We have $Z = \mathrm{bdry} Z \cup \mathrm{cl}(\mathrm{int} Z)$, with $\mathrm{bdry} Z$ nowhere dense; hence $\mathrm{cl}(\mathrm{int} Z) \in \mathfrak{Q}$. Since $\mathrm{cl}(\mathrm{int} Z) \subseteq Z'$, we have $Z' \in \mathfrak{Q}$. Finally, $\mathrm{cl}(\mathrm{int} Z)$ is regular and is contained in Z.

9.2. Theorem. *There is a minimal prime z-filter that is the intersection of its successors. Under* [CH], *it is the intersection of a decreasing ω_1-sequence of successors.*

Proof. By transfinite induction, we define a decreasing sequence $(\mathfrak{Q}_\alpha)_\alpha$ of nonminimal prime z-filters, using Corollary 8.2 and taking intersections at limit ordinals. The process must terminate with a minimal $\mathfrak{Q}_\lambda$, with λ a limit ordinal. It is clear that $\mathfrak{Q}_\lambda$ is the intersection of its successors. By Theorem 8.4, $\lambda \geqslant \omega_1$. It is easy to see that $\mathrm{card}\,\lambda \leqslant \mathrm{card}\,\mathbf{Z}(\mathbf{R}) = \mathfrak{c}$. Under [CH], $\mathrm{card}\,\lambda = \aleph_1$, so that by Theorem 8.4, λ is the limit of an increasing ω_1-sequence of lesser ordinals. The last statement now follows.

Remark 1. In Theorem 12.6 we construct, under [CH], a minimal prime z-filter that is not the intersection of its successors.

Remark 2. Theorem 13.2 will show that the intersection of a decreasing ω_1-sequence of prime z-filters is not always minimal.

9.3. *If $\mathfrak{Q}$ is minimal and E is any dense set, then $\mathfrak{Q}$ has an associated ultrafilter containing E; hence $\mathfrak{Q}$ has at least $\mathfrak{c}$ associated ultrafilters, each with a countable member.*

Proof. Let $Z \in \mathfrak{Q}$. We have $\mathrm{cl}(\mathrm{int} Z) \subseteq \mathrm{cl}(Z \cap E)$. Thus, by Theorem 9.1, $\mathrm{cl}(Z \cap E) \in \mathfrak{Q}$. By Theorem 6.2, $\mathfrak{Q}$ has an associated ultrafilter containing E. Since we can find a family of $\mathfrak{c}$ disjoint countable dense sets in $\mathbf{R}$, we obtain $\mathfrak{c}$ associated ultrafilters, each with a countable member.

9.4. [CH]. *There is a minimal prime z-filter with an associated ultrafilter that has no countable member.*

Proof. Under [CH], there exists an uncountable set E such that $E \cap T$ is countable for every nowhere dense set T (see [Si], p. 36). Let $\mathfrak{C}$ denote the filter of all sets with countable complement. Since E meets every member of $\mathfrak{C}$, we may extend $\mathfrak{C}$ to an ultrafilter $\mathfrak{U}$ that contains E; clearly $\mathfrak{U}$ has no countable member. We define $\mathfrak{Q} = \mathfrak{z}(\mathfrak{U})$; it follows that $\mathfrak{Q}$ has no nowhere dense member.

Remark. We have used only the condition C_1 of [Si], the existence of *Lusin's set*, which follows from [CH].

10. Nonclosed prime z-filters.

The main result concerning a nonclosed prime z-filter is that every predecessor is contained in an immediate predecessor (Theorem 10.6). This generalizes a property of $\mathcal{M}^p$, when p is in $\mathbf{R}$; in that case, no predecessor contains the set $\{p\}$, and the when maximal principle may be applied to the family of predecessors. For an arbitrary nonclosed prime z-filter, we use Theorem 10.3 to find a member that plays the role of $\{p\}$.

It follows from Theorem 9.1 that every nonclosed prime z-filter is nonminimal.

10.1. THEOREM. *A prime z-filter is nonclosed if and only if some associated ultrafilter has a discrete member. In this case, it has only one associated ultrafilter.*

Proof. If $\mathcal{Q}$ is nonclosed, we choose $Z \in \mathcal{Q}$ with $Z' \notin \mathcal{Q}$; it follows that the discrete set $Z-Z'$ is a member of any associated ultrafilter.

If $\mathcal{Q}$ has an associated ultrafilter $\mathcal{V}$ with a discrete member D, we have $\mathrm{cl}D \in \mathcal{Q}$ but $(\mathrm{cl}D)' \cap D = \emptyset$ so that $(\mathrm{cl}D)' \notin \mathcal{Q}$; hence $\mathcal{Q}$ is nonclosed.

Finally, let $\mathcal{Q}$ have an associated ultrafilter $\mathcal{V}$ with a discrete member D. Suppose that $\mathcal{U}$ is another ultrafilter associated with $\mathcal{Q}$. We may assume that $D \notin \mathcal{U}$. Since $\mathrm{cl}D = D \cup D'$ and $\mathrm{cl}D \in \mathcal{Q}$, we have $D' \in \mathcal{U}$ and thus $D' \in \mathcal{V}$, contradicting the fact that $D \cap D' = \emptyset$. Hence $\mathcal{V}$ is the only ultrafilter associated with $\mathcal{Q}$.

10.2. QUESTION. *If a prime z-filter has a unique associated ultrafilter, is it necessarily nonclosed?*

10.3. THEOREM. *If Z is a member of $\mathcal{Q}$, then Z' is a member of every successor of $\mathcal{Q}$.*

Proof. Let $\mathcal{Q} \subset \mathcal{F}$ and suppose $Z' \notin \mathcal{F}$. Choose an ultrafilter $\mathcal{V}$ associated with $\mathcal{F}$ and define $D = Z - Z'$; thus $D \in \mathcal{V}$. Choose $W \in \mathcal{F} - \mathcal{Q}$ and define $E = W \cap D$; thus $E \in \mathcal{V}$ and $\mathrm{cl}E \notin \mathcal{Q}$. Since D is discrete, $E \cap \mathrm{cl}(D-E) = \emptyset$; hence $\mathrm{cl}(D-E) \notin \mathcal{Q}$. Also $Z' \notin \mathcal{Q}$ and $Z = Z' \cup \mathrm{cl}E \cup \mathrm{cl}(D-E)$. Hence $Z \notin \mathcal{Q}$, contradicting our hypothesis.

