A note on Rouché's theorem

by

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In this note we will prove a generalized version of Rouché's theorem—the converse of this generalization will also be proved. Following this two closely related results will be demonstrated. These deal with analytic functions having an unequal number of zeros inside a closed curve. We hope that, in spirit at least, this note has some points of contact with Kuratowski's paper [2].

The classical Rouché's theorem (see for example [1], p. 254) states the following:

If C is a scroc (scroc = *simple closed rectifiable oriented curve) and if f and g are analytic functions on the set C^* consisting of the union of C and its interior, with $|f(z)| > |g(z)-f(z)|$ for $z \in C$, then f and g have the same number of zeros inside C.*

This theorem can be proved by observing that

$$F(t) = \frac{1}{2\pi i}\int_C \frac{f'(z)+t\big(g'(z)-f'(z)\big)}{f(z)+t\big(g(z)-f(z)\big)}\,dz$$

is continuous for $t \in [0, 1]$, and is an integer equal to the number of zeros of $f(z)+t\big(g(z)-f(z)\big)$ inside C. Therefore $F(t)$ is a constant equal to $F(0)$ and $F(1)$.

Now suppose C is a scroc and f and g are analytic functions on C^*. If G is any region of the complex plane which contains C^* and on which f and g are analytic, then a continuous complex valued function H defined on $G\times[0, 1]$ is called a *homotopy joining f and g* if it satisfies the following two conditions:

(a) *$H(z, t)$ is analytic on G for each $t \in [0, 1]$,*

(b) *$H(z, 0) = f(z)$ and $H(z, 1) = g(z)$.*

Should H also satisfy

(c) $0 \notin H(C\times[0, 1])$,

then H will be called *non-vanishing on C*.

If H has zeros on C only for $t = t_i$, where $0 < t_1 < \ldots < t_r < 1$, and if m_i is the number of zeros of $H(z, t_i)$ on C, we shall say that H *vanishes to order* $\sum_1^r m_i$ *on* C. In the case where H vanishes on C for infinitely many values of t we will say that H *vanishes to order* ∞ *on* C.

In what follows we will occasionally use an interval other than $[0, 1]$ for t and also make tacit use of the fact that homotopy is an equivalence relation.

The symbol $N_C(f)$ will designate the number of zeros, counting multiplicities, of f interior to the scroc C.

THEOREM 1. *Let C be a* scroc *and let f and g be analytic on C^*. Then there exists a homotopy joining f and g and non-vanishing on C if and only if f and g have the same number of zeros inside C and none on C.*

Proof. Let H be a homotopy joining f and g which is non-vanishing on C. We will show that

$$\frac{1}{2\pi i}\int_C \frac{H_z(z, t)}{H(z, t)}\,dz$$

is a continuous function of t: then the argument described which proves Rouché's theorem will prove that $N_C\big(H(z, 0)\big) = N_C\big(H(z, 1)\big)$, i.e. that $N_C(f) = N_C(g)$.

Let $G\times[0, 1]$ be the domain of H and let K be any compact subset of G. Since H is uniformly continuous on $K\times[0, 1]$, if $t_n\to t_0$ then $H(z, t_n)\to H(z, t_0)$ uniformly on K. By a well-known theorem of Weierstrass, $H_z(z, t_n)\to H_z(z, t_0)$ uniformly on compact subsets of G. Consequently

$$\frac{H_z(z, t_n)}{H(z, t_n)} \to \frac{H_z(z, t_0)}{H(z, t_0)} \quad \text{uniformly on } C$$

since $H(z, t_0)$ is non-vanishing on C. The integral is therefore a continuous function of t.

Now, to prove the converse, suppose that f and g have the same number of zeros inside C and none on C. The non-vanishing homotopy will be inductively constructed.

To begin consider the case where $N_C(f) = N_C(g) = 0$. In this case, let $h(z)$ be an analytic branch of $\log\big(g(z)/f(z)\big)$ on a region G containing C^*. Then the function on $G\times[0, 1]$ defined by

$$H(z, t) = f(z)\,e^{th(z)}$$

is clearly a homotopy joining f and g and is non-vanishing on C.

The case where $N_C(f) = N_C(g) = 1$ must be handled next. Let a and b be the simple zeros inside C of f and g, respectively. Then $f(z) = (z-a)f_1(z)$ and $g(z) = (z-b)g_1(z)$ where $N_C(f_1) = N_C(g_1) = 0$. Should $b = a$ then $K(z, t) = z-a$ is a homotopy joining $z-a$ to $z-a$ and non-vanishing on C. Should $b \neq a$ then, for $0 \leqslant t \leqslant 1$, let $\gamma(t)$ be a continuous curve in the interior of C for which $\gamma(0) = a$ and $\gamma(1) = b$. Then

$$K(z, t) = \frac{b-\gamma(t)}{b-a}(z-a) + \frac{\gamma(t)-a}{b-a}(z-b)$$

is a homotopy joining $z-a$ and $z-b$. If $K(z, t)$ vanished for some $z \in C$ we would have

$$[b-\gamma(t)](z-a) + [\gamma(t)-a](z-b) = (b-a)z - (b-a)\gamma(t) = 0\,.$$

This is impossible since $\gamma(t)$ is interior to C, and thus K is non-vanishing. We have previously shown there is a non-vanishing homotopy joining f_1 and g_1. Multiply it by K and we have the required non-vanishing homotopy joining f and g.