10.4. COROLLARY. *If a prime z-filter is either* (a) *the intersection of its successors, or* (b) *the union of its predecessors, then it is closed.*

Remark 1. The converse to (a) is false, even for a nonminimal prime z-filter; see Remark 2 following Theorem 15.6. The question of the converse to (b) is open, see Section 15.

Remark 2. If $(\mathcal{Q}_\beta)_{\beta<\alpha}$ is any full decreasing or increasing α-sequence, then it is easily seen that $\mathcal{Q}_\lambda$ is closed for each limit ordinal λ.

10.5. PROBLEM. *Give a necessary and sufficient condition, in terms of its members, that a prime z-filter have an immediate successor.*

10.6. THEOREM. *Let $\mathcal{Q}$ be nonclosed.*

(a) *Every predecessor of $\mathcal{Q}$ is contained in an immediate predecessor.*

(b) *$\mathcal{Q}$ has a nonclosed immediate predecessor.*

(c) *$\mathcal{Q}$ has an immediate successor.*

Proof. Choose $Z \in \mathcal{Q}$ with $Z' \notin \mathcal{Q}$. By Theorem 10.3, Z belongs to no predecessor of $\mathcal{Q}$. Thus (a) follows by Hausdorff's maximal principle. It is clear that the nowhere dense set $F = \mathrm{cl}(Z-Z')$ also belongs to $\mathcal{Q}$ but to no predecessor of $\mathcal{Q}$. By the remark following Corollary 8.2, $\mathcal{Q}$ has a predecessor $\mathcal{F}$ with a member G such that $G' = F$. By (a), $\mathcal{Q}$ has an immediate predecessor $\mathcal{F}_0$ containing $\mathcal{F}$. Since $\mathcal{F}_0$ contains G but not F, it is nonclosed; thus we have (b). Finally, it follows from Theorem 10.3 that $(\mathcal{Q}, Z')$ is the immediate successor of $\mathcal{Q}$.

Remark 1. Property (a) does not hold for all prime z-filters, see Corollary 14.3.

Remark 2. (b) may be iterated to obtain a *full* decreasing ω-sequence. By Corollary 10.4, the intersection of this sequence is closed, and thus (b) cannot be used to obtain a longer full sequence. A construction of longer full sequences will be given in Section 13.

Remark 3. $\mathcal{Q}^+ = (\mathcal{Q}, Z')$ for any $Z \in \mathcal{Q}$ with $Z' \notin \mathcal{Q}$.

10.7. QUESTION. *Is the converse of Theorem* 10.6(a) *true?* Equivalently, *is every closed nonminimal prime z-filter the union of a chain of predecessors?*

11. Remote points. In this section we give several characterizations of a *remote point* of $\beta\mathbf{R}$, i.e. a point not in the closure of any discrete subset of $\mathbf{R}$. Fine and Gillman have shown that the existence of such points follows from the continuum hypothesis. Remote points will be used to demonstrate the existence of minimal immediate predecessors in Section 12, and thus they are used for the construction of countable maximal chains in Section 13.

11.1. *A point p of $\beta\mathbf{R}$ is a remote point if and only if $\mathcal{M}^p$ has no nowhere dense member. Under* [CH], *there exist remote points in $\beta\mathbf{R}$.* [FG].

11.2. THEOREM. *For $p \in \beta\mathbf{R}$, the following are equivalent.*

(a) *p is a remote point.*

(b) *$\mathcal{M}^p$ is a minimal prime z-filter.*

(c) *$\mathcal{M}^p = \mathcal{O}^p$.*

(d) *$\mathcal{O}^p$ is prime.*

(e) *$\mathcal{O}^p_{\beta\mathbf{R}}$ is prime.*

Proof. From the characterization of minimal prime z-filters obtained in Theorem 8.1 and from 11.1 above, it follows that (a) and (b) are equivalent. The equivalence of (b) and (c) was noted in 2.1. Clearly (c) implies (d). If $\mathcal{M}^p$ is not minimal, then by Corollary 8.3, $\mathcal{M}^p$ has noncomparable predecessors, so that by 2.1, $\mathcal{O}^p$ is not prime; thus (d) implies (b). Finally, (d) and (e) are equivalent by 5.3.

Remark. The fact that (c) implies (a) is due to Donald Plank [P], 5.3.

11.3. COROLLARY. [CH]. *There exists a z-ultrafilter on* $\mathbf{R}$ *that contains no other prime z-filter.*

Remark. Hence *there exists a maximal ideal in $C(\mathbf{R})$ that contains no other prime ideal.* In contrast, it is easily shown that there is no such maximal ideal in the subring $C^*(\mathbf{R})$ of bounded continuous functions or in the ring $C([0,1])$.

11.4. PROBLEM. *When p is a remote point, determine the order type of the chain of prime z-filters on $\beta\mathbf{R}$ contained in $\mathcal{M}^p_{\beta\mathbf{R}}$.*

12. Construction of predecessors. In Theorem 12.1 we give a fundamental construction that yields immediate predecessors for a large class of prime z-filters. The construction will be used in Section 13 to construct full decreasing ω_1-sequences. Theorem 12.6 provides minimal immediate predecessors; these will be used in Section 13 to construct countable maximal chains. In Theorem 12.8, we shall construct, for any given nonminimal prime z-filter, a family of predecessors order-isomorphic with the family of *all* prime z-filters.

If $\{t_x\}_{x\in E}$ is any family of real numbers, indexed by a countable set E, then by $t_x\to 0$ we mean that for any $\varepsilon>0$, we have $|t_x|<\varepsilon$ for all but finitely many x in E.

Let T be any countable subset of $\mathbf{R}$. For each $x\in T$, let H_x be a subset of $\mathbf{R}$ with $\operatorname{cl}H_x = H_x\cup\{x\}$, and let $\operatorname{diam}H_x\to 0$. If $H=\bigcup_{x\in T}H_x$, then $\operatorname{cl}H = H\cup\operatorname{cl}T$.

If, in addition, H_x is discrete and $H_x\cap\operatorname{cl}T=\emptyset$, for each $x\in T$, then H is discrete. (The proofs are straightforward.)

12.1. THEOREM. *Let $\mathcal{Q}$ be a prime z-filter and $\mathcal{U}$ an associated ultrafilter. Let $E\in\mathcal{U}$ and define $F=\operatorname{cl}E$. For each $x\in E$, let S_x be any nonempty subset of $\mathbf{R}$ such that $S_x\cap F=\emptyset$, and suppose that $\mathcal{P}_x$ is a prime z-filter on S_x, converging to x. For each $Z\in\mathbf{Z}(\mathbf{R})$, define*

$$Z^*=\{x\in E:\ Z\cap S_x\in\mathcal{P}_x\}\,.$$

Then

$$\mathcal{P}=\{Z\in\mathbf{Z}(\mathbf{R}):\ Z^*\in\mathcal{U}\}$$

is a prime z-filter, $\mathcal{P}$ is a predecessor of $\mathcal{Q}$, and $F\notin\mathcal{P}$.

If, in addition, E is countable, $\operatorname{diam}S_x\to 0$, and for each $x\in E$, $\operatorname{cl}S_x=S_x\cup\{x\}$ and $\mathcal{P}_x$ is a z-ultrafilter on S_x, then $\mathcal{P}$ is an immediate predecessor of $\mathcal{Q}$.

Proof. For any $Z, W\in\mathbf{Z}(\mathbf{R})$, it is clear that

(a) $(Z\cap W)^* = Z^*\cap W^*$,

(b) $(Z\cup W)^* = Z^*\cup W^*$,

(c) $Z\subseteq W$ implies $Z^*\subseteq W^*$.

(d) $Z^*\subseteq Z$.

It is now easily seen that $\mathcal{P}$ is a prime z-filter and is a predecessor of $\mathcal{Q}$ not containing F.

Now assume that the additional conditions are satisfied and let $\mathcal{P}\subset\mathcal{P}'\subseteq\mathcal{Q}$. Choose $W\in\mathcal{P}'-\mathcal{P}$; since $W^*\notin\mathcal{U}$, we have $E-W^*\in\mathcal{U}$. Define $V=E-W^*$. For each $x\in V$ we have $W\cap S_x\notin\mathcal{P}_x$ so we can choose $H_x\in\mathcal{P}_x$ with $W\cap H_x=\emptyset$. Let $Z\in\mathcal{Q}$ and define $T=Z\cap V$. Define $H=\bigcup_{x\in T}H_x$; thus $W\cap H=\emptyset$ and $\operatorname{cl}H=H\cup\operatorname{cl}T$. Also $T\subseteq(\operatorname{cl}H)^*$ with $T\in\mathcal{U}$, so that $\operatorname{cl}H\in\mathcal{P}$. Thus $\mathcal{P}'$ contains $W\cap\operatorname{cl}H$, with $W\cap\operatorname{cl}H\subseteq Z$; so that $Z\in\mathcal{P}'$. Hence $\mathcal{P}'=\mathcal{Q}$. It follows that $\mathcal{P}$ is an immediate predecessor of $\mathcal{Q}$.

12.2. THEOREM. *If $\mathcal{Q}$ is nonminimal and has an associated ultrafilter with a countable member, then $\mathcal{Q}$ has a nonclosed immediate predecessor $\mathcal{P}$.*

Proof. Choose an associated ultrafilter $\mathcal{U}$ with a countable member E such that $F=\operatorname{cl}E$ is nowhere dense. For all $x\in E$, define S_x as the set of terms of a sequence of points of $\mathbf{R}-F$, converging to x, chosen in such a way that $\operatorname{diam}S_x\to 0$. Define $S=\bigcup_{x\in E}S_x$; thus $\operatorname{cl}S=S\cup F$ and S is discrete. For each $x\in E$, choose a free ultrafilter $\mathcal{P}_x$ on S_x; by the choice of S_x, $\mathcal{P}_x$ converges to x. Let $\mathcal{P}$ be the immediate predecessor of $\mathcal{Q}$ obtained by Theorem 12.1. Since $(\operatorname{cl}S)^*=E$, we have $\operatorname{cl}S\in\mathcal{P}$ with $(\operatorname{cl}S)'=F$. Thus $\mathcal{P}$ is nonclosed.

12.3. QUESTION. *Does every nonminimal prime z-filter have an immediate predecessor?*

Remark. Not every nonminimal prime z-filter satisfies the hypothesis of Theorem 12.2. To obtain an example we apply a method due to W. F. Eberlein; see [FG], 1.3. Let $\mathcal{M}$ be a z-ultrafilter containing the z-filter $\mathcal{F}$ of all closed sets with complements of finite Lebesgue measure. Since $\mathcal{F}$ has a nowhere dense member, $\mathcal{M}$ is nonminimal. If E is countable, then there exists $Z\in\mathcal{F}$ such that $E\cap Z=\emptyset$. Hence no ultrafilter associated with $\mathcal{M}$ has a countable member.

12.4. The condition on a nonminimal prime z-filter $\mathcal{Q}$ used in Theorem 12.2, that "$\mathcal{Q}$ has an associated ultrafilter with a countable member", is actually weaker than the condition "$\mathcal{Q}$ has a countable member". An example may be given as follows. Choose any uncountable nowhere dense closed set F, and choose any countable dense subset E of F. It is easy to verify that $\mathcal{D}=\{S\subseteq\mathbf{R}:\ \operatorname{cl}S$ is uncountable$\}$ is a co-deal in $2^{\mathbf{R}}$ with $E\in\mathcal{D}$. By 6.1, $\mathcal{D}$ contains an ultrafilter $\mathcal{U}$ that contains E. Define $\mathcal{Q}=z(\mathcal{U})$; since $\mathcal{Q}$ has the nowhere dense member F, it is nonminimal. It is clear that $\mathcal{Q}$ has no countable member, but $\mathcal{Q}$ has the associated ultrafilter $\mathcal{U}$ with the countable member E.