To complete the induction we will assume that the theorem is true for all f and g of the type under consideration whenever $N_C(f) = N_C(g) < k$. Suppose now that $N_C(f) = N_C(g) = k$. Let a and b be zeros inside C of f and g respectively, and write $f(z) = (z-a)f_1(z)$ and $g(z) = (z-b)g_1(z)$. Then $N_C(f_1) = N_C(g_1) < k$. The inductive hypothesis gives us a non-vanishing homotopy joining f_1 and g_1. We have also constructed a non-vanishing homotopy joining $z-a$ and $z-b$. Their products is the required non-vanishing homotopy joining f and g.

We will now consider the case where $N_C(f) \neq N_C(g)$.

THEOREM 2. *Let C be a* scroc *and let f and g be analytic functions on C^* having no zeros on C. If $N_C(f) = m$ and $N_C(g) = m+k$, then there exists a homotopy joining f and g and vanishing on C to order k. Furthermore, the homotopy can be chosen so as to have one selected zero on C of multiplicity k.*

Proof. Let z_0 be a point on C and write $g(z) = (z-r_1)\ldots(z-r_k)h(z)$ where the r_i's are zeros of g inside C and $N_C(h) = m$. By Theorem 1 there is a homotopy H_0 joining f and h which is non-vanishing on C. We will produce a homotopy H_i joining the constant function 1 and $z-r_i$ and which vanishes to order 1 at $(z, t) = (z_0, \frac{1}{2})$. Then $\prod_{i=0}^{k} H_i$ will be the homotopy sought.

Let $a = \inf_{z\in C}\{\operatorname{Re} z\}$ and let $\zeta(t)$ be a continuous curve for $\frac{1}{4} \leqslant t \leqslant 1$ for which

$$\zeta(\tfrac{1}{4}) = a-1\,, \quad \zeta(\tfrac{1}{2}) = z_0\,, \quad \zeta(1) = r_i\,, \quad \text{and} \quad \zeta(t) \notin C \text{ for } t \neq \tfrac{1}{2}\,.$$

Then

$$H_i(z, t) = \begin{cases} 1+4t(z-a)\,, & 0 \leqslant t \leqslant \frac{1}{4}\,,\\ z-\zeta(t)\,, & \frac{1}{4} \leqslant t \leqslant 1\end{cases}$$

does exactly what is required.

That the homotopy shown to exist in Theorem 2 cannot vanish to an order lower than k is asserted by the next theorem, which is essentially a converse of Theorem 2.

THEOREM 3. *Let C be a* scroc *and let f and g be analytic functions on C^* having no zeros on C. If there is a homotopy $H(z,t)$ joining f and g and vanishing to order k on C, then $|N_C(f)-N_C(g)| \leqslant k$.*

Proof. Let the domain of $H(z,t)$ be $G\times[0,1]$, where G is a region containing C^*.

Case 0. If $k=\infty$ the theorem is trivially true.

Case 1. Suppose that $H(z,t)$ has zeros on C for $t=\tilde{t}$ only, where $0<\tilde{t}<1$. Let these zeros be m in number. Set $h(z)=H(z,\tilde{t})$. Now let D and E be scrocs satisfying the following conditions:

(1) *The interior of E is contained in the interior of C.*

(2) *The interior of C is contained in the interior of D.*

(3) *The set D^* (consisting of the union of D and its interior) is contained in G.* (Hence for each t, $H(z,t)$ is analytic on D^*.)

(4) *The only zeros h has between D and E are on C and h has no zeros on D or E.*

Further by the continuity of H, there exists an $\varepsilon>0$ such that if $|t-\tilde{t}|\leqslant\varepsilon$ then $H(z,t)$ never vanishes on D or E. Let

$$h_1(z)=H(z,\tilde{t}-\varepsilon) \quad\text{and}\quad h_2(z)=H(z,\tilde{t}+\varepsilon)\,.$$

By interchanging h_1 and h_2 if necessary, we may suppose $N_C(h_1)\geqslant N_C(h_2)$. Theorem 1 tells us that

$$N_D(h_1)=N_D(h) \quad\text{and}\quad N_E(h_2)=N_E(h)\,.$$

Therefore

$$|N_C(h_1)-N_C(h_2)|=N_C(h_1)-N_C(h_2)\leqslant N_D(h_1)-N_E(h_2)$$
$$=N_D(h)-N_E(h)=m\,.$$

Again applying Theorem 1,

$$N_C(f)=N_C(h_1) \quad\text{and}\quad N_C(g)=N_C(h_2)$$

so we have

$$|N_C(f)-N_C(g)|\leqslant m\,.$$

Case 2. Suppose that $0<t_1<\ldots<t_r<1$ and $H(z,t)$ has its zeros on C for precisely these t_i's. Denote the number of zeros on C of $H(z,t_i)$ by m_i so $\sum_1^r m_i=k$. Let

$$\tau_0=0<t_1<\tau_1<t_2<\ldots<t_i<\tau_i<t_{i+1}<\ldots<t_r<\tau_r=1$$

and $h_i(z)=H(z,\tau_i)$. The h_i's have no zeros on C. By case 1,

$$|N_C(h_i)-N_C(h_{i+1})|\leqslant m_{i+1}$$

so

$$|N_C(f)-N_C(g)|=|N_C(h_0)-N_C(h_r)|\leqslant\sum_{i=0}^{r-1}|N_C(h_i)-N_C(h_{i+1})|\leqslant\sum_1^r m_i=k\,.$$

References

[1] E. Hille, *Analytic function theory*, Blaisdell, New York 1959.

[2] C. Kuratowski, *Homotopie et fonctions analytiques*, Fund. Math. 33 (1945), pp. 316-367.

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