12.5. QUESTION. *Is there a nonminimal z-ultrafilter with no countable member but with an associated ultrafilter with a countable member?*

12.6. THEOREM. [CH]. *If $\mathcal{Q}$ is nonminimal and has an associated ultrafilter with a countable member, then it has a minimal immediate predecessor.*

Proof. Choose an associated ultrafilter $\mathcal{V}$ with a countable member E such that $F = \mathrm{cl}\,E$ is nowhere dense. For each $x \in E$, choose a z-ultrafilter $\mathcal{Q}_x$ on $\mathbf{R}-\{x\}$, converging to x, that has no nowhere dense member. (It suffices to show that this may be done in the case that $x = 0$. By 11.1, choose a free z-ultrafilter $\mathcal{M}$ on $\mathbf{R}$ with no nowhere dense member, and define $\mathcal{Q}_0$ as the image of $\mathcal{M}|(\mathbf{R}-\{0\})$ under the mapping $t \to 1/t$.)

Since F is nowhere dense, we have $F \cap (\mathbf{R}-\{x\}) \notin \mathcal{Q}_x$, and hence we can choose $S_x \in \mathcal{Q}_x$ such that $S_x \cap F = \emptyset$, for each $x \in E$, in such a way that $\mathrm{diam}\,S_x \to 0$. Clearly $\mathrm{cl}\,S_x = S_x \cup \{x\}$ for all $x \in E$. We define $\mathcal{F}_x = \mathcal{Q}_x|S_x$; clearly $\mathcal{F}_x$ is a z-ultrafilter on S_x, converging to x, for each $x \in E$. Thus Theorem 12.1 yields an immediate predecessor $\mathcal{F}$ of $\mathcal{Q}$.

If $Z \in \mathcal{F}$, then $Z \cap S_x \in \mathcal{F}_x$, for some $x \in E$, and hence $Z \cap S_x \in \mathcal{Q}_x$; thus $Z \cap S_x$ has nonempty interior in $\mathbf{R}-\{x\}$ and hence Z has nonempty interior in $\mathbf{R}$. It follows that $\mathcal{F}$ is minimal.

12.7. THEOREM. *Every nonminimal prime z-filter $\mathcal{Q}$ has $2^{\mathfrak{c}}$ noncomparable nonclosed predecessors. If $\mathcal{Q}$ has an associated ultrafilter with a countable member, they may be chosen to be immediate predecessors.*

Proof. By Corollary 8.2, $\mathcal{Q}$ has a nonclosed, nonminimal predecessor. By Theorem 10.1, this predecessor has an associated ultrafilter with a countable member, thus it suffices to prove the second statement. Choose an associated ultrafilter $\mathcal{V}$ with a countable member E such that $F = \mathrm{cl}\,E$ is nowhere dense. For all $x \in E$, define S_x to be the set of terms of some sequence of points of $\mathbf{R}-F$, converging to x, chosen in such a way that $\mathrm{diam}\,S_x \to 0$. Define $S = \bigcup_{x \in E} S_x$; thus $\mathrm{cl}\,S = S \cup F$ and S is discrete. It is clear that every free ultrafilter on S_x converges to x.

Choose a fixed family of correspondences $\mathbf{N} \to S_x$, $x \in E$. For any free ultrafilter $\mathcal{V}$ on $\mathbf{N}$, let $\mathcal{V}_x$ be the corresponding ultrafilter on S_x, for each $x \in E$, and let $\mathcal{F}_{\mathcal{V}}$ be the immediate predecessor of $\mathcal{Q}$ obtained by Theorem 12.1, using the family $\{\mathcal{V}_x\}_{x \in E}$. Since $(\mathrm{cl}\,S)^* = E$, we have $\mathrm{cl}\,S \in \mathcal{F}_{\mathcal{V}}$, but $(\mathrm{cl}\,S)' = F \notin \mathcal{F}_{\mathcal{V}}$. Thus $\mathcal{F}_{\mathcal{V}}$ is nonclosed.

Let $\mathcal{U}$ and $\mathcal{V}$ be distinct free ultrafilters on $\mathbf{N}$. Thus $\mathcal{U}_x \neq \mathcal{V}_x$, for all $x \in E$. By the choice of the sets S_x, the set $S_x \cap S_y$ is finite whenever $y \neq x$. Also, each $\mathcal{U}_x$ and $\mathcal{V}_x$ contains each subset of S_x with finite complement in S_x. Thus we can choose, by induction in E, sets $U_x \in \mathcal{U}_x$ and $V_x \in \mathcal{V}_x$, for all $x \in E$, such that the sets $U = \bigcup_{x \in E} U_x$ and $V = \bigcup_{x \in E} V_x$ are disjoint. (Suppose that U_x and V_x have been chosen for all x in a finite set $J \subset E$, and y is in E but not in J. Since $T = S_y \cap \bigcup_{x \in J} S_x$ is finite, $S_y - T$ belongs to both $\mathcal{U}_y$ and $\mathcal{V}_y$. We now choose any $K \in \mathcal{U}_y$ and $L \in \mathcal{V}_y$ with $K \cap L = \emptyset$. Finally, we define $U_y = K - T$ and $V_y = L - T$.)

Define $G = \mathrm{cl}\,U$ and $H = \mathrm{cl}\,V$; hence $G \in \mathcal{F}_{\mathcal{U}}$ and $H \in \mathcal{F}_{\mathcal{V}}$. Since $G \cap H = F$, it follows that $\mathcal{F}_{\mathcal{U}}$ and $\mathcal{F}_{\mathcal{V}}$ are distinct and noncomparable.

Since there are $2^{\mathfrak{c}}$ free ultrafilters on $\mathbf{N}$, we have $2^{\mathfrak{c}}$ noncomparable nonclosed immediate predecessors of $\mathcal{Q}$.

12.8. THEOREM. *Every nonminimal prime z-filter has a family of predecessors that is order-isomorphic with the family of all prime z-filters.* (Conjectured by L. Gillman.)

Proof. We first choose, by Theorem 12.7, a family of $2^{\mathfrak{c}}$ noncomparable nonclosed predecessors of the given prime z-filter $\mathcal{Q}$, and index them by $\beta\mathbf{R}$: $\{\mathcal{Q}_p\}_{p \in \beta\mathbf{R}}$. For each $p \in \beta\mathbf{R}$, let $\boldsymbol{P}^p$ denote the family of prime z-filters contained in $\mathcal{M}^p$, thus $\boldsymbol{P}(\mathbf{R}) = \bigcup_{p \in \beta\mathbf{R}} \boldsymbol{P}^p$. Suppose that for each $p \in \beta\mathbf{R}$ we have found a family $\boldsymbol{Q}^p$ of prime z-filters contained in $\mathcal{Q}_p$ that is order-isomorphic with $\boldsymbol{P}^p$. Elements of distinct $\boldsymbol{P}^p$, as well as elements of distinct $\boldsymbol{Q}^p$, are noncomparable; thus $\bigcup_{p \in \beta\mathbf{R}} \boldsymbol{Q}^p$ is a family of predecessors of $\mathcal{Q}$ that is order-isomorphic with $\boldsymbol{P}(\mathbf{R})$. Hence it suffices to find the families $\boldsymbol{Q}^p$.

Suppose that we have found $\boldsymbol{Q}^p$ for all $p \in \beta\mathbf{R}-\mathbf{R}$. It follows that any nonminimal prime z-filter has a family of predecessors that is order-isomorphic with $\boldsymbol{P}^*(\mathbf{R})$. To obtain $\boldsymbol{Q}^p$ for $p \in \mathbf{R}$, we apply this last statement to $\mathcal{Q}_p$ and obtain a family $\boldsymbol{Q}^{*p}$ of predecessors of $\mathcal{Q}_p$ that is order-isomorphic with $\boldsymbol{P}^*(\mathbf{R})$. It follows from 7.5 that it suffices to define $\boldsymbol{Q}^p = \boldsymbol{Q}^{*p} \cup \{\mathcal{Q}_p\}$. Thus if we can find the families $\boldsymbol{Q}^p$ for $p \in \beta\mathbf{R}-\mathbf{R}$, we can also find them for $p \in \mathbf{R}$.

Hence it suffices to fix $p \in \beta\mathbf{R}-\mathbf{R}$ and find a family $\boldsymbol{Q}^p$ of predecessors of $\mathcal{Q}_p$ that is order-isomorphic with $\boldsymbol{P}^p$.

Choose a discrete member E of the ultrafilter $\mathcal{V}$ associated with $\mathcal{Q}_p$. Define $F = \mathrm{cl}\,E$, and choose a disjoint family $\{S_x\}_{x \in E}$ of sets of the form $S_x = B_0(x, r_x)$, such that $S_x \cap F = \emptyset$, in such a way that $\mathrm{diam}\,S_x \to 0$. For each $x \in E$, choose by 7.4 a family $\boldsymbol{Q}_x = \{\mathcal{F}_x \colon \mathcal{F} \in \boldsymbol{P}^p\}$ of prime z-filters on S_x that is order-isomorphic with $\boldsymbol{P}^p$. For each $\mathcal{F} \in \boldsymbol{P}^p$, let $\mathcal{F}'$ be the predecessor of $\mathcal{Q}_p$ obtained by Theorem 12.1, using the family $\{\mathcal{F}_x \colon x \in E\}$. Thus

$$\mathcal{F}' = \{Z \in \boldsymbol{Z}(\mathbf{R}) \colon \{x \in E \colon Z \cap S_x \in \mathcal{F}_x\} \in \mathcal{V}\}.$$

Let $\mathcal{F}, \mathcal{R} \in \boldsymbol{P}^p$. If $\mathcal{F} \subseteq \mathcal{R}$, then $\mathcal{F}_x \subseteq \mathcal{R}_x$, for all $x \in E$, and thus $\mathcal{F}' \subseteq \mathcal{R}'$. Now let $\mathcal{F} \not\subseteq \mathcal{R}$. Then $\mathcal{F}_x \not\subseteq \mathcal{R}_x$, for each $x \in E$, and we can choose $H_x \in \mathcal{F}_x - \mathcal{R}_x$. Define $H = \bigcup_{x \in E} H_x$ and $Z = \mathrm{cl}\,H$; thus $Z = H \cup F$. Hence $Z \cap S_x = H_x$, for all $x \in E$, so that $\{x \in E \colon Z \cap S_x \in \mathcal{F}_x\} = E$ and $\{x \in E \colon Z \cap S_x \in \mathcal{R}_x\} = \emptyset$. Thus $Z \in \mathcal{F}' - \mathcal{R}'$ and we have $\mathcal{F}' \not\subseteq \mathcal{R}'$. Hence $\mathcal{F} \to \mathcal{F}'$ is an order-isomorphism and $\boldsymbol{Q}^p = \{\mathcal{F}' \colon \mathcal{F} \in \boldsymbol{P}^p\}$ is the required family.

12.9. COROLLARY. *If there is a chain of prime z-filters of order type τ, with nonminimal intersection, there is also a chain of type $\tau \cdot \omega_1^*$.*

(For any order type σ, the order type obtained from σ by reversing the ordering is denoted by σ^*.)

13. Maximal chains. We now iterate the construction of immediate predecessors to obtain full sequences, and in Theorem 13.3 we apply the construction of minimal immediate predecessors to obtain countable maximal chains.

13.1. THEOREM. *If $\mathfrak{Q}$ has an associated ultrafilter with a countable member, there is a full decreasing ω_1-sequence, beginning with $\mathfrak{Q}$, in which each term also has an associated ultrafilter with a countable member.*

Proof. We proceed by transfinite induction. Define $\mathfrak{Q}_0 = \mathfrak{Q}$. Let $\alpha < \omega_1$ and suppose that we have a full decreasing α-sequence $(\mathfrak{Q}_\beta)_{\beta<\alpha}$ where each $\mathfrak{Q}_\beta$ is nonminimal and has an associated ultrafilter with a countable member. If α is not a limit ordinal, then by Theorem 12.2, $\mathfrak{Q}_{\alpha-1}$ has a nonminimal immediate predecessor $\mathfrak{Q}_\alpha$ that has an associated ultrafilter with a countable member. If α is a limit ordinal, define $\mathfrak{Q}_\alpha = \bigcap_{\beta<\alpha} \mathfrak{Q}_\beta$. By Theorem 8.4, $\mathfrak{Q}_\alpha$ is nonminimal and by Corollary 6.3, $\mathfrak{Q}_\alpha$ has an associated ultrafilter with a countable member. This completes the induction.

13.2. THEOREM. *If $\mathfrak{Q}$ has a countable member, there is a full decreasing ω_1-sequence, beginning with $\mathfrak{Q}$, and with nonminimal intersection.*

Proof. Define $\mathfrak{Q}_0 = \mathfrak{Q}$, choose a countable member F_0 of $\mathfrak{Q}_0$, and choose a perfect nowhere dense set K with $F_0 \subseteq K$. Proceeding by transfinite induction, let $\alpha < \omega_1$ and suppose we have a full decreasing α-sequence $(\mathfrak{Q}_\beta)_{\beta<\alpha}$, where each $\mathfrak{Q}_\beta$ has a countable member F_β with $F_\beta \subseteq K$.

First consider the case that α is not a limit ordinal. Since $F_{\alpha-1}$ is a nowhere dense subset of K, we can choose discrete sets $S_x \subseteq K$, for all $x \in F_{\alpha-1}$, such that $S_x \cap F_{\alpha-1} = \emptyset$ and $\mathrm{cl} S_x = S_x \cup \{x\}$, in such a way that $\operatorname{diam} S_x \to 0$. Define $E = F_{\alpha-1}$. For each $x \in E$, choose an ultrafilter $\mathfrak{F}_x$ on S_x that converges to x, and let $\mathfrak{Q}_\alpha$ be the immediate predecessor of $Q_{\alpha-1}$ obtained by Theorem 12.1. Define $H = \bigcup_{x\in E} S_x$ and $F_\alpha = \mathrm{cl} H$; thus $F_\alpha \in \mathfrak{Q}_\alpha$, F_α is countable, and $F_\alpha \subseteq K$.

In the case where α is a limit ordinal, define $\mathfrak{Q}_\alpha = \bigcap_{\beta<\alpha} \mathfrak{Q}_\beta$. It follows from Remark 1 following Theorem 8.4 that $\mathfrak{Q}_\alpha$ has a countable member F_α with $F_\alpha \subseteq K$.

This completes the induction. Clearly $K \in \mathfrak{Q}_\alpha$, for all $\alpha < \omega_1$; hence $\bigcap_{\alpha<\omega_1} \mathfrak{Q}_\alpha$ contains K and is nonminimal.

Remark 1. For limit ordinals λ, $(\mathfrak{Q}_\lambda)_\lambda$ is a decreasing ω_1-sequence of closed prime z-filters, each with a countable member, and each the intersection of its successors.

Remark 2. By Corollary 12.9, we obtain a decreasing ω_1^2-sequence of prime z-filters.

13.3. THEOREM. [CH]. *For $0 < \alpha < \omega_1$, there is a decreasing well-ordered maximal chain of prime z-filters of type α^* if and only if α is not λ or $\lambda+1$ for a limit ordinal λ.*

Proof. Since a maximal chain has a least element, it cannot be of type λ^* for a limit ordinal λ; by Theorem 8.4, the least element, since it is a minimal prime z-filter, is not the intersection of countably many successors, thus the chain cannot be of type $(\lambda+1)^*$.

Conversely, we first note that Corollary 11.3 yields a maximal chain of type 1^*. Now choose any point p on the line, and let $(\mathfrak{Q}_\beta)_{\beta<\omega_1}$ be the full chain constructed in Theorem 13.1, starting with $\mathcal{M}^p$. By Theorem 12.6, choose a minimal immediate predecessor $\mathfrak{F}_{\beta+1}$ of $\mathfrak{Q}_\beta$, for each $\beta < \omega_1$. If $1 < \alpha < \omega_1$ and α is not of the excluded form, then $\alpha - 2$ is also an ordinal and thus $\{Q_\beta\}_{\beta\leqslant\alpha-2} \cup \{\mathfrak{F}_{\alpha-1}\}$ is a maximal chain of type α^*.

14. Increasing sequences. In contrast with the decreasing well-ordered maximal chains found in the preceding section, we now give a construction which shows that not every maximal chain is decreasing well-ordered. The construction depends on none of the preceding except the fact that every co-ideal contains a prime z-filter.

14.1. THEOREM. *For each $\mu < \omega_1$, there is an increasing μ-sequence of prime z-filters.*

Proof. For a closed set F, $(F^\alpha)_{\alpha<\omega_1}$ will denote the sequence of derived sets of F; i.e., $F^0 = F$, $F^{\alpha+1} = (F^\alpha)'$, for all α, and $F^\alpha = \bigcap_{\beta<\alpha} F^\beta$ if α is a limit ordinal. Choose, by [H], p. 279, a countable closed set F such that $F^\alpha \neq \emptyset$, for all $\alpha < \mu$. Define $D_\alpha = F^\alpha - F^{\alpha+1}$, for each α; thus $\mathrm{cl} D_\alpha = F^\alpha$, for all α.

For any closed set Z, we define

$$Z_0 = D_0 \cap Z\,, \quad \text{and} \quad Z_\alpha = D_\alpha \cap \big(\bigcap_{\beta<\alpha} \mathrm{cl} Z_\beta\big) \quad (0 < \alpha < \mu)\,.$$

For any closed sets Z and W, and for all $\alpha < \mu$, it follows by transfinite induction that

(a) $Z_\alpha \subseteq D_\alpha \cap Z$,

(b) $Z \subseteq W$ *implies* $Z_\alpha \subseteq W_\alpha$,

(c) $(Z \cup W)_\alpha = Z_\alpha \cup W_\alpha$,

(d) $Z_\alpha = \emptyset$ *implies* $Z_\beta = \emptyset$, *for* $\alpha \leqslant \beta < \mu$,

(e) $\mathbf{R}_\alpha = D_\alpha$.

Define

$$\mathcal{E} = \{Z \in \mathbf{Z}(\mathbf{R}):\ Z_\alpha \neq \emptyset,\ \text{all}\ \alpha < \mu\}\,.$$

Clearly $D_\alpha \neq \emptyset$, for all $\alpha < \mu$, so that $\mathbf{R} \in \mathcal{E}$, by (e), and thus $\mathcal{E} \neq \emptyset$; it is also clear that $\emptyset \notin \mathcal{E}$. If $Z \in \mathcal{E}$, $W \in \mathbf{Z}(\mathbf{R})$, and $Z \subseteq W$, then, by (b), also $W \in \mathcal{E}$. Now let $Z, W \in \mathbf{Z}(\mathbf{R})$ with $Z \notin \mathcal{E}$ and $W \notin \mathcal{E}$. Choose $\alpha, \beta < \mu$ with

$Z_\alpha = \emptyset$, $W_\beta = \emptyset$. Say $\beta \geqslant \alpha$. By (d), we have also $Z_\beta = \emptyset$. (This is the reason we define the sets Z_α recursively, rather than simply $Z_\alpha = D_\alpha \cap Z$.) Thus, by (c), $(Z \cup W)_\beta = \emptyset$ so that $Z \cup W \notin \mathcal{E}$. Hence $\mathcal{E}$ is a co-ideal in $\boldsymbol{Z}(\mathbf{R})$.

Choose, by 3.2(a), any prime z-filter $\mathcal{Q}$ contained in $\mathcal{E}$. If $\alpha < \mu$, and $Z \in \mathcal{Q}$, then $Z \cap F^\alpha \supseteq Z_\alpha \neq \emptyset$, by (a); hence F^α meets every member of $\mathcal{Q}$. Define $\mathcal{Q}_\alpha = (\mathcal{Q}, F^\alpha)$, for all $\alpha < \mu$.

If $\alpha < \beta < \mu$, then $F^\alpha \supset F^\beta$ so that $F^\alpha \in (\mathcal{Q}, F^\beta) = \mathcal{Q}_\beta$ and thus $\mathcal{Q}_\alpha \subseteq \mathcal{Q}_\beta$. Hence $(\mathcal{Q}_\alpha)_{\alpha<\mu}$ is nondecreasing.

Suppose $Q_\alpha = Q_\beta$ with $\alpha < \beta < \mu$. Thus $\mathcal{Q}_\alpha = \mathcal{Q}_{\alpha+1}$ and $F^{\alpha+1} \in (\mathcal{Q}, F^\alpha)$. It follows that we may choose $Z \in \mathcal{Q}$ such that $Z \cap F^\alpha \subseteq F^{\alpha+1}$, and thus $D_\alpha \cap Z = \emptyset$. By (a), $Z_\alpha = \emptyset$, contradicting the choice of Z. Hence $(\mathcal{Q}_\alpha)_{\alpha<\mu}$ is increasing.

Remark 1. It follows from Theorem 12.8, that every nonminimal prime z-filter is an element of a maximal chain that is not decreasing well-ordered.

Remark 2. It follows from Theorem 10.6, Remark 3, that $Q_{\alpha+1} = \mathcal{Q}_\alpha^+$, for all $\alpha < \mu$. Thus $(\mathcal{Q}_n)_{n<\omega}$ is a full increasing ω-sequence.

14.2. QUESTION. *Is there an increasing ω_1-sequence of prime z-filters?*

14.3. COROLLARY. *There is a prime z-filter that has an immediate predecessor but also a predecessor contained in no immediate predecessor.*

Proof. For any limit ordinal $\lambda < \mu$, the prime z-filter $\mathcal{F}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{Q}_\alpha$ has the countable member F_0. Hence, by Theorem 12.2, $\mathcal{F}_\lambda$ has an immediate predecessor. On the other hand, it is clear that no immediate predecessor of $\mathcal{F}_\lambda$ contains the predecessor $\mathcal{Q}$.

14.4. PROBLEM. *Give a necessary and sufficient condition, in terms of its members, that a prime z-filter be the union of a chain of predecessors.*

15. Union of predecessors. In this section we study conditions that a prime z-filter $\mathcal{Q}$ be the union of its predecessors. A necessary condition is that $\mathcal{Q}$ be closed (Corollary 10.4), and in the case that $\mathcal{Q}$ has a countable member, we show that this condition is also sufficient (Theorem 15.2). For the general case, a necessary and sufficient condition is given in Theorem 15.1. It is easily seen that a prime z-filter cannot be the union of finitely many of its predecessors.

15.1. THEOREM. *Q is the union of its predecessors if and only if for every $Z \in \mathcal{Q}$ there exists $F \in \mathcal{Q}$ such that $\mathrm{cl}(Z-F) \in \mathcal{Q}$.*

Proof. This follows immediately from Corollary 4.3.

15.2. THEOREM. *Let $\mathcal{Q}$ have a countable member. Then $\mathcal{Q}$ is the union of its predecessors if and only if it is closed.*

Proof. The necessity is a special case of Corollary 10.4. Now let $\mathcal{Q}$ be closed and choose a countable member F. Let $Z \in \mathcal{Q}$ and define $W = Z \cap F$. Since W is countable, $W = \mathrm{cl}(W - W')$ so that $\mathrm{cl}(W-W') \in \mathcal{Q}$; hence also $\mathrm{cl}(Z - W') \in \mathcal{Q}$, with $W' \in \mathcal{Q}$. The conclusion now follows from Theorem 15.1.

Remark. The examples of Remark 1 following Theorem 13.2 provide a decreasing ω_1-sequence of prime z-filters, each of which is both the intersection of its successors and the union of its predecessors.

15.3. QUESTION. *Is every closed nonminimal prime z-filter the union of its predecessors?*

15.4. THEOREM. *If $\mathcal{Q}$ is nonminimal and no associated ultrafilter has a countable member, then $\mathcal{Q}$ is the union of its predecessors.*

Proof. Let $Z \in \mathcal{Q}$. Choose a countable dense subset E of Z; thus E is a member of no associated ultrafilter. By Theorem 6.2, $\mathcal{Q}$ has a member F such that $\mathrm{cl}(E \cap F) \notin \mathcal{Q}$. Since $Z = \mathrm{cl}\,E = \mathrm{cl}(E \cap F) \cup \mathrm{cl}(E-F)$, we have $\mathrm{cl}(E-F) \in \mathcal{Q}$; hence also $\mathrm{cl}(Z-F) \in \mathcal{Q}$. By Theorem 15.1, $\mathcal{Q}$ is the union of its predecessors.

15.5. COROLLARY. *There exists a nonminimal free z-ultrafilter that is the union of its predecessors.*

Proof. It suffices to consider the example given in the remark following Question 12.3.

15.6. THEOREM. *Every nonminimal prime z-filter $\mathcal{Q}$ has a closed nonminimal predecessor $\mathcal{F}$ that is the union of its predecessors. If $\mathcal{Q}$ is nonclosed or has a countable member, then $\mathcal{F}$ may be chosen to be an immediate predecessor.*

Proof. By Corollary 8.2, it suffices to prove the second statement.

First let $\mathcal{Q}$ be nonclosed. Let $\mathcal{U}$ be the associated ultrafilter and choose a discrete set E in $\mathcal{U}$. Define $F = \mathrm{cl}\,E$, and choose a family $\{S_x\}_{x\in E}$ of sets of the form $B_0(x, r_x)$, with $S_x \cap F = \emptyset$, in such a way that $\mathrm{diam}\,S_x \to 0$. For each $x \in E$, choose a free z-ultrafilter $\mathcal{F}_x$ on S_x, converging to x, such that for any countable set J, there is a member of $\mathcal{F}_x$ disjoint from J. (It suffices to show that this may be done for the case $x = 0$. Let $\mathcal{M}$ be the z-ultrafilter obtained in the remark following Question 12.3, let $\mathcal{R}$ be the image of $\mathcal{M}|(\mathbf{R}-\{0\})$ under the mapping $t \to 1/t$, and define $\mathcal{F}_0 = \mathcal{R}|S_0$.) Let $\mathcal{F}$ be the immediate predecessor of $\mathcal{Q}$ obtained by Theorem 12.1. Let $J \subseteq \mathbf{R}$ be countable. For each $x \in E$, choose $Z_x \in \mathcal{F}_x$ with $J \cap Z_x = \emptyset$. Define $W = \mathrm{cl}(\bigcup_{x\in E} Z_x)$; thus $W^* = E$ so that $W \in \mathcal{F}$. Since $J \cap W \subseteq F$ with $F \notin \mathcal{F}$, we have $\mathrm{cl}(J \cap W) \notin \mathcal{F}$, so that J is a member of no ultrafilter associated with $\mathcal{F}$. Hence $\mathcal{F}$ has no associated ultrafilter with a countable member. By 9.3, $\mathcal{F}$ is nonminimal; by Theorem 10.1, $\mathcal{F}$ is closed; and by Theorem 15.4, $\mathcal{F}$ is the union of its predecessors.

Now let $\mathcal{Q}$ have a countable member F. For each $x \in F$, choose a free z-ultrafilter $\mathcal{Q}_x$ on $T_x = B_0(x, 1)$, converging to x, such that every countable subset of T_x is disjoint from some member of $\mathcal{Q}_x$. For each $x \in F$, choose $S_x \in \mathcal{Q}_x$ such that $S_x \cap F = \emptyset$, in such a way that $\operatorname{diam} S_x \to 0$. Define $\mathcal{F}_x = \mathcal{Q}_x | S_x$. We now apply Theorem 12.1 with $E = F$, and proceed as in the first case.

Remark 1. By 12.4 and Corollary 8.2, we can find a nonminimal nonclosed prime z-filter with no countable member. By Remark 1 following Theorem 13.2, we can find a closed nonminimal prime z-filter with a countable member. Hence the two alternative conditions used are independent.

Remark 2. Theorem 15.6 yields an example of a closed nonminimal prime z-filter with an immediate successor, and provides a counterexample to the converse of Corollary 10.4(a).

15.7. Question. *When a prime z-filter is the union of all its predecessors, does it follow that it is the union of a chain of predecessors?*

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UNIVERSITY OF ROCHESTER

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A lower bound for transfinite dimension

by

David W. Henderson (Ithaca, N. Y.)

1. Introduction. In this paper, essential mappings are used to give a lower bound for the (large, strong) transfinite inductive dimension of a space. Transfinite inductive dimension (Ind) is defined by transfinite induction as follows: (See [3], p. 161).

Definition 1. (a) $\operatorname{Ind}(R) = -1$, if $R = \emptyset$. (b) $\operatorname{Ind}(R) \leqslant \alpha$ (an ordinal number) if every pair of disjoint closed subsets of R can be separated by a closed subset S such that $\operatorname{Ind}(S) < \alpha$. (*$S$ separates A and B in R* if $R - S$ is the union of disjoint open (in R) sets U, V such that $A \subset U$ and $B \subset V$.) (c) $\operatorname{Ind}(R) = \alpha$ if $\operatorname{Ind}(R) \leqslant \alpha$ and it is not true that $\operatorname{Ind}(R) < \alpha$. (d) R is said to have *transfinite dimension* (Ind) if $\operatorname{Ind}(R)$ exists.

It is known ([3], p. 209) that if a normal space R has an essential mapping onto the n-cell, I^n, then $\operatorname{Ind}(R) \geqslant n$ or $\operatorname{Ind}(R)$ does not exist. (A mapping $f\colon R \to I^n$ is *essential* if there does not exist a mapping $g\colon R \to \operatorname{Bd}(I^n)$ (Bd = Boundary) such that $f|f^{-1}(\operatorname{Bd}(I^n)) = g|f^{-1}(\operatorname{Bd}(I^n))$.) We shall construct, for each countable ordinal α, a space J^α such that (with "essential" suitably defined), if a normal space R has an essential mapping onto J^α, then $\operatorname{Ind}(R) \geqslant \alpha$ or $\operatorname{Ind}(R)$ does not exist.

Some of the ideas behind the definition of J^α and the proofs below can be found in [1], § 3, by Yu. M. Smirnov (Ю. М. Смирнов).

2. Results and questions.

Definition 2 (J^α, T^α, and p^α). For each ordinal number α, greater than or equal to 0 and less than Ω (the first uncountable ordinal), we shall define a compact metric set J^α with a compact subset T^α and a point $p^\alpha \in T^\alpha$. Let $J^0 = T^0 = p^0 =$ (a point). If α is positive and finite, then let J^α be the α-dimensional cube, T^α be the $(\alpha-1)$-dimensional sphere which is the combinatorial boundary of J^α, and p^α be any point of T^α. If α is not a limit ordinal, then define

$$J^\alpha = J^{\alpha-1} \times J^1, \quad T^\alpha = (T^{\alpha-1} \times J^1) \cup (J^{\alpha-1} \times T^1), \quad \text{and} \quad p^\alpha = p^{\alpha-1} \times p^1 .$$

If α is a limit ordinal, then let K^β, for $\beta < \alpha$, be the union of J^β and a half-open arc A^β, such that $A^\beta \cap J^\beta = p^\beta =$ (the end point of A^β); and define J^α